Observer-Based Bipartite Consensus of Linear Multi-Agent Systems With Measurement Noises

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Abstract This paper considers the bipartite consensus of linear multi-agent systems with measurement noise, in which both the cooperation and competition relations exist among agents. To attenuate the influence of measurement noises, an important function $\alpha(t)$ is introduced. Then, the output-based controller with both the noisy part and noise-free part is proposed. It is proved that the proposed controller can guarantee the mean-square bipartite consensus with the assumption in which the signed graph is connected and structurally balanced. Meanwhile, the obtained result releases the limitation on time-varying function $\alpha(t)$. Finally, an example with six agents is utilized to illustrate the effectiveness of the proposed controller.

Index Terms Multi-agent systems, bipartite consensus, measurement noises, output feedback.

I. INTRODUCTION

The consensus of multi-agent systems (MASs) has obtained widespread attention in recent years due to its wide applications in unmanned vehicles systems [1], [2], sensor networks [3], [4] and so on. The most existing results in terms of consensus are focused on the nonnegative graph [5], [6], i.e., the relationship among agents is assumed to be cooperative. However, in practical systems, the relationship among agents is not always collaborative, there may exists competitive relationship among agents, for example, the communication between two different coalition. People come form different coalitions representing different interest groups. They are competitive in their own group but are competitive for other groups. To portray both the cooperation and competition among agents, the signed graph is introduced in [7], where a positive number represents the collaboration and a negative represents the antagonism. The consensus over such graph is named bipartite consensus, i.e., the agents agree on a common value but have different symbols. Up to now, many efforts have been done on the bipartite consensus [7]–[17]. For example, reference [7] investigated the bipartite consensus problem of single-integral MASs over signed graph, which shown that the collectiveness and structurally balance are the sufficient and necessary conditions that guarantee the bipartite consensus. Then, the results were generalized to linear MASs over the signed undirected graph in [8], where the problem of bipartite consensus was converted into the problem of stabilization problem of the state-space model. The signed directed graph case was considered in [9], in which it pointed out that the graph with spanning tree is the necessary condition to guarantee the bipartite consensus. In addition, the controllers with both state feedback and output feedback were proposed in [10] by exploring the relationship between bipartite consensus and the classical consensus problems. Furthermore, the asymmetric bipartite consensus over directed networks with antagonistic interactions was considered in [11] and the bipartite output consensus of heterogeneous linear MASs with event-triggered observer was considered in [14]. More research results on bipartite consensus can be found in [12], [13], [17].

The communications among agents are often encountered with the uncertainty disturbance in practical systems. The influence of measurement noises in the classical consensus of MASs has been thoroughly considered in [18]–[23]. For example, a decreasing function $\alpha(k)$ is introduced in [18] to attenuate measurement noises for discrete-time model. Then, the results were generalized to the continuous-time case in [19], where the sufficient and necessary conditions that guarantee the consensus of single integral MASs with...
measurement noises were provided. The time-varying topologies case was considered in [20], [21] and the communication delays case was studied in [22]. In addition, the heterogeneous time-varying gain for linear MASs was proposed in [23], in which it shown that the mean square consensus with a leader can be achieved under the condition that noise-attenuation gains are infinitesimal of the same order. Although there are fruitful results for conventional consensus problem, as for bipartite consensus with measurement noises, the research is still few.

This paper aims to establish the out-put based distributed bipartite consensus protocol for linear MASs with measurement noises and signed graph. Our results accommodate to the case that both collaboration and competition relationships exist among agents. In this case, the row sum of Laplacian matrix of the graph may not be zero and all the eigenvalues of the Laplacian matrix can be positive [24], which bring difficulty in consensus analysis compared with the standard graph. Furthermore, departing from the state feedback structure existing in [24]–[27], where the state information was utilized in the controller designing, the consensus protocol in this paper only requires the measurement output information. In addition, different form the results for classical consensus of MASs with measurement noises, the error considered in this paper not only includes the difference between the state and the common value but also includes the sum of both case. All the above statements bring difficulty in our analysis.

