J. Evol. Equ. 18 (2018), 681–713

1424-3199/18/020681-33, *published online* November 24, 2017
https://doi.org/10.1007/s00028-017-0416-0

Journal of Evolution Equations

CrossMark

Obstacle problem for evolution equations involving measure data and operator corresponding to semi-Dirichlet form

TOMASZ KLIMSIAK

Abstract. In the paper, we consider the obstacle problem, with one and two irregular barriers, for semilinear evolution equation involving measure data and operator corresponding to a semi-Dirichlet form. We prove the existence and uniqueness of solutions under the assumption that the right-hand side of the equation is monotone and satisfies mild integrability conditions. To treat the case of irregular barriers, we extend the theory of precise versions of functions introduced by M. Pierre. We also give some applications to the so-called switching problem.

1. Introduction

Let E be a locally compact separable metric space, let m be a Radon measure on E with full support, and let $\{B^{(t)}, t \geq 0\}$ be a family of regular semi-Dirichlet forms on $L^2(E; m)$ with common domain V satisfying some regularity conditions. By L_t, denote the operator corresponding to the form $B^{(t)}$. In the present paper, we study the obstacle problem with one and two irregular barriers. In the case of one barrier $h : E \to \mathbb{R}$, it can be stated as follows: For given $\varphi : E \to \mathbb{R}$, $f : [0, T] \times E \times \mathbb{R} \to \mathbb{R}$ and smooth (with respect to the parabolic capacity Cap associated with $\{B^{(t)}, t \geq 0\}$) measure μ on $E_{0,T} \equiv (0, T) \times E$, find $u : \bar{E}_{0,T} \equiv (0, T] \times E \to \mathbb{R}$ such that

$$\begin{cases} -\frac{\partial u}{\partial t} - L_t u = f(\cdot, u) + \mu \text{ on the set } \{u > h\}, \quad u(T, \cdot) = \varphi, \\ -\frac{\partial u}{\partial t} - L_t u \geq f(\cdot, u) + \mu \text{ on } E_{0,T}, \\ u \geq h. \end{cases} \tag{1.1}$$

In the second part of the paper, we show how the results on (1.1) can be used to solve some system of variational inequalities associated with so-called switching problem.

Problems of the form (1.1) are at present quite well investigated in the case where L_t are local operators. Classical results for one or two regular barriers and L^2-integrable data are to be found in the monograph [1]. Semilinear obstacle problem with uniformly elliptic divergent form operators L_t and one or two irregular barriers was studied carefully in [13,14] in case of L^2-data and in [20] in case of measure data. In [2], it is considered evolutianry p-Laplacian type equation (with $p \in (1, \infty)$). In

Mathematics Subject Classification: 35K86, 35K87 an important paper [30], linear problem (1.1) with one irregular barrier, L^2-data and operators L_t associated with Dirichlet forms is considered. The aim of the present paper is to generalize or strengthen the existing results in the sense that we consider semilinear equations involving measure data with two barriers and wide class of operators corresponding to semi-Dirichlet forms. As for the obstacles, we only assume that they are quasi-càdlàg functions satisfying some mild integrability conditions. The class of quasi-càdlàg functions naturally arises in the study of evolution equations. It includes quasi-continuous functions and parabolic potentials (which in general are not quasi-continuous).

When considering problem (1.1) with measure data, one of the first difficulties one encounters is the proper definition of a solution. Because of measure data, the usual variational approach is not applicable. Moreover, even in the case of L^2-data, the variational inequalities approach does not give uniqueness of solutions (see [25]). Therefore, following [20,30], we consider so-called complementary system associated with (1.1). Roughly speaking, by a solution of this system we mean a pair (u, ν) consisting of a quasi-càdlağ function $u : \bar{E}_{0,T} \to \mathbb{R}$ and a positive smooth measure ν on $E_{0,T}$ such that

$$-\frac{\partial u}{\partial t} - L_t u = f(\cdot, u) + \nu + \mu \quad \text{on } E_{0,T}, \quad u(T, \cdot) = \varphi \quad \text{in } E, \tag{1.2}$$

$$\text{“}\nu \text{ is minimal}'', \tag{1.3}$$

$$u \geq h \quad \text{q.e.}, \tag{1.4}$$

where q.e. means quasi-everywhere with respect to the capacity Cap. Of course, in the above formulation one has to give rigorous meaning to (1.2) and (1.3). As for (1.2), we develop some ideas from the paper [15] devoted to evolution equations involving measure data and operators associated with semi-Dirichlet forms.

In the paper, we assume that μ belongs to the class $\mathbb{M}$ of smooth Borel measures with finite potential, which under additional assumption of duality for the family $\{B^{(t)}, t \geq 0\}$, takes the form

$$\mathbb{M} = \bigcup_{\rho} \mathcal{M}_\rho,$$

where $\mathcal{M}_\rho$ denotes the set of all smooth signed measures on $E_{0,T}$ such that $\|\mu\|_\rho = \int_E \rho\, d|\mu| < \infty$, and the union is taken over all strictly positive excessive functions ρ.

If $\nu \in \mathbb{M}$, then we define a solution of (1.2) as in [15]. To formulate this definition, denote by $\mathbf{M}$ a Hunt process $\{(\mathbf{X}_t)_{t\geq 0}, (P_z)_{z\in E_{0,T}}\}$ with life time ζ associated with the operator $\frac{\partial}{\partial t} + L_t$, and set $\zeta_\upsilon = \zeta \wedge (T - \upsilon(0))$, where υ is the uniform motion to the right, i.e., $\upsilon(t) = \upsilon(0) + t$ and $\upsilon(0) = s$ under P_z with $z = (s, x)$. By A^μ, A^ν denote natural additive functionals of $\mathbf{M}$ in the Revuz correspondence with μ and ν, respectively. By a solution of (1.2), we mean a function u such that for q.e. $z \in E_{0,T}$,

$$u(z) = E_z\Big(\varphi(\mathbf{X}_{\zeta_\upsilon}) + \int_0^{\zeta_\upsilon} f(\mathbf{X}_r, u(\mathbf{X}_r))\,\mathrm{d}r + \int_0^{\zeta_\upsilon} \mathrm{d}A_r^\mu + \int_0^{\zeta_\upsilon} \mathrm{d}A_r^\nu\Big). \tag{1.5}$$

Formula (1.5) may be viewed as a nonlinear Feynman–Kac formula.

Unfortunately, in general, the "reaction measure" ν need not belong to $\mathbb{M}$ (in fact, as shown in [14], in case of two barriers, it may be a nowhere Radon measure). In such case, we say that u satisfies (1.2) if u is of class (D), i.e., there is a potential on $\bar{E}_{0,T}$ (see Sect. 3.2) dominating $|u|$ on $E_{0,T}$, and there exists a local martingale M (with $M_0 = 0$) such that the following stochastic equation is satisfied under the measure P_z for q.e. $z \in E_{0,T}$:

$$
\begin{aligned}
u(\mathbf{X}_t) = \varphi(\mathbf{X}_{\zeta_\upsilon}) &+ \int_t^{\zeta_\upsilon} f(\mathbf{X}_r, u(\mathbf{X}_r))\,\mathrm{d}r \\
&+ \int_t^{\zeta_\upsilon} \mathrm{d}A_r^\mu + \int_t^{\zeta_\upsilon} \mathrm{d}A_r^\nu - \int_t^{\zeta_\upsilon} \mathrm{d}M_r, \quad t \in [0, \zeta_\upsilon].
\end{aligned} \tag{1.6}
$$

In the above definition the requirement that u is of class (D) is very important. The reason is that (1.6) is also satisfied by functions solving equation (1.2) with additional nontrivial singular (with respect to Cap) measure on its right-hand side (see [17]). These functions are not of class (D).

If $\nu \in \mathbb{M}$, then (1.6) is equivalent to (1.5). Furthermore, if the time-dependent Dirichlet form $\mathcal{E}^{0,T}$ on $L^2(E_{0,T}; m_1 := \mathrm{d}t \otimes m)$ determined by the family $\{B^{(t)}, t \geq 0\}$ has the dual Markov property and $\varphi \in L^1(E; m)$, $f(\cdot, u) \in L^1(E_{0,T}; \mathrm{d}t \otimes m)$, $\mu, \nu \in \mathcal{M}_1$ (i.e., μ, ν are bounded), then the solution u in the sense of (1.5) is a renormalized solution of (1.2) in the sense defined in [21]. In particular, this means that u has some further regularity properties and may be defined in purely analytical terms. More precisely, u is a renormalized solution if the truncations $T_k u = -k \vee (u \wedge k)$ of u belong to the space $L^2(0, T; V)$, and there exists a sequence $\{\lambda_k\} \subset \mathcal{M}_1$ such that $\|\lambda_k\|_1 \to 0$ as $k \to \infty$, and for every bounded $v \in L^2(0, T; V)$ such that $\frac{\partial v}{\partial t} \in L^2(0, T; V')$ and $v(0) = 0$ we have

$$
\mathcal{E}^{0,T}(T_k u, v) = (T_k \varphi, v(T))_{L^2} + (f(\cdot, u), v)_{L^2} + \int_{E_{0,T}} v\,\mathrm{d}\mu + \int_{E_{0,T}} v\,\mathrm{d}\nu_k \tag{1.7}
$$

for all $k \geq 0$. In case of local operators of Leray–Lions type, the above definition of renormalized solutions was proposed in [29] (for the case of elliptic equations see [4]).

