

Observations on the Darboux coordinates for rigid special geometry

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Abstract

We exploit some relations which exist when (rigid) special geometry is formulated in real symplectic special coordinates $P^I = (p^\Lambda, q_\Lambda)$, $I = 1, \dots, 2n$. The central role of the real $2n \times 2n$ matrix $M(\Re\mathcal{F}, \Im\mathcal{F})$, where $\mathcal{F} = \partial_\Lambda \partial_\Sigma F$ and F is the holomorphic prepotential, is elucidated in the real formalism. The property $M\Omega M = \Omega$, where Ω is the invariant symplectic form, is used to prove several identities in the Darboux formulation. In this setting the matrix M coincides with the (negative of the) Hessian matrix $H(S) = \frac{\partial^2 S}{\partial P^I \partial P^J}$ of a certain hamiltonian real function $S(P)$, which also provides the metric of the special Kähler manifold. When $S(P) = S(U + \bar{U})$ is regarded as a “Kähler potential” of a complex manifold with coordinates $U^I = \frac{1}{2}(P^I + iZ^I)$, it provides a Kähler metric of a hyperkähler manifold, which describes the hypermultiplet geometry obtained by c -map from the original n -dimensional special Kähler structure.

1 Introduction

Special geometry [1],[2],[3] plays an important role in the description of the moduli space of Calabi–Yau compactifications for supergravity effective actions down to $D = 4$ dimensions [4] and also for more general compactifications when fluxes [5],[6] of different nature are turned on.

More interestingly, special geometry has served as a basis to study the so-called “attractor mechanism” [7],[8],[9],[10] for black hole backgrounds preserving at most four supercharges.

In the rigid case [11],[12],[13],[14], special geometry was an important tool for the Seiberg–Witten [15] analysis of non-perturbative properties of $N = 2$ super Yang–Mills theories. An important ingredient in special geometry is the existence of a flat symplectic bundle with structure group $Sp(2n, \mathbb{R})$, ($Sp(2n + 2, \mathbb{R})$ in the local case) [3],[16],[17],[10] over the special Kähler manifold of complex dimension n , n being the number of vector multiplets in the theory.

It is the aim of the present work to elucidate some properties of such a rich structure when Darboux real special symplectic sections are adopted [18],[16],[17],[19] for the description of underlying mathematical structure. This description is particularly suitable when background charges are introduced which are related to fluxes of vector field-strength 2-forms over the space-time manifold. Such a description has recently been used to simplify the entropy area formula [20] for extremal black holes and its relation to superstring theory [21]. We will limit ourselves to giving general results for the case of rigid special geometry but many of these results can be extended to the local case which will be described elsewhere.

We consider (rigid) special geometry in real special (Darboux) coordinates $P^I = (p^\Lambda, q_\Lambda)$, $\Lambda = 1, \dots, n$ with the Kähler 2-form

$$\omega = idq_\Lambda \wedge dp^\Lambda = \frac{i}{2}dP \wedge \Omega dP \quad (1)$$

where

$$\Omega = \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$$

is the symplectic invariant form.

The special geometry data turn out to be encoded in a real “Hamiltonian” function $S(p, q)$, originally introduced by Cecotti et al. [18] and by Freed [16], which is the Legendre transform of the imaginary part of the holomorphic prepotential $F(X^\Lambda)$ of special geometry [2],[3],[22],[23],[24]. The holomorphic symplectic sections

$$V = (X^\Lambda, F_\Lambda) \quad (2)$$

are related to the real variables as follows:

$$\Re X^\Lambda = p^\Lambda \quad \Im X^\Lambda = \phi^\Lambda = \frac{\partial S(p, q)}{\partial q_\Lambda} \quad (3)$$

$$\Re F_\Lambda = q_\Lambda \quad \Im F_\Lambda = \psi_\Lambda = -\frac{\partial S(p, q)}{\partial p^\Lambda} \quad (4)$$

If we encode the imaginary parts in the symplectic real vector $I^I = (\phi^\Lambda, \psi_\Lambda)$, it turns out that

$$I^I = \Omega'^{IJ} \frac{\partial S}{\partial P^J} \quad (5)$$

with

$$\Omega'^{IJ} = -\Omega_{IJ} = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}$$

