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OBSERVATIONS IN HOMOGENEOUS MODEL UNIVERSES

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Summary

Methods are derived for the exact calculation of the luminosity distance, distance by apparent size, number counts, background radiation and the apparent angular motion of sources in an homogeneous but anisotropic model universe. These expressions can be used in any model universe for which the first integrals of the equations of the null geodesics are known.

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1. *Introduction.* In the past few years a number of authors (e.g. Zel'dovich 1964; Kompaneets & Chernov 1964; Thorne 1967a; Kantowski & Sachs 1966) have discussed some spatially homogeneous but anisotropic cosmological models. These models admit three or four parameter groups of motions transitive on space-like minimum invariant varieties. The techniques used in constructing these models are those of Bianchi (1918), Taub (1951), Petrov (1961), and Heckmann & Schücking (1962). The most convenient procedure is to identify the minimum invariant varieties of the group of motions with the hypersurfaces t = const. One can then let t, which will be called 'cosmic time' (but which will not necessarily be the proper time of the fundamental particles) measure the distance from some initial hypersurface and, then by taking advantage of the symmetries in the hypersurfaces t = const., the metric of the space time can be written in the form*

$$ds^2 = dt^2 - \gamma_{ab}(t)\, e_i{}^a e_j{}^b dx^i dx^j. \tag{1}$$

The $e_i{}^a$ (which are directly related to the vectors of the reciprocal group) are determined by the group of motions, so that the field equations reduce to a set of ordinary differential equations for the $\gamma_{ab}(t)$. The co-ordinates (t, x^i), which will be called 'cosmic co-ordinates' and denoted by (x^α) are clearly the most natural co-ordinates to use in studying these homogeneous space times.

In the present work expressions for the observations in homogeneous model universes are derived. Series approximations to these expressions have been derived by Kristian & Sachs (1966) but the only result in closed form appears to be that of Kermack, McCrea & Whittaker (1932) who showed that if u^α is the four-velocity of a source or observer and $k^\alpha = dx^\alpha/dv$ is the tangent to a light ray with v an affine parameter, then using subscripts s and o to denote measurements made at the source and observer respectively, the red shift is given by

$$1+z = (u^\alpha k_\alpha)_s/(u^\beta k_\beta)_o. \tag{2}$$

In this paper expressions are derived for the distance by apparent size, luminosity distance, number counts, background radiation intensity and apparent angular motion of sources.

2. *Application of the Ehlers-Sachs theorem.* Interpretation of many of the observations in cosmology requires a knowledge of the area of cross section of a

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pencil of null geodesics representing radiation diverging from some event in space-time. It follows from the Ehlers-Sachs theorem (Jordan, Ehlers & Sachs 1961) that to find this area one need only evaluate the integral

$$\int k^{\alpha}{}_{;\alpha}\, dv,$$

where k^{α} is the tangent vector to the central ray of the pencil. In this section a method will be derived for performing this calculation for any homogeneous space-time.

Homogeneity implies the existence of three Killing vectors, each of which must satisfy the equation

$$\xi_a{}^{\alpha} k_{\alpha} = \text{const.}, \tag{3}$$

along each (affinely parametrized) null geodesic. These three relations between the k^{α}, together with

$$k^{\alpha} k_{\alpha} = 0 \tag{4}$$

determine k^{α} up to a factor. The three constants of equations (3) serve to determine the direction and energy of radiation at an event, so if one considers radiation of a fixed frequency only, only two of the constants are independent.