To deal with the problems caused by signed graph, unavailable status information and measurement noises, the theories of matrices, graph and stochastic analysis are utilized in this paper and the corresponding contributions are listed as follows:

- This paper investigates the bipartite consensus problem of linear MASs with measurement noises. Compared with MASs with measurement noise in [24], [26], [27], where only the first-order systems are considered, our system considered here is more general.
- Different from the state-based controller for bipartite consensus proposed in [24]–[27], this paper proposed the distributed controller based on output information. In addition, the proposed observer-based controller not only contains the noise-free part but also contains the noise part.
- The sufficient condition that guarantees the bipartite consensus of linear MASs with measurement noises is proposed, in which we no longer require that the function \( \alpha(t) \) satisfies \( \int_0^\infty \alpha(s)ds = \infty \).

II. PRELIMINARIES

The consensus of MASs with \( N \) agents is considered in this paper. The dynamics of each agent is determined by:

\[
\begin{align*}
\dot{x}_i(t) &= A x_i(t) + B u_i(t), \\
y_i(t) &= C x_i(t), & i &= 1, 2, \ldots, N. 
\end{align*}
\]

where \( x_i(t) \in \mathbb{R}^n \) is the state of agent \( i \), \( u_i(t) \in \mathbb{R}^m \) and \( y_i(t) \in \mathbb{R}^r \) represent the controller and measurement output of agent \( i \), respectively. It is assumed that \((A, B)\) is controllable and \((C, A)\) is detectable. The goal of this paper is to design a distributed controller such that system (1) can achieve the mean square bipartite consensus, i.e.

\[
\lim_{t \to \infty} \mathbb{E} \left\{ \| x_i(t) - \frac{1}{N} \sum_{j=1}^{N} s_i s_j x_j(t) \|^2 \right\} = 0, \quad i = 1, 2, \ldots, N,
\]

where \( \mathbb{E} \{ \cdot \} \) denotes the mathematical expectation, \( s_i \in \{-1, 1\} \).

To describe the communication among agents, the basic knowledge of signed graph is introduced here. A signed undigraph is usually described by \( G = (V, E, A) \), where \( V = \{1, 2, \ldots, N\} \) denotes the set of vertices representing agents, \( E \subset V \times V \) represent the set of edges and \( A = \{A_{ij}\} \in \mathbb{R}^{N \times N} \) is the adjacency matrix, which is defined by: if \( A_{ij} > 0 \), the connection between \( i \) and \( j \) is cooperative; if \( A_{ij} < 0 \), the connection between \( i \) and \( j \) is competitive; if \( A_{ij} = 0 \), there is no connection between \( i \) and \( j \) . It is assumed that the graph \( G \) is simple, i.e. \( A_{ij} = 0, i \in V, N_i = \{j | (j, i) \in E\} \) denotes all neighbors of agent \( i \), \( i = 1, 2, \ldots, N \). The Laplacian matrix of \( G \) is defined by \( L = D - A \), where \( D = \text{diag}\{\sum_{j=1}^{N} |A_{ij}|, \ldots, \sum_{j=1}^{N} |A_{Nj}|\} \).

Graph \( G = (V, E, A) \), is structurally balanced if \( V \) can be partitioned into two subsets \( V_1 \) and \( V_2 \) such that \( V_1 \cap V_2 = \emptyset \), \( V_1 \cup V_2 = V \) satisfying \( A_{ij} \geq 0, i, j \in V_1, s \in \{1, 2\} \), and \( A_{ij} \leq 0, i \in V_1, j \in V_2, i \neq s \) [26]. As it is pointed out in [10], for a structurally balanced signed undigraph \( G \) there exists gauge transformation such that \( S^T A S \) is a non-negative matrix, where \( S = \text{diag}\{s_1, s_2, \ldots, s_N\}, s_i = 1, if \ i \in V_1, s_i = -1, if \ i \in V_2 \).