We now turn to condition (1.3). Intuitively, it means that ν "acts only if necessary." If h is quasi-continuous, this statement means that ν acts only when $u = h$, because then the right formulation of the minimality condition takes the form

$$
\int_{E_{0,T}} (u - h)\,\mathrm{d}\nu = 0 \tag{1.8}
$$

(see [20,30]). If h is only quasi-càdlàg, the situation is more subtle. For such h, the function u satisfying (1.5) is also quasi-càdlàg, so the left-hand side of (1.8) is well defined, because ν is a smooth measure. However, in general, the left-hand side is strictly positive. M. Pierre has shown (in the case of linear equations with L^2-data and Dirichlet forms) that for general barrier h the condition

$$\int_{E_{0,T}} (\tilde{u} - \tilde{h})\,\mathrm{d}\nu = 0 \tag{1.9}$$

is satisfied. Here $\tilde{u}$ is a precise m_1-version of u and $\tilde{h}$ is an associated precise version of h. The notions of a precise m_1-version of a potential and an associated precise version of a function h dominated by a potential (which is not necessarily m_1-version of h) were introduced in [30,31]. In the paper, in case of semi-Dirichlet forms, we use probabilistic methods to introduce another notion of a precise m_1-version $\hat{u}$ of a quasi-càdlàg function u (note that potentials are quasi-càdlàg). Since our barriers as well as solutions to (1.2)–(1.4) are quasi-càdlàg, we do not need the notion of an associated precise version. We show that if u is an L^2 potential, then

$$\hat{u} = \tilde{u} \quad \text{q.e,}$$

and for any quasi-càdlàg function u which is dominated by an L^2 potential,

$$\hat{u} \le \tilde{u} \quad \text{q.e.,} \qquad \int (\tilde{u} - \hat{u})\,\mathrm{d}\nu = 0.$$

It follows in particular that in case of L^2 data and Dirichlet form, (1.9) is equivalent to

$$\int_{E_{0,T}} (\hat{u} - \hat{h})\,\mathrm{d}\nu = 0 \tag{1.10}$$

One reason why we introduce a new notion of a precise version is that it is applicable to wider classes of operators and functions then those considered in [30,31]. The second is that our definition is more direct that the construction of a precise version given in [30,31]. Namely, in our approach by a precise version of a quasi-càdlàg function u on $\bar{E}_{0,T}$ we mean a function $\hat{u}$ on $\bar{E}_{0,T}$ such that for q.e. $z \in E_{0,T}$,

$$\hat{u}(\mathbf{X}_{t-}) = u(\mathbf{X})_{t-}, \quad t \in (0, \zeta_\upsilon)$$

As a consequence, our definition appears to be very convenient for studying obstacle problems and may be applied to quite general class of equations (possibly nonlinear with measure data and two obstacles).

In case of one obstacle, the main result of the paper says that if φ, $f(\cdot, 0)$ satisfy mild integrability conditions, and f is monotone and continuous with respect to u, then for every $\mu \in \mathbb{M}$ there exists a unique solution (u, ν) of (1.1), i.e., a unique pair (u, ν) consisting of a quasi-càdlàg function u on $\bar{E}_{0,T}$ and positive smooth measure ν on $E_{0,T}$ such that (1.6), (1.10) and (1.4) are satisfied. We also give conditions under which $\nu \in \mathbb{M}$ or $\nu \in \mathcal{M}_1$, i.e., when equivalent to (1.6) formulations (1.5) or (1.7) may be used. Moreover, we show that $u_n \nearrow u$ q.e., where u_n is a solution of the following Cauchy problem

$$-\frac{\partial u_n}{\partial t} - L_t u_n = f(\cdot, u_n) + n(u_n - h)^- + \mu, \quad u_n(T, \cdot) = \varphi. \tag{1.11}$$

Our probabilistic approach allows us to prove similar results also for two quasi-càdlàg barriers h_1, h_2 satisfying some separation condition. In the case of two barriers, the measure ν appearing in the definition of a solution is a signed smooth measure. Moreover, we replace the minimality condition (1.9) by

$$\int_{E_{0,T}} (\hat{u} - \hat{h}_1)\, d\nu^+ = \int_{E_{0,T}} (\hat{h}_2 - \hat{u})\, d\nu^- = 0, \tag{1.12}$$

and replace condition (1.4) by $h_1 \le u \le h_2$ q.e. We show that under the same conditions on φ, f, μ as in the case of one barrier, there exists a unique solution (u, ν) of the obstacle problem with two barriers. We also show that $\bar{u}_n \nearrow u$ and $\lambda_n \nearrow \nu^-$, where $(\bar{u}_n, \lambda_n)$ is a solution of problem of the form (1.1), but with one upper barrier h_2 and f replaced by

$$f_n(t, x, y) = f(t, x, y) + n(y - h_1(t, x))^-.$$

We prove these results under two different separation conditions. The first one, more general, may be viewed as some analytical version of the Mokobodzki condition considered in the literature devoted to reflected stochastic differential equations (see, e.g., [16]). The second one, which is simpler and usually easier to check, as yet, has not been considered in the literature on evolution equations. It says that

$$h_1 < h_2, \quad \hat{h}_1 < \hat{h}_2 \quad \text{q.e.} \tag{1.13}$$

If h_1, h_2 are quasi-continuous, then (1.13) reduces to the condition $h_1 < h_2$ q.e., because quasi-continuous functions are precise (see [14] for this case).

Note also that at the end of Sect. 6 we show that the study of the obstacle problem with one merely measurable barrier (or two measurable barriers satisfying the Mokobodzki condition) can be reduced to the study of the obstacle problem with quasi-càdlàg barriers. It should be stressed, however, that when dealing with merely measurable barriers, our definition of a solution is weaker. Namely, instead of (1.4) we only require that $u \ge h$ m_1-a.e. (or $h_1 \le u \le h_2$ m_1-a.e. in case of two barriers).

In the last part of the paper, we use our results on (1.1) to study so-called switching problem (see Sect. 7). This problem is closely related to system of quasi-variational inequalities, which when written as a complementary system has the form

$$-\frac{\partial u^j}{\partial t} - L_t u^j = f^j(t, x, u) + \nu^j + \mu^j, \tag{1.14}$$

$$\int_{E_{0,T}} (\hat{u}^j - \hat{H}^j(\cdot, u))\, d\nu^j = 0, \tag{1.15}$$

$$u^j \geq H^j(\cdot, u) \quad \text{q.e.}, \tag{1.16}$$

where

$$H^j(z, y) = \max_{i \in A_j} h_{j,i}(z, y^i), \quad z \in E_{0,T}, \ y \in \mathbb{R}^N.$$

In (1.14)–(1.16), we are given $f^j : E_{0,T} \times \mathbb{R}^N \to \mathbb{R}^N$, $h_{j,i} : E_{0,T} \times \mathbb{R} \to \mathbb{R}$, $\mu^j \in \mathbb{M}$ and sets $A_j \subset \{1, \dots, j-1, j+1, \dots, N\}$, and we are looking for a pair $(u, \nu) = ((u^1, \dots, u^N), (\nu^1, \dots, \nu^N))$ satisfying (1.14)–(1.16) for $j = 1, \dots, N$. Note that in (1.16) the barrier H^j depends on u.

Systems of the form (1.14)–(1.16) were subject to numerous investigations, but only in the framework of viscosity solutions and for local operators (see [6,7,9–12,24]) or for special class of nonlocal operators associated with a random Poisson measure (see [23]). In the paper, we prove an existence result for (1.14)–(1.16). In the important special case where

$$h_{j,i}(z, y) = -c_{j,i}(z) + y^i,$$

we show that there is a unique solution of (1.14)–(1.16) and, moreover, that u is the value function for the optimal switching problem related to (1.14)–(1.16).

2. Preliminaries

In the paper, E is a locally compact separable metric space and m a Radon measure on E such that $\mathrm{supp}[m] = E$. For $T > 0$, we set $E_{0,T} = (0, T) \times E$, $\bar{E}_{0,T} = (0, T] \times E$.

Recall (see [26]) that a form (B, V) is called semi-Dirichlet on $L^2(E; m)$ if V is a dense linear subspace of $L^2(E; m)$, B is a bilinear form on $V \times V$, and the following conditions (B1)–(B4) are satisfied:

(B1) B is lower bounded, i.e., there exists $\alpha_0 \geq 0$ such that $B_{\alpha_0}(u, u) \geq 0$ for $u \in V$, where $B_{\alpha_0}(u, v) = B(u, v) + \alpha_0(u, v)$,

(B2) B satisfies the sector condition, i.e., there exists $K > 0$ such that

$$|B(u, v)| \leq K B_{\alpha_0}(u, u)^{1/2} B_{\alpha_0}(v, v)^{1/2}, \quad u, v \in V,$$

(B3) B is closed, i.e., for every $\alpha > \alpha_0$ the space V equipped with the inner product $B^{(s)}_\alpha(u, v) = \frac{1}{2}(B_\alpha(u, v) + B_\alpha(v, u))$ is a Hilbert space,

(B4) B has the Markov property, i.e., for every $a \geq 0$, $B(u \wedge a, u \wedge a) \leq B(u \wedge a, u)$ for all $u \in V$.