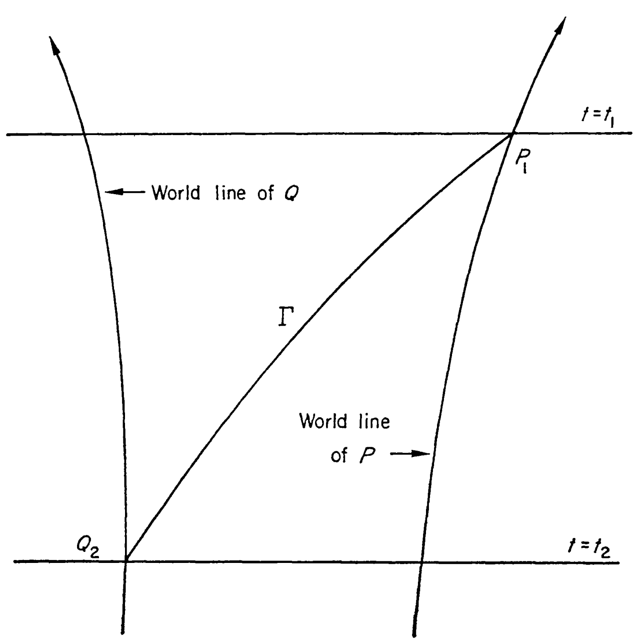

Fig. 1

In the cosmic co-ordinate system (1), unfortunately, the constants of equation (3) will in general vary from geodesic to geodesic within the pencil, and this makes the calculation of the partial derivatives $\partial k^{\alpha}/\partial x^{\alpha}$ difficult. It is therefore convenient to introduce a second co-ordinate system. Let P be an observer moving along some arbitrarily chosen world line and suppose that along this world line is propagated an orthonormal triad of space-like vectors, relative to which P measures local phenomena. It is convenient, though not essential, to suppose that the triad is Fermi propagated along the world line. Let Q be a source moving along some other world line and suppose that Q_2, the event on the world line of Q with cosmic time

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co-ordinate t_2, lies on the past null cone of P_1, the event on the world line of P with cosmic time co-ordinate t_1. In other words, suppose P observes Q_2 at time t_1. Assign to Q_2 geodesic polar co-ordinates $(\bar{x}^\alpha) = (t_2, \theta, \phi, t_1)$ where θ and ϕ are the polar angles of the direction in which P observes Q to lie, relative to the orthonormal triad. This co-ordinate system will be referred to as ' observer's polar co-ordinates '; except for a change in parametrization they are the same as the ' optical co-ordinates ' of Joseph (1958). Note that in observer's polar co-ordinates Q_2 lies on a null geodesic Γ with $\bar{x}^a = \text{const.}$

If one imagines photons to be streaming towards P from all events in the space-time, one may think of them as forming a fluid whose streamlines are the null geodesics. The cosmic co-ordinates are Eulerian co-ordinates for this fluid, while the observer's polar co-ordinates are the Lagrangian co-ordinates. It is well known that in the study of compressible flow the dilatation of an infinitesimal volume is given by the Jacobian of the transformation from Eulerian to Lagrangian co-ordinates; it will now be shown that a similar result holds for the area of cross-section of a pencil of null geodesics.

Let v be an affine parameter along Γ and again let $k^\alpha = dx^\alpha/dv$. On Γ, $d\bar{x}^a/dv = 0$ so that

$$\frac{d}{dv} = k^0 \frac{\partial}{\partial \bar{x}^0}. \tag{5}$$

Then

$$\frac{d}{dv}\left(\frac{\partial x^a}{\partial \bar{x}^b}\right) = k^0 \frac{\partial}{\partial \bar{x}^b}\left(\frac{\partial x^a}{\partial \bar{x}^0}\right),$$

$$= k^0 \frac{\partial}{\partial \bar{x}^b}\left(\frac{1}{k^0}\frac{dx^a}{dv}\right) = \frac{\partial k^a}{\partial \bar{x}^b} - \frac{k^a}{k^0}\frac{\partial k^0}{\partial \bar{x}^b}. \tag{6}$$

Let J be the Jacobian of the transformation from cosmic to observer's polar co-ordinates; then

$$J \equiv \frac{\partial(x^\alpha)}{\partial(\bar{x}^\beta)} \equiv \epsilon_{\alpha\beta\gamma\delta}\frac{\partial x^\alpha}{\partial \bar{x}^1}\frac{\partial x^\beta}{\partial \bar{x}^2}\frac{\partial x^\gamma}{\partial \bar{x}^3}\frac{\partial x^\delta}{\partial \bar{x}^0} = \epsilon_{abc}\frac{\partial x^a}{\partial \bar{x}^1}\frac{\partial x^b}{\partial \bar{x}^2}\frac{\partial x^c}{\partial \bar{x}^3} \tag{7}$$

and with

$$A = \epsilon_{abc}\left[\frac{\partial k^a}{\partial \bar{x}^1}\frac{\partial x^b}{\partial \bar{x}^2}\frac{\partial x^c}{\partial \bar{x}^3} + \frac{\partial x^a}{\partial \bar{x}^1}\frac{\partial k^b}{\partial \bar{x}^2}\frac{\partial x^c}{\partial \bar{x}^3} + \frac{\partial x^a}{\partial \bar{x}^1}\frac{\partial x^b}{\partial \bar{x}^2}\frac{\partial k^c}{\partial \bar{x}^3}\right] \tag{8}$$