One may notice that equation (2) equals to the following two equations

\[
\begin{align*}
&\lim_{t \to \infty} \mathbb{E} \left\{ \| x_i(t) - \frac{1}{N} \sum_{j=1}^{N} s_j x_j(t) \|^2 \right\} = 0, & i & \in V_1, \\
&\lim_{t \to \infty} \mathbb{E} \left\{ \| x_i(t) + \frac{1}{N} \sum_{j=1}^{N} s_j x_j(t) \|^2 \right\} = 0, & i & \in V_2,
\end{align*}
\]

which implies that terminal state of agent may convergence to \( \frac{1}{N} \sum_{j=1}^{N} s_j x_j(t) \) or \( -\frac{1}{N} \sum_{j=1}^{N} s_j x_j(t) \). That is why the consensus in the sense is called bipartite consensus. If \( V_1 = \emptyset \) or \( V_2 = \emptyset \), bipartite consensus reduces to the traditional consensus in [23], where \( \lim_{t \to \infty} \mathbb{E} \{x_i(t) - x_j(t)\} = 0 \).

A. PROBLEM STATEMENT

III. MAIN RESULTS

Since the full information of systems’ state cannot always be measured, to achieve the bipartite consensus, a novel observer-based distributed controller is proposed.
as follows:

\[
\begin{align*}
    u_i(t) &= \alpha(t)K_1 \sum_{j \in N_i} \beta_{ij} \rho_j^{-1} z_j(t) \\
    \dot{z}_i(t) &= (A + BK_2)\hat{y}_i(t) + \alpha(t) \sum_{j \in N_i} (c_i j^{-1} \cdot z_j(t)), \\
    \dot{\hat{y}}_i(t) &= (A + BK_2)\hat{y}_i(t) + \alpha(t) \sum_{j \in N_i} (c_i j^{-1} \cdot z_j(t)), \\
    \dot{\check{y}}_i(t) &= C\check{y}_i(t),
\end{align*}
\]

where \(v_i(t) \in \mathbb{R}^n\) is the internal state of controller, \(\hat{x}_i(t) \in \mathbb{R}^n\) is the observed state of \(x_i(t)\), \(\bar{y}_i(t) \in \mathbb{R}^p\) is the measurement output of the observer, \(K_1 \in \mathbb{R}^{m \times n}\) and \(K_2 \in \mathbb{R}^{m \times n}\) are the feedback gain matrices, \(F \in \mathbb{R}^{n \times l}\) is the observed feedback gain matrix, \(\beta_{ij} = \beta_{ji}\) and \(\beta_{ij} \rho_j^{-1} = \beta_{ji} I_n\), and \(c\) are positive constant, the time-varying function \(\alpha(t) > 0\), \(z_j(t) = |A_{ij}|(\text{sign}(A_{ij})\hat{z}_i(t) + \rho_j \bar{y}_j(t) - \check{z}_i(t))\), \(c_i j^{-1} = |A_{ij}|(\text{sign}(A_{ij})\bar{y}_j(t) + \rho_j \bar{y}_j(t) - v_i(t))\). \(i, j = 1, 2, \ldots, N\) are the measurement noises which are the independent standard white noises with \(\int_0^\infty \eta_i(s)ds = w_i(t)\).

**Remark 1:** The controller (4) includes two parts: noise part \(K_1\sum_{j \in N_i} \beta_{ij} \rho_j^{-1} z_j(t)\) and noise-free part \(\sum_{j \in N_i} \beta_{ij} \rho_j^{-1} (t - \bar{y}_j(t) - c_i j^{-1})\). The noise-free part not only provide benefit in the convergence analysis but also lessen the limitation on the time-varying function \(\alpha(t)\), i.e. we do not require that \(\alpha(t)\) satisfies \(\int_0^\infty \alpha(s)ds = \infty\).

Some assumptions are listed as follows

**Assumption 1:** The signed undirected graph \(G = (V, E, A)\) is connected and structurally balanced.

**Assumption 2:** The \((A, B)\) is assumed to be controllable and \((C, A)\) to be detectable.

**Assumption 3:** The piecewise continuous function \(\alpha(t) : R^+ \to R^+\) is uniform bounded and satisfies \(\int_0^\infty \alpha^2(s)ds < \infty\).

**Remark 2:** Assumption 1 provides the fundamental requirement on the communication graph, which is standard assumption in dealing with the bipartite consensus of MASs over the signed graph [24]-[27]. Assumption 2 is the general assumption for control system. The role of function \(\alpha(t)\) in Assumption 3 is to decrease the influence of noises when \(t \to \infty\). Different from the assumption in [24]-[27], here we no require \(\alpha(t)\) satisfy \(\int_0^\infty \alpha(s)ds = \infty\).