Condition (B4) is called the Markov property, because it is equivalent to the fact that the semigroup $\{T_t, t \geq 0\}$ associated with (B, V) is sub-Markov (see [26, Theorem 1.1.5]). Recall that (B, V) is said to have the dual Markov property if

(B5) for every $a \geq 0$,

$$B(u \wedge a, u \wedge a) \leq B(u, u \wedge a), \quad u \in V.$$

Note that (B5) is equivalent to the fact that the dual semigroup $\{\hat{T}_t, t \geq 0\}$ associated with (B, V) is sub-Markov (see [26, Theorem 1.1.5]). For the notions of transiency and regularity, see [26, Sections 1.2, 1.3].

In the paper, $\{B^{(t)}, t \in \mathbb{R}\}$ is a family of regular semi-Dirichlet forms on $L^2(E; m)$ satisfying (B2), (B3) with some constants K, α_0 not depending on t. We also assume that for all $u, v \in V$ the mapping $\mathbb{R} \ni t \mapsto B^{(t)}(u, v)$ is measurable, and for some $\lambda \geq 1$,

$$\lambda^{-1} B^{(0)}(u, u) \leq B^{(t)}(u, u) \leq \lambda B^{(0)}(u, u), \quad u \in V, \quad t \in \mathbb{R}.$$

Let $(\mathcal{E}, D[\mathcal{E}])$ denote the time-dependent semi-Dirichlet form on $L^2(E_1; m_1)$ ($E_1 := \mathbb{R} \times E$) associated with the family $\{B^{(t)}, t \in \mathbb{R}\}$ (see [26, Section 6.1]), and Cap denotes the associated capacity. We say that some property holds quasi-everywhere (q.e. for short) if it holds outside some set $B \subset E_1$ such that $\mathrm{Cap}(B) = 0$. Capacity Cap on $E_{0,T}$ is equivalent to the capacity considered in [30,31] (see [32]) in the context of parabolic variational inequalities.

Let μ be a signed measure on E_1. By $|\mu|$, we denote the variation of μ, i.e., $|\mu| = \mu^+ + \mu^-$, where μ^+ (resp. μ^-) denote the positive (resp. negative) part of μ. Recall that a Borel measure on E_1 is called smooth if μ charges no exceptional sets and there exists an increasing sequence $\{F_n\}$ of closed subsets of E_1 such that $|\mu|(F_n) < \infty$ for $n \geq 1$ and $\mathrm{Cap}(K \setminus F_n) \to 0$ for every compact $K \subset E_1$.

It is known (see [26, Section 6.3]) that there exists a unique Hunt process $\mathbf{M} = \{(\mathbf{X})_{t \geq 0}, (P_z)_{z \in E_1}\}$ with life time ζ associated with the form $(\mathcal{E}, D[\mathcal{E}])$. Moreover,

$$\mathbf{X}_t = (\upsilon(t), X_{\upsilon(t)}),$$

where υ is the uniform motion to the right, i.e., $\upsilon(t) = \upsilon(0) + t$ and $\upsilon(0) = s$, P_z-a.s. for $z = (s, x)$ (see [26, Theorem 6.3.1]). In the sequel, for fixed $T > 0$ we set

$$\zeta_\upsilon = \zeta \wedge (T - \upsilon(0)).$$

Let us recall that from [26, Lemma 6.3.2] it follows that a nearly Borel set $B \subset E_1$ is of capacity zero iff it is exceptional, i.e., $P_z(\exists_{t>0};\ \mathbf{X}_t \in B) = 0$, q.e. Recall also (see [26, Section 6]) that there is one-to-one correspondence, called Revuz duality, between positive smooth measures and positive natural additive functionals (PNAFs) of $\mathbf{M}$. For a positive smooth measure μ, we denote by A^μ the unique PNAF in the Revuz duality with μ. For a signed smooth measure μ, we put $A^\mu = A^{\mu^+} - A^{\mu^-}$. For a fixed positive measurable function f and a positive Borel measure μ, we denote by $f \cdot \mu$ the measure defined as

$$(f \cdot \mu)(\eta) = \int_{E_1} \eta f \, \mathrm{d}\mu, \quad \eta \in \mathcal{B}^+(E_1).$$

By S_0, we denote the set of all measures of finite energy integrals, i.e., the set of all smooth measures μ having the property that there is $K \geq 0$ such that

$$\int_E |\tilde{\eta}| \, d|\mu| \leq K \|\eta\|_{\mathcal{W}}, \quad \eta \in \mathcal{W},$$

where $\tilde{\eta}$ is a quasi-continuous m_1-version of η (for the existence of such version see [26, Theorem 6.2.11]). By $\mathbb{M}$, we denote the set of all smooth measures on $E_{0,T}$ such that $E_z A^{|\mu|}_{\zeta_\upsilon} < \infty$ for q.e. $z \in E_{0,T}$. $\mathbb{M}_c$ is the set of those $\mu \in \mathbb{M}$ for which A^μ is continuous.

For a given positive smooth measure μ on $\bar{E}_{0,T}$, we set

$$R^{0,T}\mu(z) = E_z A^\mu_{\zeta_\upsilon}, \quad z \in E_{0,T}.$$

Set

$$\mathcal{V}_{0,T} = L^2(0,T;V), \qquad \mathcal{W}_{0,T} = \left\{u \in \mathcal{V}_{0,T} : \frac{\partial u}{\partial t} \in \mathcal{V}'_{0,T}\right\},$$
$$\mathcal{W}_T = \left\{u \in \mathcal{W}_{0,T} : u(T) = 0\right\}, \qquad \mathcal{W}_0 = \left\{u \in \mathcal{W}_{0,T} : u(0) = 0\right\}$$

and

$$\mathcal{E}^{0,T}(u,v) = \begin{cases} \int_0^T \langle -\frac{\partial u}{\partial t}(t), v(t)\rangle\,\mathrm{d}t + \int_0^T B^{(t)}(u(t),v(t))\,\mathrm{d}t, & (u,v) \in \mathcal{W}_T \times \mathcal{V}_{0,T}, \\ \int_0^T \langle u(t), \frac{\partial v}{\partial t}(t)\rangle\,\mathrm{d}t + \int_0^T B^{(t)}(u(t),v(t))\,\mathrm{d}t, & (u,v) \in \mathcal{V}_{0,T} \times \mathcal{W}_0, \end{cases}$$

where $\langle\cdot,\cdot\rangle$ is the duality pairing between V' and V (V' stands for the dual of V). It is known (see [33, Example I.4.9(iii)]) that $\mathcal{E}^{0,T}$ is a generalized semi-Dirichlet form. The operator associated with $\mathcal{E}^{0,T}$ has the form

$$\mathcal{L} = -\frac{\partial}{\partial t} - L_t, \quad D(\mathcal{L}) = \left\{u \in \mathcal{W}_T : \mathcal{L}u \in L^2(E_{0,T}; m_1)\right\},$$

where $(L_t, D(D_t))$ is the operator associated with $(B^{(t)}, V)$, $t \in [0,T]$. By $(G^{0,T}_\alpha)_{\alpha>0}$, we denote the (unique) strongly continuous contraction resolvent on $L^2(0,T; L^2(E;m))$ associated with $\mathcal{E}^{0,T}$ (see Propositions I.3.4 and I.3.6 in [33]).

3. Precise versions of quasi-càdlàg functions

3.1. Precise versions of parabolic potentials in the sense of Pierre and its probabilistic interpretation

We first recall the notion of a precise version of a parabolic potential introduced in [31].

DEFINITION. A measurable function $u \in \mathcal{V}_{0,T} \cap L^\infty(0,T;L^2(E;m))$ is called a parabolic potential if for every nonnegative $v \in \mathcal{W}_0$,

$$\int_0^T \left\langle \frac{\partial v}{\partial t}(t), u(t)\right\rangle \mathrm{d}t + \int_0^T B^{(t)}(u(t),v(t))\,\mathrm{d}t \geq 0.$$

The set of all parabolic potentials will be denoted by $\mathcal{P}^2$.

PROPOSITION 3.1. *Let $u \in \mathcal{P}^2$. Then, there exists a unique positive $\mu \in S_0$ on $\bar{E}_{0,T}$ such that*

$$u(z) = E_z A^{\mu}_{\zeta_\upsilon} \tag{3.1}$$

for m_1-a.e. $z \in E_{0,T}$.

Proof. By [31, Theorem III.1], there exists a positive measure $\mu \in S_0$ on $\bar{E}_{0,T}$ such that

$$\int_t^T \left(\frac{\partial v}{\partial t}(s), u(s)\right) \mathrm{d}s + \int_t^T B^{(s)}(u(s), v(s))\,\mathrm{d}s = \int_{(t,T]\times E} v\,\mathrm{d}\mu - (v(t), u(t))_{L^2}$$

for all $v \in \mathcal{W}_{0,T}$ and $t \in [0, T]$. This and [15, Theorem 3.7] yield (3.1). □

In the sequel, for $u \in \mathcal{P}^2$ we set $\mathcal{E}(u) = \mu$, where $\mu \in S_0$ is the measure from Proposition 3.1.