In this note we will show the special property played by the $2n \times 2n$ real symmetric positive-definite matrix [23],[8]

$$M(\Re\mathcal{F}, \Im\mathcal{F}) = \begin{pmatrix} \Im\mathcal{F} + \Re\mathcal{F}\Im\mathcal{F}^{-1}\Re\mathcal{F} & -\Re\mathcal{F}\Im\mathcal{F}^{-1} \\ -\Im\mathcal{F}^{-1}\Re\mathcal{F} & \Im\mathcal{F}^{-1} \end{pmatrix} \quad (6)$$

which in the real formulation turns out to be related to the Hessian matrix

$$H_{IJ}(S) = \frac{\partial^2 S}{\partial P^I \partial P^J} \quad (7)$$

namely

$$M(p, q) = -H(S) \quad (8)$$

Note that M is positive as a consequence of the fact that $\Im F$ is positive [23], [8] which is required by the positivity of the metric (see Eq. (42)). The matrix M is known to play a special role when a background (symplectic) charge vector $Q = (m^\Lambda, e_\Lambda)$ is introduced and a ‘‘central charge’’ holomorphic function

$$Z = \langle Q, V \rangle = Q^T \Omega V = X^\Lambda e_\Lambda - m^\Lambda F_\Lambda \quad (9)$$

is defined.

In rigid special geometry¹ the following identity holds (for the local case, see later):

$$Q + i\Omega M Q = -2ig^{\Lambda\bar{\Sigma}} \partial_\Lambda V \bar{\partial}_\Sigma \bar{Z} \quad (10)$$

Multiplying by $Q^T \Omega$ on the left we get the ‘‘central charge’’ potential function

$$\frac{1}{2} Q^T M Q = g^{\Lambda\bar{\Sigma}} \partial_\Lambda Z \bar{\partial}_\Sigma \bar{Z} \quad (11)$$

where $g^{\Lambda\bar{\Sigma}}$ is the inverse of the metric defined below.

The Hessian (7) gives also the metric of the original Kähler manifold

$$g_{\Lambda\bar{\Sigma}} dz^\Lambda \otimes d\bar{z}^\Sigma = \frac{\partial^2 K}{\partial X^\Lambda \partial \bar{X}^\Sigma} dX^\Lambda \otimes d\bar{X}^\Sigma \quad (12)$$

where special coordinates $z^\Lambda = X^\Lambda$ have been adopted and

$$K = -i(\bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda) = -i\langle V, \bar{V} \rangle$$

¹See ref. [25] for an intrinsic definition.

namely

$$g_{\Lambda\bar{\Sigma}}dz^\Lambda \otimes d\bar{z}^\Sigma = -2H_{IJ}dP^I \otimes dP^J \quad (13)$$

Another interesting observation is related to the c -map hypermultiplet geometry as defined in references [18],[26].

By adopting real symplectic coordinates $Z^I = (\zeta^\Lambda, \tilde{\zeta}_\Lambda)$ for the (other half) hypermultiplet coordinates, the hyperkähler metric has the form [18],[26],[27]

$$g_{\Lambda\bar{\Sigma}}dz^\Lambda \otimes d\bar{z}^\Sigma + 2M_{IJ}(z, \bar{z})dZ^I \otimes dZ^J \quad (14)$$

Adopting Darboux coordinates for the Kähler manifold \mathcal{M} , and because of (8) and (13), we then have

$$-2\frac{\partial^2 S}{\partial P^I \partial P^J}(dP^I \otimes dP^J + dZ^I \otimes dZ^J) \quad (15)$$

By complexifying the Darboux coordinates as $U^I = \frac{1}{2}(P^I + iZ^I)$ we see that (14) is a Kähler metric with Kähler potential [17]

$$K(U, \bar{U}) = -8S(U + \bar{U}) \quad (16)$$

Note that, as expected from the results of [26], the metric (14) has $2n$ real isometries

$$U^I \longmapsto U^I + ia^I \quad (17)$$

In the “complex” formulation of Cecotti et al. [18], where the (second half of the) hypermultiplet coordinates were denoted by W_Λ , the same isometries were acting as

$$W_\Lambda \longmapsto W_\Lambda + i\alpha_\Lambda \quad (18)$$

$$W_\Lambda \longmapsto W_\Lambda + (\Im \mathcal{F})_{\Lambda\Sigma} \beta^\Sigma \quad (19)$$

and the Kähler potential was [18]

$$N(X^\Lambda, W_\Lambda) = K(X, \bar{X}) + (\Im \mathcal{F}^{-1})^{\Lambda\Sigma}(W_\Lambda + \bar{W}_\Lambda)(W_\Sigma + \bar{W}_\Sigma) \quad (20)$$

In the local case a particular choice of real coordinates has recently been used [20],[19] in conjunction with the modification of the black hole entropy formula due to higher curvature corrections and also in relating the entropy area formula with the topological string partition function [21].