An useful lemma is given as follow:

**Lemma 1:** [10] Suppose that the undirected signed graph \(G\) is connected. Then it is structurally balanced, if and only if 0 is a simple eigenvalue of \(L\) and all its other eigenvalues are positive.

Applying controller (4) to system (1), we have

\[
\begin{align*}
    \dot{\hat{z}}_i(t) &= (I_N \otimes (A - FC))\hat{z}_i(t) \\
    \dot{x}_i(t) &= (I_N \otimes A - \alpha(t) + c\beta_{ji}(L \otimes BK_1))x_i(t) \\
    &\quad + ((\alpha(t) + c\beta_{ji}(L \otimes BK_1))\eta_i(t)) \\
    &\quad \times \Phi_i(t) + (\alpha(t)I_N \otimes BK_1)D_1 \eta_i(t) \\
    \dot{v}_i(t) &= (I_N \otimes (A + BK_2))v_i(t) - \alpha(t)(L \otimes I_p)v_i(t) \\
    &\quad - x_i(t) + e_i(t) - \alpha(t)D_2 \eta_i(t)
\end{align*}
\]

where \(e_i(t) = x_i(t) - \hat{x}_i(t), e_i(t) = e_1(t) e_2(t) \cdots e_N(t)\), \(\hat{x}_i(t) = (\hat{x}_1^T(t) \hat{x}_2^T(t) \cdots \hat{x}_N^T(t))^T, D_1 = \text{diag}([d_1, \ldots, d_N], d_i = (A_{ij}\beta_{ji}\cdots A_{Nj}\beta_{nj})D_2 = \text{diag}([d_1, \ldots, d_N]), v_i(t) = (v_1^T(t) v_2^T(t) \cdots v_N^T(t))^T, \Phi_i(t) = (A_{ij}\beta_{ji}\cdots A_{Nj}\beta_{nj}), \Phi_i\rho_j^{-1} = \rho_{ji}, \Phi = (\Phi_j),\) where \(\Phi_{ij} = \sum_{j \in N_i} |A_{ij}|\beta_{ji}\rho_j^{-1}, \Phi_{ii} = -|A_{ij}|\beta_{ji}\rho_j^{-1}, i \neq j, n = (\eta_i^T \eta_j^T \cdots \eta_N^T)^T \in R^{N^2}.

Denote \(\bar{e}_i(t) = ((I_N - \frac{1}{N}ss^T) \otimes I_p)e_i(t), \tilde{e}_i(t) = ((I_N - \frac{1}{N}ss^T) \otimes I_p)\bar{e}_i(t), \check{e}_i(t) = ((I_N - \frac{1}{N}ss^T) \otimes I_p)\tilde{e}_i(t),\) where \(s = (s_1 s_2 \cdots s_N).\) Since \((I_N - \frac{1}{N}ss^T) = L(I_N - \frac{1}{N}ss^T) = L, (s^T \otimes I_n)\Phi = (\Phi(s \otimes I_n) = 0,\)

\[
\begin{align*}
    \dot{\bar{e}}_i(t) &= (I_N \otimes (A - FC))e_i(t) \\
    \dot{\tilde{e}}_i(t) &= [(I_N \otimes A - (\alpha(t) + c\beta_{ji}(L \otimes BK_1)))\bar{e}_i(t) \\
    &\quad + ((\alpha(t) + c\beta_{ji}(L \otimes BK_1))\eta_i(t)) \times \Phi_i(t)]dt + (\alpha(t)I_N \otimes BK_1)D_1 \eta_i(t) \\
    \dot{\check{e}}_i(t) &= [(I_N \otimes (A + BK_1) - \alpha(t)(L \otimes I_p))\tilde{e}_i(t) + (\alpha(t)I_N \otimes BK_1)D_2 \eta_i(t) \\
    &\quad - \alpha(t)D_2 \eta_i(t)]dt + (\alpha(t)I_N \otimes BK_1)D_2 \eta_i(t)
\end{align*}
\]

where \(D_1 = ((I_N - \frac{1}{N}ss^T) \otimes I_p)\bar{e}_i(t), i = 1, 2, 2\) According the property of signed graph, we know there exists non-singular matrix \(U = (s U_1)\) such that \(U^T L U = \Lambda = \text{diag}([\lambda_1(\Lambda_1, \Lambda)], \Lambda = \text{diag}([\lambda_2(\Lambda_2, \Lambda_3(\Lambda_2, \cdots, \Lambda_N(\Lambda_2, \Lambda_2)\Lambda_2))\Lambda_2, \Lambda_2)\Lambda_2, \Lambda_2)\Lambda_2, \Lambda_2)\).