DEFINITION. A measurable function u on $E_{0,T}$ is called precise (in the sense of M. Pierre) if there exists a sequence $\{u_n\}$ of quasi-continuous parabolic potentials such that $u_n \searrow u$ q.e. on $E_{0,T}$.

The following result has been proved in [31].

THEOREM 3.2. *Each $u \in \mathcal{P}^2$ has a precise m_1-version.*

In what follows, we denote by $\tilde{u}$ a precise version of $u \in \mathcal{P}^2$ in the sense of Pierre. It is clear that $\tilde{u}$ it is defined q.e. In [31], it is proved that the mapping $t \mapsto \tilde{u}(t, \cdot) \in L^2(E; m)$ is left continuous and has right limits, whereas in [15, Proposition 3.4] it is proved that u defined by (3.1) has the property that the mapping $t \mapsto u(t, \cdot) \in L^2(E; m)$ is right continuous and has left limits. Since for any $B \in \mathcal{B}(E)$ and $t \in [0, T]$, $\mathrm{Cap}(\{t\} \times B) = 0$ if and only if $m(B) = 0$ (see [31, Proposition II.4]), it follows that in general $\mathrm{Cap}(\{u \neq \tilde{u}\}) > 0$. In the sequel, for given $u \in \mathcal{P}^2$ we will always consider its version defined by (3.1).

Let us recall that function $x : [a, b] \to \mathbb{R}$ is called càdlàg (resp. càglàd) iff x is right continuous (resp. left continuous) and has left (resp. right) limits.

LEMMA 3.3. *Assume that $\{x_n\}$ is a decreasing sequence of càglàd functions on $[0, T]$ such that $x_n(t) \searrow x(t)$, $t \in [0, T]$, and $x(t) = -a(t) + b(t)$, $t \in [0, T]$, for some nondecreasing function a and càglàd function b on $[0, T]$. Then, a and x are càglàd functions.*

Proof. The proof is analogous to that of [28, Lemma 2.2], so we omit it. □

LEMMA 3.4. *Assume that for each $n \in \mathbb{N}$,*

$$Y^n_t = Y^n_0 - A^n_t + M^n_t, \quad t \in [0, T], \tag{3.2}$$

where A^n is a predictable increasing process with $A^n_0 = 0$, and M^n is a local martingale with $M^n_0 = 0$. If Y^n is positive, $Y^n_t \searrow Y_t$, $t \in [0, T]$, and $E \sup_{t \le T}(|Y^1_t|^2 + |Y_t|^2) < \infty$, then $\hat{Y}_t := \lim_{n\to\infty} Y^n_{t-}$, $t \in [0, T]$, is a càglàd process.

Proof. By [22, Theorem 3.1],

$$E|A^n|_T^2 + E[M^n]_T \le cE \sup_{t\in[0,T]} |Y_t^n|^2 \le cE \sup_{t\in[0,T]} \Big(|Y_t^1|^2 \vee |Y_t|^2\Big).$$

In particular, $\sup_n E|M_T^n|^2 < \infty$. Therefore, there exists $X \in L^2(\mathcal{F}_T)$ such that $M_T^n \to X$ weakly in $L^2(\mathcal{F}_T)$. Let N be a càdlàg version of $E(X|\mathcal{F}_t)$, $t \in [0,T]$. Then, for every $\tau \in \mathcal{T}$, $M_\tau^n \to N_\tau$ weakly in $L^2(\mathcal{F}_T)$. Indeed, if $Z \in L^2(\mathcal{F}_T)$, then

$$\begin{aligned} EM_\tau^n Z &= E(E(M_T^n|\mathcal{F}_\tau)\cdot Z) = E(M_T^n E(Z|\mathcal{F}_\tau)) \\ &\to EXE(Z|\mathcal{F}_\tau) = E(E(X|\mathcal{F}_\tau)Z) = EN_\tau \cdot Z. \end{aligned} \tag{3.3}$$

Since $\mathbf{M}$ is a Hunt process each $\mathcal{F}$-martingale M has the property that $M_{\tau-} = M_\tau$ for all predictable $\tau \in \mathcal{T}$ (see [3, Proposition 2]). Therefore, for any predictable $\tau \in \mathcal{T}$,

$$A_{\tau-}^n = Y_0^n - Y_{\tau-}^n + M_{\tau-}^n = Y_0^n - Y_{\tau-}^n + M_\tau^n. \tag{3.4}$$

Set

$$\hat{A}_t = Y_0 - \hat{Y}_t + N_{t-}, \quad t \in [0,T].$$

Of course, $\hat{A}$ is predictable. By (3.3) and (3.4), for any predictable $\tau \in \mathcal{T}$,

$$A_{\tau-}^n \to Y_0 - \hat{Y}_\tau + N_\tau = Y_0 - \hat{Y}_\tau + N_{\tau-} = \hat{A}_\tau$$

weakly in $L^2(\mathcal{F}_T)$. From the above convergence, $\hat{A}_\sigma \le \hat{A}_\tau$ for all predictable $\sigma, \tau \in \mathcal{T}$ such that $\sigma \le \tau$. Therefore, applying the predictable cross-sectional theorem (see [5, [Theorem 86, p. 138]) we conclude that $\hat{A}$ is an increasing process. Consequently, by Lemma 3.3, $\hat{Y}$ is càglàd. $\square$

Now we are ready to prove the main result of this section. In the sequel, for a function u on $\bar{E}_{0,T}$ we set $u(\mathbf{X})_{t-} = Y_{t-}$ and $u(\mathbf{X})_{t+} = Y_{t+}$, where $Y_t = u(\mathbf{X}_t)$, $t \in (0,T]$.

THEOREM 3.5. *Assume that $u \in \mathcal{P}^2$. Then, for q.e. $z \in E_{0,T}$,*

$$\tilde{u}(\mathbf{X}_{t-}) = u(\mathbf{X})_{t-}, \quad t \in (0, \zeta_\upsilon], \quad P_z\text{-a.s.} \tag{3.5}$$

Proof. By the definition of a precise version of a potential, there exists a sequence $\{u_n\}$ of quasi-continuous parabolic potentials such that $u_n \searrow \tilde{u}$ q.e. on $E_{0,T}$. Since $\varphi(u) \in \mathcal{P}^2$ for any bounded concave function φ on $\mathbb{R}^+$ and any $u \in \mathcal{P}^2$, we may assume that u_n, u are bounded. Since $u_n, u \in \mathcal{P}^2$, then by Proposition 3.1 and the strong Markov property there exist measures $\mu_n, \mu \in \mathbb{M}$ and martingales M^n, M such that

$$u_n(\mathbf{X}_t) = \int_t^{\zeta_\tau} dA_r^{\mu_n} - \int_t^{\zeta_\tau} dM_r^n, \quad t \in [0, \zeta_\tau]$$

and

$$u(\mathbf{X}_t) = \int_t^{\zeta_\tau} \mathrm{d}A_r^\mu - \int_t^{\zeta_\tau} \mathrm{d}M_r, \quad t \in [0, \zeta_\tau]. \tag{3.6}$$

Since $u_n \searrow \tilde{u}$ q.e. on $E_{0,T}$, we have

$$u_n(\mathbf{X}_{t-}) \searrow \tilde{u}(\mathbf{X}_{t-}), \quad t \in (0, \zeta_\tau], \quad P_z\text{-a.s.}$$

for q.e. $z \in E_{0,T}$. But $u_n(\mathbf{X}_{t-}) = u_n(\mathbf{X})_{t-}$, $t \in (0, \zeta_\tau]$, P_z-a.s. for q.e. $z \in E_{0,T}$, because u_n is quasi-continuous. Therefore, applying Lemma 3.4 shows that $\tilde{u}(\mathbf{X}_{t-})$ is càglàd. We are going to show that

$$\tilde{u}(\mathbf{X}_-)_{t+} = u(\mathbf{X}_t), \quad t \in (0, \zeta_\tau). \tag{3.7}$$

Since $\tilde{u} = u$ m_1-a.e.,

$$\begin{aligned} 0 = R^{0,T}|\tilde{u} - u|(z) &= E_z \int_0^{\zeta_\upsilon} |\tilde{u}(\mathbf{X}_t) - u(\mathbf{X}_t)|\,\mathrm{d}t \\ &= E_z \int_0^{\zeta_\upsilon} |\tilde{u}(\mathbf{X}_{t-}) - u(\mathbf{X}_t)|\,\mathrm{d}t \\ &= E_z \int_0^{\zeta_\upsilon} |(\tilde{u}(\mathbf{X}_-)_{t+} - u(\mathbf{X}_t)|\,\mathrm{d}t \end{aligned}$$

for q.e. $z \in E_{0,T}$. Hence,

$$\tilde{u}(\mathbf{X}_-)_{t+} = u(\mathbf{X}_t) \quad \text{for a.e. } t \in (0, \zeta_\tau) \tag{3.8}$$

for q.e. $z \in E_{0,T}$. Since $\tilde{u}(\mathbf{X}_-)_{t+}$ and $u(\mathbf{X}_t)$ are càdlàg processes, (3.8) implies (3.7). In turn, (3.7) implies (3.5), because $\tilde{u}(\mathbf{X}_-)$ is càglàd. □

3.2. Probabilistic approach to precise versions of quasi-càdlàg functions

DEFINITION. We say that $u : \bar{E}_{0,T} \to \mathbb{R}$ is quasi-càdlàg if the process $u(\mathbf{X})$ is càdlàg on $[0, \zeta_\upsilon]$ under the measure P_z for q.e. $z \in E_{0,T}$.