The paper is organized as follows. In section 2 we give an explicit derivation of the special Kähler metric in the Darboux coordinates [16],[17] and, by using the properties of the M matrix we arrive at equations (7), (13), (12). In section 3 we discuss real special coordinates in connection with the central charge function. We finally comment on some central charge relations and some metric differential identities for local special geometry.

2 Complex and real special coordinates

2.1 Rigid real sections and the functional $S(p, q)$

Let (X^Λ, F_Λ) be the rigid holomorphic symplectic sections depending on the holomorphic coordinates z^Λ [24]

$$X^\Lambda = X^\Lambda(z) \quad F_\Lambda = F_\Lambda(z) \quad (21)$$

Under some general assumptions we can take X^Λ as the holomorphic coordinates $X^\Lambda = z^\Lambda$ such that

$$F_\Lambda(X) = \frac{\partial F(X)}{\partial X^\Lambda} \quad (22)$$

for a suitable function $F(X)$.

The holomorphicity condition reads

$$\bar{\partial}_\Sigma X^\Lambda = 0 \quad \bar{\partial}_\Sigma F_\Lambda = 0 \quad (23)$$

Decomposing (X^Λ, F_Λ) in terms of real and imaginary parts we can write the holomorphic symplectic sections in terms of real variables

$$X^\Lambda = p^\Lambda + i\phi^\Lambda \quad F_\Lambda = q_\Lambda + i\psi_\Lambda \quad (24)$$

Define the function $L = \Im F > 0$, the imaginary part of $F(X)$. Then it can be proved that $(q_\Lambda, \phi^\Lambda)$ and $(p^\Lambda, \psi_\Lambda)$ are pairs of conjugate variables for L

$$q_\Lambda = \frac{\partial L}{\partial \phi^\Lambda} \quad (25)$$

$$\psi_\Lambda = \frac{\partial L}{\partial p^\Lambda} \quad (26)$$

We perform a Legendre transform on L of the form

$$S(p, q) = q_\Lambda \cdot \phi^\Lambda(p, q) - L(p, \phi(p, q)) \quad (27)$$

where we have to invert equation (25) to write $\phi = \phi(p, q)$. Then, the next set of equations follows:

$$\phi^\Lambda = \frac{\partial S}{\partial q_\Lambda} \quad (28)$$

$$-\psi_\Lambda = \frac{\partial S}{\partial p^\Lambda} \quad (29)$$

Our change of coordinates $(p^\Lambda, \phi^\Lambda) \mapsto (p^\Sigma, q_\Sigma)$ is of the form

$$p^\Sigma = \delta_\Lambda^\Sigma p^\Lambda \quad q_\Sigma = q_\Sigma(p^\Lambda, \phi^\Lambda) \quad (30)$$

and, accordingly, the inverse change of coordinates $(p^\Sigma, q_\Sigma) \mapsto (p^\Lambda, \phi^\Lambda)$ is

$$p^\Lambda = \delta_\Sigma^\Lambda p^\Sigma \quad \phi^\Lambda = \phi^\Lambda(p^\Sigma, q_\Sigma) \quad (31)$$

In the following we use the notation

$$\left. \frac{\partial f(x, y)}{\partial x} \right|_y = \frac{\partial f}{\partial x} \quad \left. \frac{\partial f(x, y)}{\partial y} \right|_x = \frac{\partial f}{\partial y}$$

where the role of $f(x, y)$ will be played by $q_\Sigma(p, \phi)$, $\phi^\Lambda(p, q)$ as defined in Eqs. (30), (31).

Let \mathbb{J} be the Jacobian matrix. It follows from $\mathbb{J}(\mathbb{J})^{-1} = \mathbf{1} = (\mathbb{J})^{-1}\mathbb{J}$ that

$$\frac{\partial q_\Sigma}{\partial \phi^\Lambda} \frac{\partial \phi^\Lambda}{\partial q_\Gamma} = \delta_\Gamma^\Sigma = \frac{\partial q_\Sigma}{\partial \phi^\Lambda} \frac{\partial^2 S}{\partial q_\Lambda \partial q_\Gamma} \quad (32)$$

i.e.

$$\frac{\partial q_\Sigma}{\partial \phi^\Lambda} = \left(\frac{\partial^2 S}{\partial q_\Lambda \partial q_\Sigma} \right)^{-1} = \frac{\partial^2 L}{\partial \phi^\Sigma \partial \phi^\Lambda} \quad (33)$$

and we also have

$$\frac{\partial q_\Sigma}{\partial p^\Lambda} \delta_\Gamma^\Lambda + \frac{\partial q_\Sigma}{\partial \phi^\Lambda} \frac{\partial \phi^\Lambda}{\partial p^\Gamma} = \frac{\partial q_\Sigma}{\partial p^\Lambda} \delta_\Gamma^\Lambda + \left(\frac{\partial^2 S}{\partial q_\Sigma \partial q_\Lambda} \right)^{-1} \frac{\partial^2 S}{\partial q_\Lambda \partial p^\Gamma} = 0 \quad (34)$$