and

$$B = -\frac{\epsilon_{abc}}{k^0}\left[k^a \frac{\partial k^0}{\partial \bar{x}^1}\frac{\partial x^b}{\partial \bar{x}^2}\frac{\partial x^c}{\partial \bar{x}^3} + k^b \frac{\partial x^a}{\partial \bar{x}^1}\frac{\partial k^0}{\partial \bar{x}^2}\frac{\partial x^c}{\partial \bar{x}^3} + k^c \frac{\partial x^a}{\partial \bar{x}^1}\frac{\partial x^b}{\partial \bar{x}^2}\frac{\partial k^0}{\partial \bar{x}^3}\right], \tag{9}$$

$$\frac{dJ}{dv} = A + B. \tag{10}$$

From the relations

$$0 = \frac{d\bar{x}^a}{dv} = \frac{\partial \bar{x}^a}{\partial x^b}k^b + \frac{\partial \bar{x}^a}{\partial x^0}k^0, \tag{11}$$

$$\frac{\partial \bar{x}^1}{\partial x^a} = J^{-1}\epsilon_{abc}\frac{\partial x^b}{\partial \bar{x}^2}\frac{\partial x^c}{\partial \bar{x}^3}, \tag{12}$$

it follows that

$$B = -\frac{J}{k^0}\left[k^a \frac{\partial k^0}{\partial \bar{x}^1}\frac{\partial \bar{x}^1}{\partial x^a} + k^b \frac{\partial k^0}{\partial \bar{x}^2}\frac{\partial \bar{x}^2}{\partial x^b} + k^c \frac{\partial k^0}{\partial \bar{x}^3}\frac{\partial \bar{x}^3}{\partial x^c}\right],$$

$$= J \frac{\partial k^0}{\partial \bar{x}^b}\frac{\partial \bar{x}b}{\partial x^0} \tag{13}$$

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and since $\partial x^0/\partial \bar{x}^0 = 1$,

$$B = J\left(k^0{}_{,0} - \frac{\partial k^0}{\partial \bar{x}^0}\right) = J\left(k^0{}_{,0} - \frac{1}{k^0}\frac{dk^0}{dv}\right). \tag{14}$$

Also, from equation (12),

$$\frac{\partial k^a}{\partial x^a} = \frac{\partial k^a}{\partial \bar{x}^\beta}\frac{\partial \bar{x}^\beta}{\partial x^a} = \frac{\partial k^a}{\partial \bar{x}^b}\frac{\partial \bar{x}^b}{\partial x^a} = J^{-1}A. \tag{15}$$

From equations (10), (14) and (15),

$$\frac{1}{J}\frac{dJ}{dv} = k^a{}_{,a} + k^0{}_{,0} - \frac{1}{k^0}\frac{dk^0}{dv} \tag{16}$$

and since $\Gamma_{\alpha\beta}{}^{\alpha} = \partial(\log\sqrt{-g})/\partial x^\beta$ it follows that

$$\exp\left\{\int_{t_1}^{t_2} k^\alpha{}_{;\alpha}\, dv\right\} = \frac{J_2 t_2'\sqrt{-g_2}}{J_1 t_1'\sqrt{-g_1}} \equiv \mathscr{J}_2/\mathscr{J}_1, \tag{17}$$

where a subscript 1 indicates that the quantity is to be evaluated at time $t = t_1$ and so on; a prime denotes differentiation with respect to v.

This result holds generally. If, as in the case of homogeneous models, the explicit form of k^α is known, an explicit calculation of $\mathscr{J}$ is quite straightforward.

As the observer P moves along his world line, the change with respect to P's proper time τ_P of the cosmic co-ordinates of a source Q is given by

$$\frac{dx^\alpha}{d\tau_P} = \frac{\partial x^\alpha}{\partial t_1}\frac{dt_1}{d\tau_P} + \frac{\partial x^\alpha}{\partial \theta}\frac{d\theta}{d\tau_P} + \frac{\partial x^\alpha}{\partial \phi}\frac{d\phi}{d\tau_P} + \frac{\partial x^\alpha}{\partial t_2}\frac{dt_2}{d\tau_P}. \tag{18}$$

The three non-trivial equations (18) may now be solved for the three unknowns $dt_1/d\tau_P$, $d\theta/d\tau_P$, $d\phi/d\tau_P$. The coefficients are found in the following manner.