Of course, any quasi-continuous function is quasi-càdlàg. In the sequel the class of smooth measures μ on $\bar{E}_{0,T}$ for which $E_z A^{|\mu|}_{\zeta_\upsilon} < \infty$ q.e. on $E_{0,T}$ we denote by $\mathbb{M}_T$. We say that u is a potential on $\bar{E}_{0,T}$ iff for a positive $\mu \in \mathbb{M}_T$.

$$u(z) = E_z A^\mu_{\zeta_\upsilon}$$

for q.e. $z \in E_{0,T}$. By [15, Proposition 3.4], each potential on $\bar{E}_{0,T}$ is quasi-càdlàg. By Proposition 3.1, each $u \in \mathcal{P}^2$ is a potential on $\bar{E}_{0,T}$.

THEOREM 3.6. *Let u be a quasi-càdlàg function on $\bar{E}_{0,T}$. Then, there exists a unique (q.e.) function $\hat{u}$ such that for q.e. $z \in E_{0,T}$,*

$$\hat{u}(\mathbf{X}_{t-}) = u(\mathbf{X})_{t-}, \quad t \in (0, \zeta_\upsilon), \quad P_z\text{-a.s.} \tag{3.9}$$

Proof. By [26, Theorem 3.3.6], there exists a measure $\hat{m}_1$ equivalent to m_1 such that $(\mathbf{X}, P_z)$ has the dual Hunt process $(\hat{\mathbf{X}}, \hat{P}_z)$ with respect to $\hat{m}_1$. Therefore, by [8, Theorem 16.4] applied to the process $u(X)_-$ there exists a Borel measurable function $\hat{u}$ satisfying (3.9). Uniqueness follows from the very definition of exceptional sets and [8, Proposition 11.1]. □

DEFINITION. For a quasi-càdlàg function u on $\bar{E}_{0,T}$ we call the function $\hat{u}$ from Theorem 3.6 a precise m_1-version of u.

COROLLARY 3.7. *For $u \in \mathcal{P}^2$, $\tilde{u} = \hat{u}$ q.e.*

REMARK 3.8. It is clear that $\hat{u}$ is an m_1-version of u (see the reasoning before (3.8)).

In the sequel, we will need the following result.

PROPOSITION 3.9. *Let μ be a positive smooth measure on $E_{0,T}$, and let u be a positive quasi-càdlàg function on $E_{0,T}$. Then*

$$\int_0^{\zeta_\upsilon} [u(\mathbf{X})]_{t-}\, \mathrm{d}A_t^\mu = \int_0^{\zeta_\upsilon} \hat{u}(\mathbf{X}_t)\, \mathrm{d}A_t^\mu, \quad P_z\text{-a.s.} \tag{3.10}$$

for q.e. $z \in E_{0,T}$. Moreover,

$$E_z \int_0^{\zeta_\upsilon} u(\mathbf{X}_t)\, \mathrm{d}A_t^\mu = 0 \ \ \textit{for q.e.} \ \ z \in E_{0,T}$$

if and only if $\int_{E_{0,T}} u\, d\mu = 0$.

Proof. By Theorem 3.6, $\int_0^{\zeta_\upsilon}[u(\mathbf{X})]_{t-}\, \mathrm{d}A_t^\mu = \int_0^{\zeta_\upsilon} \hat{u}(\mathbf{X}_{t-})\, \mathrm{d}A_t^\mu$, P_z-a.s. It is well known that $\mathbf{X}$ has only totally inaccessible jumps (as a Hunt process), which combined with the fact that A^μ is predictable gives (3.10). The second part of the proposition follows directly from the Revuz duality. □

3.3. Associated precise versions in the sense of Pierre

Let u be a measurable function on $\bar{E}_{0,T}$ bounded by some element of $\mathcal{W}$. In [30], M. Pierre introduced the so-called associated precise m_1-version $\tilde{u}$ of u. By the definition, $\tilde{u}$ is the unique quasi-u.s.c. function such that

$$\{v \in \mathcal{W}_{0,T} + \mathcal{P}^2 : \tilde{v} \geq u \text{ q.e.}\} = \{v \in \mathcal{W}_{0,T} + \mathcal{P}^2 : \tilde{v} \geq \tilde{u} \text{ q.e.}\}$$

In general, it is not true that $\tilde{u} = u$ m_1-a.e. However, if u is quasi-càdlàg, then

$$\hat{u} \leq \tilde{u} \quad \text{q.e.} \tag{3.11}$$

Indeed, by [30, Proposition IV-I] there exists a sequence $\{u_n\} \subset \mathcal{W}_{0,T}$ such that $\inf_{n\geq 1} u_n = \tilde{u}$ q.e. Since $u \leq u_n$ q.e. and u is quasi-càdlàg, we have

$$u(\mathbf{X}_t) \leq u_n(\mathbf{X}_t), \quad t \in [0, \zeta_\upsilon),$$

which implies

$$\hat{u}(\mathbf{X}_{t-}) = u(\mathbf{X})_{t-} \le u_n(\mathbf{X})_{t-} = u_n(\mathbf{X}_{t-}), \quad t \in (0, \zeta_\upsilon).$$

Taking infimum, we get

$$\hat{u}(\mathbf{X}_{t-}) \le \tilde{u}(\mathbf{X}_{t-}), \quad t \in (0, \zeta_\upsilon).$$

By the above and [8, Proposition 11.1], we get (3.11). Set

$$u_i(t,x) = u_i(t) = \begin{cases} 1, \ t \in I_i, \\ 0, \ t \in [0,T] \backslash I_i, \end{cases} \tag{3.12}$$

where $I_1 = [0,T] \cap \mathbb{Q}$, $I_2 = [\frac{T}{2}, T]$, $I_3 = (\frac{T}{2}, T]$. Then, $\tilde{u}_1 \equiv 1$, so $\tilde{u}_1$ is not an m_1-version of u_1. Moreover, $\tilde{u}_2 = u_2$ and $\hat{u}_2 = u_3$ (u_3 is not quasi-u.s.c.), so in this case $\hat{u}_2 < \tilde{u}_2$ on the set $\{\frac{T}{2}\} \times E$. Since $\mathrm{Cap}(\{t\} \times B) = 0$ if and only if $m(B) = 0$, it follows that in general, (3.11) cannot be replaced by "$\hat{u} = \tilde{u}$ q.e."

4. Reflected BSDEs

In what follows, $(\Omega, \mathcal{F}, P)$ is a probability space, $\mathbb{F} = \{\mathcal{F}_t, t \ge 0\}$ is a filtration satisfying the usual conditions, and T is an arbitrary, but fixed bounded $\mathbb{F}$-stopping time. By $\mathcal{T}$, we denote the set of all $\mathbb{F}$-stopping times with values in $[0,T]$. For $\sigma, \gamma \in [0,T]$ such that $\sigma \le \gamma$, we denote by $\mathcal{T}_\gamma$ (resp. $\mathcal{T}_{\sigma,\gamma}$) the set of all $\tau \in \mathcal{T}$ such that $P(\tau \in [\gamma, T]) = 1$ (resp. $P(\tau \in [\sigma, \gamma]) = 1$).

By $\mathcal{M}$ (resp. $\mathcal{M}_{loc}$), we denote the space of martingales (resp. local martingales) with respect to $\mathbb{F}$. $[M]$ denotes the quadratic variation process of $M \in \mathcal{M}_{loc}$. By $\mathcal{M}_0$ (resp. $\mathcal{M}^p$), we denote the subspace of $\mathcal{M}$ consisting of all M such that $M_0 = 0$ (resp. $E[M]_T^{p/2} < \infty$).

By $\mathcal{V}$ (resp. $\mathcal{V}^+$), we denote the space of all $\mathbb{F}$-progressively measurable processes (resp. increasing processes) V of finite variation such that $V_0 = 0$. $\mathcal{V}^p$ is the subspace of $\mathcal{V}$ consisting of V such that $E|V|_T^p < \infty$, where $|V|_t$ denotes the variation of V on the interval $[0,t]$. ${}^p\mathcal{V}$ is the space of all predictable processes in $\mathcal{V}$.

By $\mathcal{S}^p$, we denote the space of $\mathbb{F}$-progressively measurable processes Y such that $E \sup_{t \le T} |Y_t|^p < \infty$. $L^p(\mathcal{F})$ is the space of $\mathbb{F}$-progressively measurable processes X such that $E \int_0^T |X_t|^p \, \mathrm{d}t < \infty$. $L^p(\mathcal{F}_T)$ is the space of $\mathcal{F}_T$-measurable random variables X such that $E|X|^p < \infty$.