\[
\begin{align*}
    \dot{\check{e}}_i(t) &= ((I_N - \frac{1}{N}ss^T) \otimes I_p)\check{e}_i(t) \\
    \dot{\tilde{e}}_i(t) &= [(I_N - \frac{1}{N}ss^T) \otimes I_p)\tilde{e}_i(t) \\
    &\quad - (\alpha(t)I_N \otimes BK_1)\tilde{e}_i(t)]dt + (\alpha(t)I_N \otimes BK_1)\tilde{e}_i(t)]dt
\end{align*}
\]

and

\[
\begin{align*}
    \dot{\check{w}}_i(t) &= (I_N - \frac{1}{N}ss^T)\check{w}_i(t) \\
    \dot{\tilde{w}}_i(t) &= (I_N - \frac{1}{N}ss^T)\tilde{w}_i(t) \\
    \dot{\tilde{w}}_i(t) &= (I_N - \frac{1}{N}ss^T)\tilde{w}_i(t) \\
    \dot{\check{w}}_i(t) &= (I_N - \frac{1}{N}ss^T)\check{w}_i(t)
\end{align*}
\]
Denote \( \xi(t) = (\xi^T_1(t) \xi^T_2(t))^T \), so we have
\[
d\xi(t) = \tilde{A}\xi(t)dt + \alpha(t)\tilde{B}\xi(t)dt + \tilde{C}e(t)dt + \alpha(t)\tilde{D}dW(t)
\]
where
\[
\tilde{A} = \begin{pmatrix} I_{N-1} \otimes A - c\beta_0(\Lambda \otimes BK_1) & c(I_{N-1} \otimes BK_1)\Phi \\ 0 & I_{N-1} \otimes (A + BK_2) \end{pmatrix}, \\
\tilde{B} = \begin{pmatrix} \beta_0(\Lambda \otimes BK_1) & 0 \\ (\Lambda \otimes I_n) & -(\Lambda \otimes I_n) \end{pmatrix}, \\
\tilde{C}_1 = \begin{pmatrix} \beta_0(\Lambda \otimes BK_1) \\ (\Lambda \otimes I_n) \end{pmatrix}, \\
\tilde{C}_2 = \begin{pmatrix} \beta_0(\Lambda \otimes BK_1) \\ 0 \end{pmatrix}, \\
\tilde{D} = \begin{pmatrix} I_{N-1} \otimes BK_1 & D_1 \end{pmatrix}.
\]

**Theorem 1:** Consider multi-agent system (1) with the controller (4). Let assumptions 1-3 hold, \( F \) be any gain matrices such that \( A - FC \) is Hurwitz matrices, \( P_i, i = 1, 2 \) are the solution of the Algebraic Riccati equations (ARE)
\[
P_iA + A^T P_i - P_iBB^T P_i + (\tilde{\alpha} + c)P_i + k_i I_n = 0,
\]
for \( i = 1, 2 \), and \( k_1 > 0 \) and \( k_2 > 0 \) satisfy
\[
\begin{align*}
&k_1 > 0, \\
&k_1 - k_2^{-1}(Q_1^T Q_1 + \tilde{d}^2 Q_2^T Q_2) \geq 0,
\end{align*}
\]

where \( \tilde{\alpha} \) is the boundness of \( \alpha(t) \), i.e., \( \alpha(t) \leq \tilde{\alpha} \), \( \tilde{Q}_1 = (c \otimes P_1BB^T P_1)\Phi \), \( \tilde{Q}_2 = \Lambda \otimes P_1 \), then the mean square bipartite consensus of MASs can be achieved if controller gain \( K = B^T P_1 \), \( K_2 = -\frac{1}{2} B^T P_2 \), \( \epsilon_B > 0 \).