In fact, equations (32) to (34) follow not only from the Jacobian but also from the holomorphicity conditions (23). These also imply

$$\frac{\partial q_\Sigma}{\partial \phi^\Lambda} = -\frac{\partial \psi_\Sigma}{\partial p^\Lambda} - \frac{\partial q_\Gamma}{\partial p^\Lambda} \frac{\partial \psi_\Sigma}{\partial q_\Gamma} \quad (35)$$

where $\psi = \psi(p, q)$ and its partial derivatives are computed following the notation explained before. The analyticity of $L = \Im F$, for F holomorphic, also imply

$$\frac{\partial^2 L}{\partial \phi^\Lambda \partial \phi^\Sigma} + \frac{\partial^2 L}{\partial p^\Lambda \partial p^\Sigma} = 0 \quad \frac{\partial^2 L}{\partial \phi^\Lambda \partial p^\Sigma} = \frac{\partial^2 L}{\partial p^\Lambda \partial \phi^\Sigma}$$

In terms of S Eq. (35) can be written as

$$\left(\frac{\partial^2 S}{\partial q_\Sigma \partial q_\Lambda} \right)^{-1} = \frac{\partial^2 S}{\partial p^\Sigma \partial p^\Lambda} + \frac{\partial q_\Gamma}{\partial p^\Lambda} \frac{\partial^2 S}{\partial q_\Gamma \partial p^\Sigma} \quad (36)$$

These relations will be useful when simplifying the metric in the next section.

2.2 Kähler potential and metric

In this section we show that the Kähler potential K can be expressed directly in terms of $S(p, q)$. Then, the metric can also be expressed directly in terms of $S(p, q)$, since

$$g = g_{\Lambda\bar{\Sigma}} dX^\Lambda \otimes d\bar{X}^\Sigma \quad (37)$$

with

$$g_{\Lambda\bar{\Sigma}} = \frac{\partial^2 K}{\partial X^\Lambda \partial \bar{X}^\Sigma} \quad (38)$$

In the rigid case the Kähler potential is

$$K = i(X^\Lambda \bar{F}_\Lambda - \bar{X}^\Lambda F_\Lambda) \quad (39)$$

The Kähler form then is

$$\omega = -\frac{1}{4} \partial_\Lambda \bar{\partial}_\Sigma K dX^\Lambda \wedge d\bar{X}^\Sigma = -\frac{1}{2} \Im F_{\Lambda\Sigma} dX^\Lambda \wedge d\bar{X}^\Sigma \quad (40)$$

so that

$$\omega = idq_\Lambda \wedge dp^\Lambda \quad (41)$$

is a symplectic form. The metric will be

$$g_{\Lambda\bar{\Sigma}} = 2\Im F_{\Lambda\Sigma} > 0 \quad (42)$$

Using equations (32) to (36), one arrives at

$$g_{\Lambda\bar{\Sigma}} = -2 \left(\frac{\partial^2 S}{\partial p^\Lambda \partial p^\Sigma} + \frac{\partial q_\Gamma}{\partial p^\Sigma} \frac{\partial^2 S}{\partial q_\Gamma \partial p^\Lambda} \right) = -2 \left(\frac{\partial^2 S}{\partial q_\Lambda \partial q_\Sigma} \right)^{-1} \quad (43)$$

Note that the symmetry properties $F_{\Lambda\Sigma} = F_{\Sigma\Lambda}$ imply

$$\frac{\partial q_\Lambda}{\partial p^\Sigma} = \frac{\partial q_\Sigma}{\partial p^\Lambda} \quad (44)$$

$$\frac{\partial q_\Gamma}{\partial p^\Lambda} \frac{\partial^2 S}{\partial q_\Gamma \partial p^\Sigma} = \frac{\partial q_\Gamma}{\partial p^\Sigma} \frac{\partial^2 S}{\partial q_\Gamma \partial p^\Lambda} \quad (45)$$

$$\frac{\partial q_\Gamma}{\partial p^\Lambda} \frac{\partial^2 S}{\partial q_\Gamma \partial p^\Sigma} = \frac{\partial q_\Gamma}{\partial p^\Sigma} \frac{\partial^2 S}{\partial q_\Gamma \partial p^\Lambda} \quad (46)$$

which will help in the simplifications.