Let $f : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ be a measurable function such that $f(\cdot, y)$ is $\mathbb{F}$-progressively measurable for every $y \in \mathbb{R}$, ξ be a $\mathcal{F}_T$-measurable random variable and V be a càdlàg process of finite variation such that $V_0 = 0$.

DEFINITION. We say that a pair of processes (Y, M) is a solution of the backward stochastic differential equation with terminal condition ξ and right-hand side $f + \mathrm{d}V$ (BSDE(ξ, $f + \mathrm{d}V$) for short) if

(a) Y is an $\mathbb{F}$-adapted càdlàg process of Doob's class (D), $M \in \mathcal{M}_{0,loc}$,
(b) $[0,T] \ni t \to f(t, Y_t) \in L^1(0,T)$ and

$$Y_t = \xi + \int_t^T f(r, Y_r)\,\mathrm{d}r + \int_t^T \mathrm{d}V_r - \int_t^T \mathrm{d}M_r, \quad t \in [0,T], \quad P\text{-a.s.}$$

Let L, U be two càdlàg $\mathbb{F}$-adapted processes such that $L_t \le U_t$, $t \in [0,T]$, and $L_T \le \xi \le U_T$.

DEFINITION. We say that a triple of processes (Y, M, K) is a solution of the reflected backward stochastic differential equation with terminal condition ξ, right-hand side $f + \mathrm{d}V$ and lower barrier L ($\underline{\mathrm{R}}$BSDE(ξ, $f + \mathrm{d}V$, L) for short) if

(a) Y is an $\mathbb{F}$-adapted càdlàg process of Doob's class (D), $M \in \mathcal{M}_{0,loc}$, $K \in {}^p\mathcal{V}^+$,
(b) $Y_t \ge L_t$, $t \in [0,T]$,
(c) $\int_0^T (Y_{t-} - L_{t-})\,\mathrm{d}K_t = 0$, P-a.s.,
(d) $[0,T] \ni t \to f(t, Y_t) \in L^1(0,T)$ and

$$Y_t = \xi + \int_t^T f(r, Y_r)\,\mathrm{d}r + \int_t^T \mathrm{d}V_r + \int_t^T \mathrm{d}K_r - \int_t^T \mathrm{d}M_r, \quad t \in [0,T], \quad P\text{-a.s.}$$

DEFINITION. We say that a triple of processes (Y, M, K) is a solution of the reflected backward stochastic differential equation with terminal condition ξ, right-hand side $f + \mathrm{d}V$ and upper barrier L ($\overline{\mathrm{R}}$BSDE(ξ, $f + \mathrm{d}V$, U) for short) if the triple $(-Y, -M, K)$ is a solution of $\underline{\mathrm{R}}$BSDE($-\xi$, $\tilde{f} - \mathrm{d}V$, $-L$), where $\tilde{f}(t, y) = -f(t, -y)$.

DEFINITION. We say that a triple of processes (Y, M, R) is a solution of the reflected BSDE with terminal condition ξ, right-hand side $f + \mathrm{d}V$, lower barrier L and upper barrier U (RBSDE(ξ, $f + \mathrm{d}V$, L, U) for short) if

(a) Y is an $\mathbb{F}$-adapted càdlàg process of class (D), $M \in \mathcal{M}_{0,loc}$, $R \in {}^p\mathcal{V}$,
(b) $L_t \le Y_t \le U_t$, $t \in [0,T]$, P-a.s.
(c) $\int_0^T (Y_{t-} - L_{t-})\,\mathrm{d}R_t^+ = \int_0^T (U_{t-} - Y_{t-})\,\mathrm{d}R_t^- = 0$, P-a.s.
(d) $[0,T] \ni t \mapsto f(t, Y_t) \in L^1(0,T)$ and

$$Y_t = \xi + \int_t^T f(r, Y_r)\,\mathrm{d}r + \int_t^T dV_r + \int_t^T dR_r - \int_t^T dM_r, \quad t \in [0,T], \quad P\text{-a.s.}$$

REMARK 4.1. Observe that if (Y, M, K) is a solution of $\underline{\mathrm{R}}$BSDE(ξ, $f + \mathrm{d}V$, L) and $A \in {}^p\mathcal{V}^+$ has the property that $\mathrm{d}A \le \mathrm{d}K$, then the triple $(Y, M, K - A)$ is a solution of $\underline{\mathrm{R}}$BSDE(ξ, $f + \mathrm{d}V + \mathrm{d}A$, L).

In the sequel, we will need the following lemma.

LEMMA 4.2. *Assume that*

(i) *there is* $\mu \in \mathbb{R}$ *such that for a.e.* $t \in [0,T]$ *and all* $y, y' \in \mathbb{R}$,

$$(f(t,y) - f(t,y'))(y - y') \le \mu |y - y'|^2,$$

(ii) $[0, T] \ni t \mapsto f(t, y) \in L^1(0, T)$ *for every* $y \in \mathbb{R}$,
(iii) $\mathbb{R} \ni y \mapsto f(t, y)$ *is continuous for a.e.* $t \in [0, T]$,
(iv) $\xi \in L^1(\mathcal{F}_T)$, $V \in \mathcal{V}^1$, $f(\cdot, 0) \in L^1(\mathcal{F})$.

Let $f_n = f \vee (-n)$, *and let* (Y^n, M^n, R^n) *be a solution of* RBSDE$(\xi, f_n + dV, L, U)$. *Then*

$$Y_t^n \searrow Y_t, \quad M_t^n \to M_t, \quad t \in [0, T], \quad dR^{+,n} \nearrow dR^+, \quad dR^{-,n} \searrow dR^-,$$

where (Y, M, R) *is a solution of* RBSDE$(\xi, f + dV, L, U)$.

Proof. By [16, Proposition 2.14, Proposition 3.1, Theorem 3.3], $Y^n \geq Y^{n+1} \geq Y$, $\mathrm{d}R^{+,n} \leq \mathrm{d}R^{+,n+1} \leq \mathrm{d}R^+, \mathrm{d}R^{-,n} \geq \mathrm{d}R^{-,n+1}$ for $n \geq 1$. Set $\bar{Y}_t = \lim_{n\to\infty} Y_t^n$, $K_t = \lim_{n\to\infty} R_t^{+,n}$, $A_t = \lim_{n\to\infty} R_t^{-,n}$, $\bar{R}_t = K_t - A_t$, $t \in [0, T]$. Without loss of generality, we may assume that $\mu \leq 0$ in (i). Then,

$$f\left(r, Y_r^1\right) \leq f_n\left(r, Y_r^n\right) \leq f_1(r, Y_r), \quad r \in [0, T].$$

From this, we conclude that the sequence $\{M^n\}$ is locally uniformly integrable. Hence, $\bar{M}_t := \lim_{n\to\infty} M_t^n$, $t \in [0, T]$, is a local martingale. We shall show that the triple $(\bar{Y}, \bar{M}, \bar{R})$ is a solution of RBSDE$(\xi, f + \mathrm{d}V, L, U)$. It is clear that $\bar{R}$ is a càdlàg process of finite variation. Moreover, by (ii) and (iii),

$$\bar{Y}_t = \xi + \int_t^T f(r, \bar{Y}_r)\,\mathrm{d}r + \int_t^T d\bar{R}_r + \int_t^T \mathrm{d}V_r - \int_t^T d\bar{M}_r, \quad t \in [0, T].$$

It is also clear that $\bar{Y}$ is of class (D) and $L \leq \bar{Y} \leq U$. By the Hahn–Saks theorem,

$$\int_0^T (\bar{Y}_t - L_t)\,\mathrm{d}K_t^c = \lim_{n\to\infty} \int_0^T (\bar{Y}_t - L_t)\,\mathrm{d}R_t^{+,n,c} = 0. \tag{4.1}$$

Assume that $\Delta K_t > 0$. Then, there exists n_0 such that $\Delta R_t^{+,n} > 0$ for $n \geq n_0$. Since $\int_0^T (Y_{t-}^n - L_{t-})\,\mathrm{d}R_t^{+,n} = 0$ and $Y_t^n \geq L_t$, it follows that $Y_{t-}^n = L_{t-}$ for $n \geq n_0$. Consequently, $\bar{Y}_{t-} \leq Y_{t-}^n = L_{t-}$, which implies that $\bar{Y}_{t-} = L_{t-}$. We have proved that $\sum_t (\bar{Y}_{t-} - L_{t-})\Delta K_t = 0$, which when combined with (4.1) shows that

$$\int_0^T \left(\bar{Y}_{t-} - L_{t-}\right)\,\mathrm{d}K_t = 0.$$

Also observe that

$$\int_0^T \left(U_t - \bar{Y}_t\right)\,\mathrm{d}A_t \leq \int_0^T \left(U_t - Y_t^n\right)\,\mathrm{d}R_t^{n,-} = 0. \tag{4.2}$$

Thus, the triple $(\bar{Y}, \bar{M}, \bar{R})$ is a solution of RBSDE$(\xi, f + \mathrm{d}V, L, U)$. From this and the minimality of the Jordan decomposition of the measure R, it follows that $R^+ = K$, $R^- = A$. □

5. PDEs with one reflecting barrier

In this section, $T > 0$ is a real number, $\varphi : E \to \mathbb{R}$, $f : \bar{E}_{0,T} \times \mathbb{R} \to \mathbb{R}$ are measurable functions, and $h : \bar{E}_{0,T} \to \mathbb{R}$ is a quasi-càdlàg function such that $\hat{h}(T, \cdot) \le \varphi$.