Since

$$u^\alpha = dx^\alpha/d\tau \tag{19}$$

and

$$1 + z = d\tau_Q/d\tau_P, \tag{20}$$

where τ_Q is the proper time of the source Q,

$$\frac{dx^\alpha}{d\tau_P} = (u^\alpha)_Q(1+z). \tag{21}$$

Also from equation (5) it follows that

$$\frac{\partial x^a}{\partial t_2} = (k^a/k^0)_Q. \tag{22}$$

Then since $\partial x^0/\partial \bar{x}^a = 0$ and since $t_2 = x^0$ may be used as a (non-affine) parameter along the null geodesics

$$\frac{\partial}{\partial \theta}[k^a/k^0] = \frac{\partial}{\partial \theta}[dx^a/dt_2] = \frac{d}{dt_2}[\partial x^a/\partial \theta] \tag{23}$$

so that by differentiating with respect to θ the differential equation for x^a as a function of t_2 (*not* of v) one obtains a first order ordinary differential equation for $\partial x^a/\partial \theta$; an equation for $\partial x^a/\partial \phi$ is obtained similarly.

If one now solves equations (18) by determinants for $dt_1/d\tau_P$ and then uses the

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fact that $dt_1/d\tau_P$ is simply $(u^0)_P$ one obtains

$$(1+z)\begin{vmatrix} \left(u^1-\frac{k^1}{k^0}u^0\right)_2 & \frac{\partial x^1}{\partial\theta} & \frac{\partial x^1}{\partial\phi} \\ \left(u^2-\frac{k^1}{k^0}u^0\right)_2 & \frac{\partial x^2}{\partial\theta} & \frac{\partial x^2}{\partial\phi} \\ \left(u^3-\frac{k^3}{k^0}u^0\right)_2 & \frac{\partial x^3}{\partial\theta} & \frac{\partial x^3}{\partial\phi} \end{vmatrix} = \begin{vmatrix} \frac{\partial x^1}{\partial t_1} & \frac{\partial x^1}{\partial\theta} & \frac{\partial x^1}{\partial\phi} \\ \frac{\partial x^2}{\partial t_1} & \frac{\partial x^2}{\partial\theta} & \frac{\partial x^2}{\partial\phi} \\ \frac{\partial x^3}{\partial t_1} & \frac{\partial x^3}{\partial\theta} & \frac{\partial x^3}{\partial\phi} \end{vmatrix} (u^0)_P = (u^0)_P J. \tag{24}$$

Thus the Jacobian can be evaluated without the $\partial x^a/\partial t_1$ being known.

(These quantities are needed to find the components of the observed angular velocity of Q about P, $d\theta/d\tau_P$ and $d\phi/d\tau_P$. In this case one must consider the congruence of null geodesics which intersect the world line of P at constant observed angles θ, ϕ. Then one must consider an infinitesimal displacement vector $q^\alpha = (u^\alpha)\, d\tau_P$ and Lie propagate it back along the congruence to time $t = t_2$ by solving the equation

$$Dq^\alpha/Dv = k^\alpha{}_{;\,\beta}\, q^\beta. \tag{25}$$

The $\partial x^\alpha/\partial t_1$ are then given by

$$\frac{\partial x^\alpha}{\partial t_1} = \left(\frac{\partial x^\alpha}{\partial \tau_P}\right)_{\theta,\ \phi,\ t_2} \Big/ \frac{dt_1}{d\tau_P}, \tag{26}$$

where

$$\left(\frac{\partial x^\alpha}{\partial \tau_P}\right)_{\theta,\ \phi,\ t_2} d\tau_P = \left[q^\alpha - \frac{k^\alpha}{k^0}q^0\right]_Q. \tag{27}$$

In general this calculation is very laborious, particularly since the congruence k^α is not the same as the one used for the other calculations.)

In the next three sections, the results here obtained will be used to obtain expressions for the distance by apparent size, luminosity distance, number counts, and background radiation density.