The change of variables in the differentials gives

$$\begin{aligned} dX^\Lambda \otimes d\bar{X}^\Sigma &= \left(\delta_\Gamma^\Lambda \delta_\Delta^\Sigma + \frac{\partial^2 S}{\partial q_\Lambda \partial p^\Gamma} \frac{\partial^2 S}{\partial q_\Sigma \partial p^\Delta} + i \left(\frac{\partial^2 S}{\partial q_\Lambda \partial p^\Gamma} \delta_\Delta^\Sigma - \frac{\partial^2 S}{\partial q_\Sigma \partial p^\Delta} \delta_\Gamma^\Lambda \right) \right) dp^\Gamma \otimes dp^\Delta \\ &+ \left(\frac{\partial^2 S}{\partial q_\Lambda \partial q_\Delta} \frac{\partial^2 S}{\partial q_\Sigma \partial p^\Gamma} + \frac{\partial^2 S}{\partial q_\Lambda \partial p^\Gamma} \frac{\partial^2 S}{\partial q_\Sigma \partial q_\Delta} + i \left(\frac{\partial^2 S}{\partial q_\Lambda \partial q_\Delta} \delta_\Gamma^\Sigma - \frac{\partial^2 S}{\partial q_\Sigma \partial q_\Delta} \delta_\Gamma^\Lambda \right) \right) dp^\Gamma \otimes dq_\Delta \\ &+ \left(\frac{\partial^2 S}{\partial q_\Lambda \partial q_\Gamma} \frac{\partial^2 S}{\partial q_\Sigma \partial q_\Delta} \right) dq_\Gamma \otimes dq_\Delta \end{aligned} \quad (47)$$

Finally, using again equations (32) to (36), and (44) to (46) the following expression is obtained:

$$g(p, q) = -2 \left(\frac{\partial^2 S}{\partial p^\Lambda \partial p^\Sigma} dp^\Lambda \otimes dp^\Sigma + 2 \frac{\partial^2 S}{\partial p^\Lambda \partial q_\Sigma} dp^\Lambda \otimes dq_\Sigma + \frac{\partial^2 S}{\partial q_\Lambda \partial q_\Sigma} dq_\Lambda \otimes dq_\Sigma \right) \quad (48)$$

Equation (48) implies that the Hessian matrix (7) is negative-definite.

Comparison of (48) with a different evaluation of the metric in Darboux coordinates (62) in section 2.4, will allow us to prove that $M = -H$, as asserted in the introduction.

2.3 The Kähler form

In rigid special geometry the symplectic holomorphic vector $V = (X^\Lambda, F_\Lambda)$ defines the Kähler form through the formula [16]

$$\Omega_{IJ}dV^I \wedge d\bar{V}^J = -dX^\Lambda \wedge d\bar{F}_\Lambda + dF_\Lambda \wedge d\bar{X}^\Lambda \quad (49)$$

by writing $X^\Lambda = p^\Lambda + i\phi^\Lambda$, $F_\Lambda = q_\Lambda + i\psi_\Lambda$, we find

$$-2dp^\Lambda \wedge dq_\Lambda - 2d\phi^\Lambda \wedge d\psi_\Lambda \quad (50)$$

On the other hand, if we compute (using the property $dX^\Lambda \wedge dF_\Lambda = 0$)

$$\begin{aligned} d(X^\Lambda + \bar{X}^\Lambda) \wedge d(F_\Lambda + \bar{F}_\Lambda) &= dX^\Lambda \wedge d\bar{F}_\Lambda + d\bar{X}^\Lambda \wedge dF_\Lambda \\ &= (dX^\Lambda \wedge d\bar{F}_\Lambda - dF_\Lambda \wedge d\bar{X}^\Lambda) = 4dp^\Lambda \wedge dq_\Lambda \end{aligned} \quad (51)$$

the following relation must hold,

$$dp^\Lambda \wedge dq_\Lambda = d\phi^\Lambda \wedge d\psi_\Lambda \quad (52)$$

We postpone in proving (52) but simply observe that

$$\omega = \frac{i}{4}\Omega_{IJ}dV^I \wedge d\bar{V}^J = \frac{i}{2}\Omega_{IJ}dP^I \wedge dP^J = idq_\Lambda \wedge dp^\Lambda \quad (53)$$