DEFINITION. We say that a quasi-càdlàg function u on $\bar{E}_{0,T}$ is a solution of the Cauchy problem

$$-\frac{\partial u}{\partial t} - L_t u = f(\cdot, u) + \mu, \quad u(T, \cdot) = \varphi$$

(PDE(φ, $f + \mathrm{d}\mu$) for short) if for q.e. $z \in E_{0,T}$,

$$u(z) = E_z \varphi(\mathbf{X}_{\zeta_\upsilon}) + E_z \int_0^{\zeta_\upsilon} f(\mathbf{X}_r, u(\mathbf{X}_r))\, \mathrm{d}r + E_z \int_0^{\zeta_\upsilon} \mathrm{d}A_r^\mu.$$

DEFINITION. We say that a pair (u, ν), where $u : \bar{E}_{0,T} \to \mathbb{R}$ is a quasi-càdlàg function and ν is a positive smooth measure on $E_{0,T}$, is a solution of problem (1.1), i.e., obstacle problem with data φ, f, μ and lower barrier h (we denote it by $\underline{\mathrm{O}}\mathrm{P}(\varphi, f + \mathrm{d}\mu, h)$), if

(a) (1.6) is satisfied for some local martingale M (with $M_0 = 0$),
(b) $u \ge h$ q.e.,
(c) $\int_{E_{0,T}} (\hat{u} - \hat{h})\, \mathrm{d}\nu = 0$.

DEFINITION. We say that a pair (u, ν) is a solution of the obstacle problem with data φ, f, μ and upper barrier h (we denote it by $\overline{\mathrm{O}}\mathrm{P}(\varphi, f + \mathrm{d}\mu, h)$), if $(-u, \nu)$ is a solution of $\underline{\mathrm{O}}\mathrm{P}(-\varphi, \tilde{f} - \mathrm{d}\mu, -h)$, where $\tilde{f}(t, x, y) = -f(t, x, -y)$.

We will need the following duality condition considered in [15].

(Δ) For some $\alpha \ge 0$, there exists a nest $\{F_n\}$ on $E_{0,T}$ such that for every $n \ge 1$ there is a nonnegative $\eta_n \in L^2(E_{0,T}; m_1)$ such that $\eta_n > 0$ m_1-a.e. on F_n and $\hat{G}_\alpha^{0,T} \eta_n$ is bounded, where $\hat{G}_\alpha^{0,T}$ is the adjoint operator to $G_\alpha^{0,T}$.

Note that (Δ) is satisfied if for some $\gamma \ge 0$ the form $\mathcal{E}_\gamma$ has the dual Markov property (see [15, Remark 3.9]).

Following [15], we say that $f : \bar{E}_{0,T} \to \mathbb{R}$ is quasi-integrable if $f \in \mathcal{B}(E_{0,T})$ and $P_z(\int_0^{\zeta_\tau} |f|(\mathbf{X}_r)\, \mathrm{d}r < \infty) = 1$ for q.e. $z \in E_{0,T}$.

The set of all quasi-integrable functions will be denoted by $qL^1(E_{0,T}; m_1)$. Note that by [15, Remark 5.1], under condition (Δ), if f satisfies the condition

$$\begin{aligned} &\forall\, \varepsilon > 0\ \exists\, F_\varepsilon \subset E_{0,T},\ \ F_\varepsilon\text{-closed},\ \ \mathrm{Cap}(E_{0,T} \setminus F_\varepsilon) \\ &\quad < \varepsilon,\ \ \mathbf{1}_{F_\varepsilon} f \in L^1(E_{0,T}; m_1), \end{aligned} \tag{5.1}$$

then f is quasi-integrable. Also note that by [15, Proposition 3.8], under condition (Δ),

$$\mathcal{M}_1 \subset \mathbb{M}.$$

We say that a function $u : \bar{E}_{0,T} \to \mathbb{R}$ is of class (D) if process $u(\mathbf{X})$ is of class (D), i.e., family $\{u(\mathbf{X}_\tau),\ \tau \le \zeta_\upsilon,\ \tau - \text{stopping time}\}$ is uniformly integrable under measure P_z for q.e. $z \in E_{0,T}$.

REMARK 5.1. If $E_z|u(\mathbf{X}_{\zeta_\upsilon})| < \infty$ for q.e. $z \in E_{0,T}$ (this holds for example if $u(T,\cdot) \in L^1(E;m)$, see [15, Proposition 3.8]) and there exists a potential v on $\bar{E}_{0,T}$ such that $|u| \le v$ q.e. on $E_{0,T}$ then u is of class (D). Indeed, let $v = R^{0,T}\beta$ for some positive $\beta \in \mathbb{M}_T$. By [15, Proposition 3.4],

$$v(\mathbf{X}_t) = v(\mathbf{X}_0) - \int_0^t \mathrm{d}A_r^\beta + \int_0^t \mathrm{d}M_r, \quad t \in [0, \zeta_\upsilon]$$

for some martingale M. Of course, the process $v(\mathbf{X})$ is of class (D), and $|u|(\mathbf{X}_t) \le v(\mathbf{X}_t)$, $t \in [0, \zeta_\upsilon)$. Since $E_z|u|(\mathbf{X}_{\zeta_\upsilon}) < \infty$ for q.e. $z \in E_{0,T}$, it follows that $u(\mathbf{X})$ is of class (D), too.

Let us consider the following assumptions.

(H1) $\mu,\ f(\cdot,0)\cdot m_1 \in \mathbb{M},\ E_z|\varphi(\mathbf{X}_{\zeta_\upsilon})| < \infty$ for q.e. $z \in E_{0,T}$,
(H2) there exists $\lambda \in \mathbb{R}$ such that for all $y, y' \in \mathbb{R}$ and $(t,x) \in E_{0,T}$,

$$(f(t,x,y) - f(t,x,y'))(y-y') \le \lambda|y-y'|^2,$$

(H3) the mapping $y \mapsto f(t,x,y)$ is continuous for every $(t,x) \in E_{0,T}$,
(H4) $f(\cdot,y) \in qL^1(E_{0,T};m_1)$ for every $y \in \mathbb{R}$,
(H5) h^+ is of class (D).

REMARK 5.2. It is an elementary check (see [15, (3.7)]) that $E_z|\varphi(\mathbf{X}_{\zeta_\upsilon})| = E_z A^{|\beta|}_{\zeta_\upsilon}$ (on $E_{0,T}$), where $\beta = \delta_{\{T\}} \otimes (\varphi \cdot m)$, so by [15, Proposition 3.8] under condition (Δ) if $\varphi \in L^1(E;m)$, then $E_z|\varphi(\mathbf{X}_{\zeta_\upsilon})| < \infty$ for q.e. $z \in E_{0,T}$.

THEOREM 5.3. *Assume* (H2). *Then, there exists at most one solution of* $\underline{\mathrm{O}}\mathrm{P}(\varphi, f + d\mu, h)$.

Proof. Let (u_1,ν_1), (u_2,ν_2) be two solutions of $\underline{\mathrm{O}}\mathrm{P}(\varphi, f+\mathrm{d}\mu, h)$. Set $u = u_1 - u_2$, $\nu = \nu_1 - \nu_2$. By the Tanaka–Meyer formula, for any stopping time τ such that $\tau \le \zeta_\upsilon$ we have

$$\begin{aligned}|u(\mathbf{X}_t)| \le |u(\mathbf{X}_\tau)| &+ \int_t^\tau (f(r,u_1(\mathbf{X}_r)) - f(r,u_2(\mathbf{X}_r)))\mathrm{sgn}(u)(\mathbf{X}_r)\,\mathrm{d}r \\ &+ \int_t^\tau \mathrm{sgn}(u)(\mathbf{X}_{r-})\,\mathrm{d}A_r^\nu - \int_t^\tau \mathrm{sgn}(u)(\mathbf{X}_{r-})\,\mathrm{d}M_r, \quad t \in [0,\tau]\end{aligned}$$

for q.e. $z \in E_{0,T}$. Observe that by the minimality condition and Proposition 3.9,

$$\begin{aligned}\int_t^\tau \operatorname{sgn}(u)(\mathbf{X}_{r-})\,\mathrm{d}A^\nu_r &= \int_t^\tau \mathbf{1}_{\{u\neq 0\}}\frac{1}{|u|}u(\mathbf{X}_{r-})\,\mathrm{d}A^\nu_r\\ &\le \int_t^\tau \mathbf{1}_{\{u\neq 0\}}\frac{1}{|u|}(u_1-h)(\mathbf{X}_{r-})\,\mathrm{d}A^{\nu_1}_r\\ &= \int_t^\tau \mathbf{1}_{\{u\neq 0\}}\frac{1}{|u|}(\hat u_1(\mathbf{X}_r)-\hat h(\mathbf{X}_r))\,\mathrm{d}A^{\nu_1}_r = 0. \end{aligned}\tag{5.2}$$