**Proof:** Construct a candidate Lyapunov function as
\[
V(t) = \xi^T(t)(P\tilde{A} + \tilde{A}^TP + \alpha(t)(P\tilde{B} + \tilde{B}^TP))\xi(t) + 2\epsilon_B^2(t)(P\tilde{C}e(t)) + \alpha^2(t)Tr(D^T D)dt + 2\alpha(t)\xi^T(t)(P\tilde{D})dW(t).
\]

It is clear that
\[
2\xi^T(t)(P\tilde{C}e(t)) \leq (\alpha(t) + c)\xi^T(t)(P\xi(t) + \alpha(t)e^2(t)\tilde{C}_1^T P\tilde{C}_1 e(t) + ce^2(t)\tilde{C}_2^T P\tilde{C}_2 e(t) \\
\leq (\tilde{\alpha} + c)\xi^T(t)(P\xi(t) + \tilde{\alpha}e^2(t)\tilde{C}_1^T P\tilde{C}_1 e(t) + ce^2(t)\tilde{C}_2^T P\tilde{C}_2 e(t).
\]

\[
\begin{align*}
\xi^T(t)(P\tilde{A} + \tilde{A}^TP + \alpha(t)(P\tilde{B} + \tilde{B}^TP))\xi(t) \\
\leq -2\xi^T(t)(P\tilde{C}e(t)) + \alpha^2(t)Tr(D^T D)dt + 2\alpha(t)\xi^T(t)(P\tilde{D})dW(t)
\end{align*}
\]

For any \( T \), define a stopping time \( \tau_{m} \) \( = \inf\{t \in [0, T] | V(t) \geq \tilde{m}\} \), where \( \tilde{m} > 0 \). According to (12), we have
\[
\mathbb{E}\{V(t) \wedge \tau_{m} \} - \mathbb{E}\{V(0)\} \leq -\mathbb{E}\{\gamma \int_{t}^{T}(a\tilde{e}(s))^{2}(P\tilde{D})dW(s)\} + \mathbb{E}\{\int_{t}^{T}a^2(s)Tr(D^T D)ds\}
\]

\[
\leq -\mathbb{E}\{\gamma \int_{t}^{T}(V(s))_{\chi_{\{t \leq \tau_{m}\}}}ds\} + \mathbb{E}\{\int_{t}^{T}a^2(s)Tr(D^T D)ds\}
\]

\[
\leq -\mathbb{E}\{\gamma \int_{t}^{T}V(s)_{\chi_{\{t \leq \tau_{m}\}}}ds + \int_{t}^{T}\tilde{a}(s)e^2(s)\tilde{C}_1^T P\tilde{C}_1 e(s)ds + \int_{t}^{T}ce^2(s)\tilde{C}_2^T P\tilde{C}_2 e(s)ds\}
\]

**FIGURE 1.** The communication topology among agents.
The observer error of agent 1.

\[ + \int_{t_0}^{t} 2\alpha(s)e^{T}(s)(\mathcal{P}\mathcal{D})\chi_{\{t \leq t_0^{b}\}}dW(t) \]
\[ \leq \text{Tr}(\mathcal{D}^{T}\mathcal{D})\mathbb{E}\{\int_{t_0}^{t} \alpha^2(s)ds\} + \|e(0)\|^{2}(\bar{\alpha}^{T}\mathcal{P}\mathcal{C}_1 \]
\[ + c\|\mathcal{C}_1^{T}\mathcal{P}\mathcal{C}_2\mathbb{E}\{\int_{t_0}^{t} e^{-2\lambda_{s}s}ds\} \]