2.4 The Kähler metric

We now consider the Kähler metric in complex coordinates. Let us first note a basic identity satisfied by the complex symplectic differential

$$dV = (dX^\Lambda, dF_\Lambda) = (dX^\Lambda, F_{\Lambda\Sigma}dX^\Sigma)$$

and the M matrix defined in (6). It is easy to see that

$$MdV = i\Omega dV \quad (54)$$

and also to recall that M is real, symmetric and symplectic, i.e. that it satisfies

$$M\Omega M = \Omega \quad (55)$$

The basic identity (54) allows us to prove that

$$\begin{aligned} d\bar{V}M(\mathcal{F})dV &= id\bar{V}\Omega dV = i\langle d\bar{V}, dV \rangle = i(d\bar{X}^\Lambda, d\bar{F}_\Lambda) \begin{pmatrix} -dF_\Lambda \\ dX^\Lambda \end{pmatrix} \\ &= i(dX^\Lambda d\bar{F}_\Lambda - d\bar{X}^\Lambda dF_\Lambda) = idX^\Lambda d\bar{X}^\Sigma (\bar{F}_{\Lambda\Sigma} - F_{\Lambda\Sigma}) = 2dX^\Lambda d\bar{X}^\Sigma \Im F_{\Lambda\Sigma} \end{aligned} \quad (56)$$

Therefore the (positive-definite) Kähler metric is

$$2\Im F_{\Lambda\Sigma} = \partial_\Lambda \bar{\partial}_\Sigma K \quad K = -i\langle V, \bar{V} \rangle \quad (57)$$

Let us now consider the Darboux (special) coordinates

$$P^I = (\Re X^\Lambda, \Re F_\Lambda) = (p^\Lambda, q_\Lambda)$$

and

$$I^I = (\Im X^\Lambda, \Im F_\Lambda) = (\phi^\Lambda, \psi_\Lambda)$$

such that

$$dV^I = dP^I + idI^I \quad d\bar{V}^I = dP^I - idI^I$$

In components we have

$$dV^I d\bar{V}^J = (dP^I + idI^I)(dP^J - idI^J) = (dP^I dP^J + dI^I dI^J) + i(dI^I dP^J - dP^I dI^J) \quad (58)$$

$$M_{IJ} dV^I d\bar{V}^J = M_{IJ}(dP^I dP^J + dI^I dI^J) \quad (59)$$

We now compute $dI^I dI^J$ by using the following property

$$I^I = \begin{pmatrix} \phi^\Lambda \\ \psi_\Lambda \end{pmatrix} = \begin{pmatrix} \frac{\partial S}{\partial q_\Lambda} \\ -\frac{\partial S}{\partial p^\Lambda} \end{pmatrix} = \Omega'^{IJ} \frac{\partial S}{\partial P^J} = -\Omega \frac{\partial S}{\partial P} \quad (60)$$

where $\Omega'^{IJ} = -\Omega_{IJ}$ so that $\Omega' \Omega = \mathbf{1}$. It then follows that

$$dI^I = \Omega' H dP \quad (61)$$

where H is the (real symmetric) Hessian of $S(P)$. By inserting (61) into (59) we obtain

$$M_{IJ} dV^I \otimes d\bar{V}^J = (M - H \Omega M \Omega H)_{IJ} dP^I \otimes dP^J \quad (62)$$

by comparing Eq. (48) with (62) we get

$$M_{IJ} dV^I \otimes d\bar{V}^J = -2H_{IJ} dP^I \otimes dP^J \quad (63)$$

Eqs. (62) and (63) then imply that $M = -H$. The same argument can be used for the Kähler form to prove equation (52). In fact

$$\Omega_{IJ} dV^I \wedge d\bar{V}^J = \Omega_{IJ}(dP^I \wedge dP^J + dI^I \wedge dI^J) \quad (64)$$

by use of (60), (61) we get

$$\Omega_{IJ} dV^I \wedge d\bar{V}^J = dP^I \wedge dP^J (\Omega - M \Omega \Omega \Omega M)_{IJ} \quad (65)$$

and, using $\Omega^3 = -\Omega$, $M \Omega M = \Omega$, we get

$$\Omega_{IJ} dV^I \wedge d\bar{V}^J = 2\Omega_{IJ} dP^I \wedge dP^J \quad (66)$$

which is nothing but equation (53)

2.5 The hyperkähler metric

The Darboux coordinates give also a striking simplification [17] of the metric of the real $4n$ -manifold of the hypermultiplet geometry, which is obtained by the c -map introduced by [18] and which associates to any special Kähler manifold of dimension n an hyperkähler manifold of complex dimension $2n$. Furthermore, as discussed in [26], this manifold has at least $2n$ isometries ($2n+3$ in the local case which come from the c -map construction [26]). The hypermultiplet geometry, as described in [18], has a metric of the form² [18],[26],[27]

$$g_{\Lambda\bar{\Sigma}}dz^\Lambda \otimes d\bar{z}^\Sigma + 2M_{IJ}(z, \bar{z})dZ^I \otimes dZ^J \quad (67)$$

where $Z^I = (\zeta^\Lambda, \bar{\zeta}_\Lambda)$ are $2n$ real coordinates associated to a symplectic real vector Z and $g_{\Lambda\bar{\Sigma}}$ is the original Kähler metric.