3. *Distance.* Suppose an extended object whose area is known to be A is observed to subtend a solid angle $\delta\Omega$. Then the distance by apparent size ξ of the object is defined by

$$A = \xi^2 \delta\Omega. \tag{28}$$

In order to be able to write down an expression for ξ it is necessary to adapt equation (17) to deal with the case of a pencil of geodesics diverging from a single event P_1. Let

$$A(t_2) = \mathrm{K}^{-2}\mathscr{J}_2\delta\Omega, \tag{29}$$

then the area of cross section at parameter distance dv from P_1 is given by

$$A(dv) = (u^\alpha k_\alpha)_1{}^2\,(dv)^2\,\delta\Omega = \mathrm{K}^{-2}\mathscr{J}(dv)\,\delta\Omega. \tag{30}$$

Hence

$$\mathrm{K} = [(u^\alpha k_\alpha)^{-1}\, d\sqrt{\mathscr{J}}/dv]_1, \tag{31}$$

since $\mathscr{J}(0) = 0$. Thus a source which emits radiation at time t_s and is observed at time t_o is at a distance by apparent size

$$\xi = \sqrt{\mathscr{J}_s}(u^\alpha k_\alpha)_o/(d\sqrt{\mathscr{J}}/dv)_o \tag{32}$$

Let the luminosity distance, D, be defined by the property that the observed

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intensity of a light source is inversely proportional to the square of D. Penrose (1966) has shown that*

$$D = (1+z)^2 \xi. \tag{33}$$

Then from equations (32) and (33) it follows that

$$D = \frac{\sqrt{\mathscr{J}}_s(1+z)^2 (u^\alpha k_\alpha)_o}{(d\sqrt{\mathscr{J}}/dv)_o}. \tag{34}$$

4. *Number counts.* The cross-sectional area of a pencil of null geodesics diverging from an observer is given by equations (28) and (32), and the length of the (perpendicular, by the Ehlers-Sachs theorem) vector along the space projection of k^α into the instantaneous three-space of a source with velocity $(u^\alpha)_s$ and corresponding to a parameter distance dv is $(u^\alpha k_\alpha)_s dv$; therefore the three-volume enclosed by the pencil of geodesics and containing sources whose universal time co-ordinates lie in the interval (t_s, t_s+dt_s) is

$$dV = (u^\alpha k_\alpha/t')_s\, \xi^2 \delta\Omega dt_s \tag{35}$$

and the number of sources contained in the solid angle whose universal time co-ordinates lie in the given interval is

$$dN = \psi(t)\, n_o(\rho_s/\rho_o)\, (u^\alpha k_\alpha/t')_s\, \xi^2 \delta\Omega dt_s \tag{36}$$

where ρ is the density of matter, n_o the present number density of sources, and the function $\psi(t)$ represents the possibility that at different epochs the ratio between the density of matter and the number density of sources may be quite different from what it is now.

5. *Background radiation.* Suppose that the average intensity of radiation emitted by a source at time t in the frequency interval $(\nu, \nu+d\nu)$ is $L(\nu, t)\, d\nu$. Then the total flux received by an observer at time t_o from sources lying within the solid angle $\delta\Omega$ and having emission times in the range (t_s, t_s+dt_s) is

$$L(\nu_s, t_s) D^{-2}\, (dN/dt_s)\, d\nu_s\, dt_s \frac{\delta\Omega}{4\pi} \tag{37}$$

so that the total incident flux at frequency ν_o is given by

$$I(\nu_o, t_o)\, d\nu_o = \int_{t_o}^{t_m} L((1+z)\, \nu_o, t_s)\, D^{-2}(dN/dt_s)\, (1+z)\, dt_s d\nu_o \tag{38}$$

where t_m is the earliest epoch from which radiation is received. On account of equations (34) and (36), equation (38) may be written in the form

$$I(\nu_o, t_o)\, d\nu_o = (u^\alpha k_\alpha)_o \int_{t_o}^{t_m} \frac{n(t)\, F((1+z)\, \nu_o, t)}{t'(1+z)^2}\, dt d\nu_o \tag{39}$$

where $F(\nu, t) = \psi(t)\, L(\nu, t)$ is the ‘ radiation density function ’ and $n(t) = n_o \rho_s/\rho_o$.

In order to evaluate the above expression it is not necessary to know the Jacobian of equation (17). Hence this result is useful even for models for which the explicit form of the k^α is not known. On the other hand, the usefulness of the result is limited by its dependence on the radiation density function which may be very difficult to estimate; the difficulty is greater in anisotropic models because it is not possible to adopt the usual device of studying the dependence of this function on the red shift, which is directly observable, rather than on the emission time.