where \( \chi_{\{t \leq t_0^{b}\}} \) is indicative function. Equation (14) implies that \( \mathbb{E}[V(t ∧ t_0^{b})] \leq M_1 \), where \( M_1 \) is a positive constant which related with \( t_0, T \). It is clear that \( \lim_{t \to \infty} t ∧ t_0^{b} = t \). According to Fatou lemma, we have \( \sup_{0 \leq s \leq T} \mathbb{E}[V(s)] \leq M_1 \). So we have \( \int_{t_0}^{t} \alpha^2(s)V(s)ds \leq \sup_{0 \leq s \leq T} \mathbb{E}[V(s)] \int_{t_0}^{s} \alpha^2(s)ds \) \( < \infty \), which yields that \( \mathbb{E}[\int_{t_0}^{t} \alpha^2(s)||e^{T}(s)(\mathcal{P}\mathcal{D})||^{2}ds] \leq \mathbb{E}[\int_{t_0}^{t} \alpha^2(s)||\mathcal{P}\mathcal{D}||^{2}V(s)\chi_{\{s \leq t_0\}}ds] \) \( < \infty \), for \( t \in [t_0, T] \). Since \( T \) can be arbitrary, we have \( \mathbb{E}[\int_{t_0}^{t} \alpha^2(s)||e^{T}(s)(\mathcal{P}\mathcal{D})||^{2}ds] \) \( < \infty \), \( t \geq t_0 \), which implies that equation (13) holds. According to the comparison lemma and equation (12), we have

\[ \mathbb{E}[V(t)] \]
\[ \leq e^{-\gamma(t-t_0)}\mathbb{E}[V(t_0)] + \text{Tr}(\mathcal{D}^{T}\mathcal{D})\int_{t_0}^{t} \alpha^2(s)e^{-\gamma(t-s)}ds \]
\[ + c\int_{t_0}^{t} e^{-\gamma(t-s)}e^{T}(s)\bar{\alpha}^{T}\mathcal{C}_2e(s)ds \]
\[ + \int_{t_0}^{t} e^{-\gamma(t-s)}\alpha e^{T}(s)\bar{\alpha}^{T}\mathcal{C}_1e(s)ds \]

Because that \( \int_{t_0}^{t} \alpha^2(s)ds \) \( < \infty \), we have that there exists \( t^* \) such that \( \int_{t_0}^{t^*} \alpha^2(s)ds < \epsilon \) for any given \( \epsilon > 0 \). For \( t > t^* \), we have

\[ 0 \leq \int_{t_0}^{t} \alpha^2(s)e^{-\gamma(t-s)}ds \]
\[ = \int_{t_0}^{t^*} \alpha^2(s)e^{-\gamma(t-s)}ds + \int_{t_0}^{t} \alpha^2(s)e^{-\gamma(t-s)}ds \]
\[ \leq e^{-\gamma(t-t^*)} \int_{t_0}^{t^*} \alpha^2(s)ds + \int_{t_0}^{t} \alpha^2(s)ds. \]

\textbf{FIGURE 2.} The observer error of agent 1.

\textbf{FIGURE 3.} (a) The trajectory of the consensus error \( x_{11}(t) - \frac{1}{6} \sum_{j=1}^{n} s_{i} s_{j} x_{ij}(t) \), \( i = 1, 2, 3, 4, 5, 6 \). (b) The trajectory of the consensus error \( x_{22}(t) - \frac{1}{6} \sum_{j=1}^{n} s_{i} s_{j} x_{2j}(t) \), \( i = 1, 2, 3, 4, 5, 6 \). (c) The trajectory of the consensus error \( x_{33}(t) - \frac{1}{6} \sum_{j=1}^{n} s_{i} s_{j} x_{3j}(t) \), \( i = 1, 2, 3, 4, 5, 6 \). (d) The trajectory of the consensus error \( x_{44}(t) - \frac{1}{6} \sum_{j=1}^{n} s_{i} s_{j} x_{4j}(t) \), \( i = 1, 2, 3, 4, 5, 6 \).
Because $\varepsilon$ is arbitrary, we have $\lim_{t \to \infty} \int_0^t \alpha^i(s)e^{-\gamma(t-s)}ds = 0$. According to generalization of mean value theorem for integrals, there exists $s^*$ such that

$$
\int_0^t e^{-\gamma(t-s)}e^T(s)\tilde{C}_t\mathcal{P}\tilde{C}_ie(s)ds = e^T(s^*)\tilde{C}_t\mathcal{P}\tilde{C}_ie(s^*)\int_0^t e^{-\gamma(t-s)}ds
$$

$$
= e^T(s^*)\tilde{C}_t\mathcal{P}\tilde{C}_ie(s^*)(1 - e^{-\gamma t}), \quad i = 1, 2, \ldots
$$

Because that $\lim_{t \to \infty} e(t) = 0$ and $\lim_{t \to \infty} (1 - e^{-\gamma t}) = 1$, let $t \to \infty$, we have $\lim_{t \to \infty} \int_0^t e^{-\gamma(t-s)}e^T(s)\tilde{C}_t\mathcal{P}\tilde{C}_ie(s)ds = 0, \quad i = 1, 2, \ldots$. According to (15)-(17), one has $\lim_{t \to \infty} \mathbb{E}[V(t)] = 0$, which implies $\lim_{t \to \infty} \mathbb{E}[\|e_i(t)\|^2] = 0$.