By adopting Darboux coordinates and noticing that $M = -H$ equation (66) takes the simple form

$$-2\frac{\partial^2 S}{\partial P^I \partial P^J}(dP^I \otimes dP^J + dZ^I \otimes dZ^J) = \frac{\partial^2 K}{\partial U^I \partial \bar{U}^J}dU^I \otimes d\bar{U}^J \quad (68)$$

It is immediate to see that (68) is a Kähler metric, for a complex $2n$ -dimensional manifold with $2n$ complex coordinates given by

$$U^I = \frac{1}{2}(P^I + iZ^I) \quad (69)$$

and Kähler potential given by (16). Interestingly enough the hamiltonian function $S(P)$ has a double role. In the original symplectic manifold \mathcal{M} of complex dimension n , its Hessian matrix in Darboux coordinates is the metric on the manifold. Considered as a function of “complex coordinates” U^I , it is the Kähler potential on the cotangent bundle (of the special manifold \mathcal{M}) with real dimension $4n$.

Note the obvious $2n$ isometries

$$U^I \longmapsto U^I + ia^I \quad (70)$$

as implied by the analysis of [26].

3 Central charges and special coordinate identities in complex and real coordinates

In the present section we discuss identities and relations of special geometry in presence of a background charge real symplectic vector $Q = (m^\Lambda, e_\Lambda)$. In terms of the special geometry complex sections the “central charge” function is given by

$$Z = \langle Q, V \rangle = Q^T \Omega V \quad (71)$$

²Note that because of property (55) if we lower the indices Z_I the second term in (67) becomes $2(M^{-1})^{IJ}dZ_I \otimes dZ_J$. This agrees with the hypermultiplet metric as given in [19]

where scalar symplectic products are understood. In local special geometry Z is a section of a $U(1)$ bundle over \mathcal{M} and it is conveniently written in terms of symplectic sections over $U(1)$ [24],[23],[8]

$$Z^L = \langle Q, V^L \rangle$$

$$V^L = (L^\Lambda, M_\Lambda) \quad i\langle V^L, \bar{V}^L \rangle = 1 \quad (72)$$

so that

$$K = -\log(i(\bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda)) \quad (73)$$

In the rigid case the Kähler potential is rather given by

$$K = -i\langle V, \bar{V} \rangle = -i(\bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda) \quad (74)$$

so that

$$K_{\Lambda\bar{\Sigma}} = 2\Im F_{\Lambda\Sigma} > 0$$

Note that in the local case $\Im\mathcal{F}_{\Lambda\Sigma}$ is a matrix of lorentzian signature [23] with n positive and one negative eigenvalues, while the matrix \mathcal{N}

$$\mathcal{N}_{\Lambda\Sigma} = \bar{h}_{I\Lambda}(\bar{f}^{-1})^I_\Sigma$$

(where $\bar{h}_{I\Lambda}$, $\bar{f}_{I\Lambda}$ are $(n+1) \times (n+1)$ matrices which are the components of the sections $\bar{D}_i \bar{V}$, $I = 1, \dots, n$, and V , $I = 0$) is negative-definite (in the rigid case $\mathcal{N} \mapsto \bar{\mathcal{F}}$ so that $\Im\mathcal{N} \mapsto -\Im\mathcal{F}$, and $\Im\mathcal{F}$ becomes positive-definite and is an $n \times n$ instead of an $(n+1) \times (n+1)$ matrix).

In the local case, if we define the $U(1)$ covariant derivatives $D_i Z^L$ ($\bar{D}_i Z^L = 0$), $D_i V^L$ in terms of the $U(1)$ sections Z^L and V^L , then the following identity is true³ [28]:

$$Q - i\Omega M(\mathcal{N})Q = -2i\bar{V}^L Z^L - 2ig^{i\bar{j}} D_i V^L \bar{D}_{\bar{j}} \bar{Z}^L \quad (75)$$

with the (local) special geometry identity

$$M(\mathcal{N})V^L = i\Omega V^L \quad (76)$$

where $M(\mathcal{N})$ is the same matrix as in (6) but with $\mathcal{F} \mapsto \mathcal{N}$.