* This result was originally conjectured by Kristian & Sachs (1966).

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The simplest plausible assumption about the radiation density function is to assume that there is no time dependence apart from a possible cut-off at time t_m and that measurements are being made in the radio frequency range, so that the rate of radiation into a given frequency interval $d\nu_s$ is given by

$$B(\nu_s) = (\nu_s/\nu_o)^x B(\nu_o) \tag{40}$$

where x is the spectral index and $B(\nu_o)$ is the rate of emission at some arbitrarily chosen frequency ν_o. Equation (39) may then be written

$$I(\nu_o, t_o)\, d\nu_o = (u^\alpha k_\alpha)_o\, B(\nu_o) \int_{t_o}^{t_m} n(t)\,(1+z)^{x-2}\,(t')^{-1}\, dt d\nu_o. \tag{41}$$

McVittie & Wyatt (1959) concluded that in isotropic universes the energy received at radio frequencies which had been emitted at optical frequencies could safely be neglected; there seems to be no reason why this should not be the case more generally and so the use of the radiation density function given by equation (40) throughout the entire range of integration is probably justified.

A quite different type of background radiation from that from discrete sources is that observed by Penzias & Wilson (1965) and others and which appears to be black body in nature, corresponding to a temperature of about 3 °K. As Dicke *et al.* (1965) showed, the existence of such a universal radiation field is to be expected if the Universe originated in a ' hot big bang '. If the observed field does in fact comprise the remnants of an original fireball, then it is clearly of considerable cosmological significance. In particular Thorne (1967a) has shown how the apparent isotropy of the black body background can provide strong evidence for the isotropy of the universe. In order to show this, Liouville's theorem is applied to derive an expression for the ratio of radiation temperatures in any two directions (Thorne 1967b). The methods of the present work make possible a physically intuitive derivation of the same result.

Following Thorne's basic approach, suppose that up to a certain time t_s in the evolution of the universe the micro-wave radiation was kept isotropic in distribution and black body in spectrum by interactions with matter and that since that time it has propagated freely through space-time. The time t_s is determined by the relation

$$\int_{t_s}^{t_o} [\lambda(t)]^{-1}\, dt = 1 \tag{42}$$

where t_o is the present age of the Universe and $\lambda(t)$ is the mean free path for Thomson scattering.

Since at time t_s the radiation is, by hypothesis, isotropic with respect to the matter, the problem is in fact equivalent to that of studying radiation actually emitted by the matter. An observer at time t_o observes in a solid angle $\delta\Omega$ what corresponds to radiation emitted instantaneously from an area $\xi^2\delta\Omega$. Since the radiation was black body at time t_s the energy flux he receives within the range of emitted frequency $(\nu_s, \nu_s+d\nu_s)$ is

$$\frac{2\pi h\nu_s^3}{\exp\{h\nu_s/kT_s\}-1} \cdot \frac{\xi^2 d\nu_s \delta\Omega}{D^2} \tag{43}$$

which, on account of equation (33) may be written

$$\frac{2\pi h\nu_o^3}{\exp\{h\nu_s/kT_s\}-1} \cdot d\nu_o\delta\Omega \tag{44}$$

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from which we see that the radiation he observes is also black body but corresponds to a different temperature

$$T_0(\theta, \phi) = T_s(\nu_0/\nu_s) = T_s(1+z)^{-1} \tag{45}$$

where $1+z$ is the red shift which would be observed for a source which is viewed in the given direction and which emitted radiation at time t_s.

6. *Conclusion.* Expressions have been derived for the common observations in cosmology; many others can be derived in a straightforward manner from those given here. These expressions enable exact calculations to be performed and so, where they can be used, have a considerable advantage over the series expansions of Kristian & Sachs (1966). Some general results obtained with their help will be published elsewhere. Of course most of them can only be applied to models for which the first integrals of the equations of the null geodesics are known explicitly and they often require the integration of eleven or more simultaneous ordinary differential equations. It follows that in studying any anisotropic model universe the natural approach will be to apply the series expansions first and to use the method of the present work only if the model is not eliminated by the criteria suggested by Kristian & Sachs. A computer program has been written to perform the necessary calculations and some typical results are in preparation.

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