Remark 3: From the proof of Theorem 1, one may notice that $\alpha(t)$ plays an important role in dealing with the measurement noises part and the consensus analysis. Due to the existence of noises-free part in the controller, here we reduce the requirements of $\alpha(t)$, i.e. we do not require $\int_0^\infty \alpha(s)ds = \infty$. Compared with state feedback results in [25]-[27], [29], controller based on output measurement information in this paper is practical.

Remark 4: The requirement of structure balance of $\mathcal{G}$ is important for the bipartite consensus for MASs with cooperative and competitive communication. If only cooperative relationships exist in the MASs, the structure balance is satisfied, which implies the results can be extend to the unsigned graph case.

IV. SIMULATION

Consider a linear multi-agent systems with six nodes and suppose that the system of each agent is described by

$$
\begin{cases}
\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \\
y_i(t) = Cx_i(t), \quad i = 1, 2, \ldots, N
\end{cases}
$$

where

$$
A = \begin{pmatrix}
0 & 10 & 0 & 0 \\
-20 & 0 & 10 & 0 \\
0 & 0 & 0 & 10 \\
0 & 0 & -10 & 0
\end{pmatrix}, \quad B = \begin{pmatrix}
0 \\
1 \\
0 \\
0
\end{pmatrix},
$$

$$
C = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{pmatrix}.
$$

the communication topology among agents is shown in FIGURE 1, where the solid edges are utilized to describe the cooperative relationships and the dash edges are the competitive case. The Laplacian matrix $L$ of graph $\mathcal{G}$ is

$$
\begin{pmatrix}
2 & -1 & 0 & 0 & 0 & -1 \\
-1 & 2 & 1 & 0 & 0 & 0 \\
0 & 1 & 2 & -1 & 0 & 0 \\
0 & 0 & -1 & 2 & 1 & 0 \\
0 & 0 & 0 & 1 & 2 & -1 \\
-1 & 0 & 0 & 0 & -1 & 2
\end{pmatrix}.
$$

It is clear that the graph $\mathcal{G}$ connective and can be divided into two subsets $\mathcal{V}_1 = \{1, 2, 5, 6\}$ and $\mathcal{V}_2 = \{3, 4\}$, which implies that $\mathcal{G}$ is structurally balanced. The gauge transformation matrix is $S = \text{diag}(1, 1, -1, -1, 1, 1)$, and the eigenvalues of the Laplacian matrix $L = \{0, 1, 1, 3, 3, 4\}$. Take $c = 5$, $\beta_p = \frac{1}{4}$, and $K_1 = \{15.6579, 21.0901, -5.7519, 14.5213\}$, $K_2 = \{-105.6927, -60.0162, 29.5310, -80.3165\}$. The time-varying function $\alpha(t)$ is chosen as $\alpha(t) = \frac{1}{\tau(t)}$ which is not satisfied $\int_0^\infty \alpha(s)ds = \infty$. Under the proposed controller (4), the observer error $s(t) - \tilde{s}(t)$ of agent 1 is shown in FIGURE 2. The tracking errors $x_i(t) - \tilde{s} \sum_{j=1}^6 s_i y_j(t)$, $i = 1, 2, 3, 4, 5, 6$ of agents are shown in FIGURE 3.

V. CONCLUSION

This paper investigates the consensus of linear multi-agent systems with measurement noises and antagonistic communication topology. To solve this proposed problem, a observer-based controller is given. It is shown that the condition for the signed undirected graph with connected and structurally balanced is the sufficient condition that guarantees the mean-square bipartite consensus of multi-agent systems. In the future, the interesting topics such as the bipartite consensus under switching topology graph, containment control problems for multi-agent system with measurement noises and signed graph, the event-triggered controller for multi-agent system with signed graph will be considered.

REFERENCES


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