Note that if in (75) we take the scalar product with Q , since $\langle Q, Q \rangle = 0$ we obtain the black-hole potential as [8]

$$-\frac{1}{2}Q^T M(\mathcal{N})Q = |Z^L|^2 + D_i Z^L \bar{D}_{\bar{j}} \bar{Z}^L g^{i\bar{j}} = V_{BH} \quad (77)$$

On the other hand, by multiplying by V and using the fact that $MV = i\Omega V$, we obtain

$$\langle Q, V \rangle = Z$$

³Taking the real part of (75) we obtain the identity used in [29],[30],[31]. This identity has recently been generalized in the presence of more general fluxes in Ref. [32].

The rigid formula that replaces (75) is

$$Q + i\Omega M(\mathcal{F})Q = -2ig^{i\bar{j}}\partial_i V \bar{\partial}_{\bar{j}} \bar{Z} \quad (78)$$

which implies the rigid formula

$$V = \frac{1}{2}Q^T M(\mathcal{F})Q = g^{i\bar{j}}\partial_i Z \bar{\partial}_{\bar{j}} \bar{Z} \quad (79)$$

Note that (79), with respect to (77), loses the graviphoton charge contribution and it is identical to the $N = 1$ rigid formula for chiral multiplets of superpotential Z .

This formula coincides with the superpotential contribution to the $N = 2$ potential considered in Ref. [33]. From (78) we also see that at a supersymmetric extremum $\partial_i Z = 0$ implies $Q = 0$, something which is different from the local case.

The local supersymmetric attractor point

$$D_i Z^L = 0$$

gives, instead, the so called ‘‘BPS attractor equations’’:

$$Q = -i(\bar{V}^L Z^L - V^L \bar{Z}^L) \quad (80)$$

$$\Omega M(\mathcal{N})Q = \bar{V}^L Z^L + V^L \bar{Z}^L \quad (81)$$

The rigid identities (78) can be written for real Darboux symplectic special coordinates by noticing that

$$Z = \langle Q, V \rangle = p^\Lambda e_\Lambda - m^\Lambda q_\Lambda + i(\phi^\Lambda - m^\Lambda \psi_\Lambda) = Z(Q, P, I) \quad (82)$$

By using now the property that

$$I = \Omega' \frac{\partial S}{\partial P}$$

we get

$$\frac{\partial Z}{\partial P^I} = -\Omega Q + iH Q = \tau Q \quad (83)$$

with $\tau = -\Omega + iH$, $\tau = -\tau^\dagger$.

Note that, as a consequence of the fact that $H\Omega H = \Omega$ we have

$$\tau\Omega\tau = 2\tau \quad \tau\Omega\tau^T = 0$$

The potential becomes

$$V(P, Q) = -\frac{1}{2}Q^T H Q \quad (84)$$

3.1 Local special geometry

We finally discuss further identities of local special geometry. Since

$$V = (L^\Lambda, M_\Lambda = \mathcal{N}_{\Lambda\Sigma} L^\Sigma) \quad (85)$$

and also

$$M_\Lambda = F_{\Lambda\Sigma}L^\Sigma \quad (86)$$

(whenever $F_\Lambda = \partial_\Lambda F$) it is also true that, in addition to (76), we have

$$M(\mathcal{F})V^L = i\Omega V^L \quad (87)$$

by further use of the identity

$$dM_\Lambda = F_{\Lambda\Sigma\Delta}X^\Sigma dL^\Delta + F_{\Lambda\Sigma}dL^\Sigma = F_{\Lambda\Sigma}dL^\Sigma \quad (88)$$

(because $F_{\Lambda\Sigma\Delta}X^\Sigma = 0$) it is also true, as in the rigid case, that

$$M(\mathcal{F})dV^L = i\Omega dV^L \quad (89)$$

By using the definition

$$D_i V^L = \partial V^L + \frac{1}{2}K_i V^L \quad (90)$$

we have

$$M(\mathcal{F})D_i V^L = i\Omega D_i V^L \quad (\bar{D}_i V^L = 0) \quad (91)$$

where D_i is a $U(1)$ covariant derivative.

The Kähler metric on the manifold is given by [24]

$$g^L = i\langle \bar{D}\bar{V}^L, DV^L \rangle = i\bar{D}\bar{V}^L \Omega DV^L \quad (92)$$

By using the previous relations we get the equivalent expression

$$g^L = (\bar{D}\bar{V}^L M(\mathcal{F})DV^L) \quad (93)$$

Formula (93) is the local analogue of (56) and it will be useful to formulate local special geometry in Darboux real special coordinates.

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