Let $\{\tau_k\}$ be a chain (i.e., an increasing sequence of stopping times with the property $P_z(\liminf_{k\to\infty}\{\tau_k = T\}) = 1$, q.e.) such that M^{τ_k} is a martingale for each $k \ge 1$. Then, by (H2) and (5.2),

$$E_z|u(\mathbf{X}_t)| \le E_z|u(\mathbf{X}_{\tau_k})| + \lambda E_z\int_t^{\tau_k}|u(\mathbf{X}_r)|\,\mathrm{d}r. \tag{5.3}$$

Letting $k\to\infty$ in (5.3) and using the fact that u is of class (D) we get

$$E_z|u(\mathbf{X}_t)| \le \lambda E_z\int_t^{\zeta_\upsilon}|u(\mathbf{X}_r)|\,\mathrm{d}r.$$

Applying Gronwall's lemma shows that $E_z|u|(\mathbf{X}_t)| = 0$, $t\in[0,\zeta_\upsilon]$ for q.e. $z\in E_{0,T}$. This implies that $u = 0$ q.e. From this, (1.6) and the uniqueness of the Doob–Meyer decomposition we conclude that $A^\nu = 0$, which forces $\nu = 0$. □

THEOREM 5.4. *Assume* (H1)–(H5). *Then, there exists a solution of* $\underline{\mathrm{O}}\mathrm{P}(\varphi, f + d\mu, h)$.

Proof. By [15, Theorem 5.8], there exists a unique solution u_n of (1.11). Set

$$f_n(t,x,y) = f(t,x,y) + n(y-h(t,x))^-.$$

By the definition of a solution of (1.11) and [15, Proposition 3.4], there exists a martingale M^n such that the pair $(u_n(\mathbf{X}), M^n)$ is a solution of $\mathrm{BSDE}(\varphi(\mathbf{X}_{\zeta_\upsilon}), f_n(\mathbf{X},\cdot)+\mathrm{d}A^\mu)$ under the measure P_z for q.e. $z\in E_{0,T}$. By [16, Theorem 4.1] (see also [34]), there exists a solution (Y^z, M^z, K^z) of $\underline{\mathrm{R}}\mathrm{BSDE}(\varphi(\mathbf{X}), f(\mathbf{X},\cdot)+\mathrm{d}A^\mu, h(\mathbf{X}))$ under the measure P_z for q.e. $z\in E_{0,T}$, and

$$u_n(\mathbf{X}_t)\nearrow Y^z_t,\quad t\in[0,\zeta_\upsilon],\quad P_z\text{-a.s.} \tag{5.4}$$

From (5.4), it follows that $u_n \le u_{n+1}$ q.e. (since the exceptional sets coincide with the sets of zero capacity) for $n\ge 1$ and

$$Y^z_t = u(\mathbf{X}_t),\quad t\in[0,\zeta_\upsilon],\quad P_z\text{-a.s.}$$

for q.e. $z\in E_{0,T}$, where $u := \sup_{n\ge1}u_n$. This implies that u is quasi-càdlàg. What is left to show that there exists a smooth measure ν such that $K^z = A^\nu$ for q.e. $z\in E_{0,T}$. Since the pointwise limit of additive functionals is an additive functional, we may assume by Lemma 4.2 and [16, Theorem 2.13] that $EK^z_{\zeta_\upsilon} < \infty$ for q.e. $z\in E_{0,T}$. By [16, Proposition 4.3] and Theorem 3.6, for every predictable stopping time τ,

$$\Delta K^z_\tau = \Big(u(\mathbf{X}_\tau) - \hat h(\mathbf{X}_\tau) + \Delta A^\mu_\tau\Big)^-.$$

Hence,

$$J_t = \sum_{s \le t} \Delta K^z_t$$

is a PNAF (without jump in ζ_υ since $\hat{h}(T, \cdot) \le \varphi$), which implies that there exists $\beta \in \mathbb{M}$ such that $J = A^\beta$ (by the Revuz duality). Set

$$C^z_t = K^z_t - A^\beta_t, \quad t \in [0, \zeta_\upsilon].$$

It is clear that the process C^z is continuous. By Remark 4.1, the triple (Y^z, M^z, C^z) is a solution of $\underline{\mathrm{R}}$BSDE$(\varphi(\mathbf{X}), f(\mathbf{X}, \cdot) + \mathrm{d}A^\mu + \mathrm{d}A^\beta, h(\mathbf{X}))$. By [15, Theorem 5.8], there exists a solution v_n of the equation

$$-\frac{\partial v_n}{\partial t} - L_t v_n = f_n(\cdot, v_n) + \mu + \beta, \quad v_n(T, \cdot) = \varphi.$$

Therefore, by the definition of solution and [15, Proposition 3.4] there exists a martingale N^n such that the pair $(v_n(\mathbf{X}), N^n)$ is a solution of BSDE$(\varphi(\mathbf{X}), f_n(\mathbf{X}, \cdot) + \mathrm{d}A^\mu + \mathrm{d}A^\beta)$ under the measure P_z for q.e. $z \in E_{0,T}$. Let $C^n_t = \int_0^t n(v_n(\mathbf{X}_r) - h(\mathbf{X}_r))^- \,\mathrm{d}r$, $t \in [0, \zeta_\upsilon]$. By [16, Theorem 2.13], the sequence $\{C^n\}$ converges uniformly on $[0, \zeta_\upsilon]$ in probability P_z for q.e. $z \in E_{0,T}$. Since C^n is a PCAF for each $n \ge 1$, the process C defined by

$$C_t = \lim_{n \to \infty} C^n_t, \quad t \in [0, \zeta_\upsilon],$$

is a PCAF. Therefore, there exists a measure $\alpha \in \mathbb{M}$ such that $C = A^\alpha$. It is clear that $C^z = A^\alpha$ for q.e. $z \in E_{0,T}$. Finally, $K^z = A^\nu$ for q.e. $z \in E_{0,T}$, where $\nu = \alpha + \beta$. $\square$

PROPOSITION 5.5. *Assume that* φ_i, f_i, μ_i, $i = 1, 2$, *satisfy* (H1)–(H4) *and* h_1, h_2 *are quasi-càdlàg. Let* (u_i, ν_i) *be a solution to* $\underline{\mathrm{O}}$P$(\varphi_i, f_i + \mathrm{d}\mu_i, h_i)$. *Assume that* $\varphi_1 \le \varphi_2$ *m-a.e.,* $f_1(\cdot, y) \le f_2(\cdot, y)$ m_1*-a.e. for every* $y \in \mathbb{R}$, $\mathrm{d}\mu_1 \le \mathrm{d}\mu_2$ *and* $h_1 \le h_2$ *q.e. Then,*

$$u_1 \le u_2 \quad \text{q.e. on } E_{0,T},$$

and if $h_1 = h_2$ *q.e., then* $\mathrm{d}\nu_1 \ge \mathrm{d}\nu_2$.

Proof. By the proof of Theorem 5.4, $u_{i,n} \nearrow u_i$ q.e., where $u_{i,n}$ is a solution to PDE$(\varphi_i, f_{n,i} + \mathrm{d}\mu_i)$ with

$$f_{i,n}(t, x, y) = f_i(t, x, y) + n(y - h_i(t, x))^-.$$

By [15, Corollary 5.9], $u_{1,n} \le u_{2,n}$ q.e. Hence, $u_1 \le u_2$ q.e. As for the second assertion of the theorem, by Lemma 4.2 we may assume that $\nu_1, \nu_2 \in \mathbb{M}$. By the proof of Theorem 5.4,

$$A^{\nu_i}_t = A^{\alpha_i}_t + A^{\beta^i}_t, \quad t \in [0, \zeta_\upsilon], \quad P_z\text{-a.s.}$$

for q.e. $z \in E_{0,T}$ and some positive $\beta_i \in \mathbb{M}$, $\alpha_i \in \mathbb{M}_c$ such that A^{β_i} is purely jumping and

$$\Delta A_t^{\beta_i} = \Big(u_i(\mathbf{X}_t) - \hat{h}(\mathbf{X}_t) + \Delta A_t^{\mu_i}\Big)^- .$$

We already know that $u_1 \le u_2$ q.e. Moreover, by the assumptions and Revuz duality, $\mathrm{d}A^{\mu_1} \le \mathrm{d}A^{\mu_2}$. Therefore, $\mathrm{d}A^{\beta_1} \ge \mathrm{d}A^{\beta_2}$. Furthermore, by the proof of Theorem 5.4,

$$A_t^{\alpha_{i,n}} \to A_t^{\alpha_i}, \quad t \in [0, \zeta_\upsilon], \quad P_z\text{-a.s.}$$

for q.e. $z \in E_{0,T}$, where $\alpha_{i,n} = n(v_{i,n} - h)^- \cdot m_1$ and $v_{i,n}$ is a solution to PDE$(\varphi_i, f_n + \mathrm{d}\mu_i + \mathrm{d}\beta_i)$. By [15, Corollary 5.9], $v_{1,n} \le v_{2,n}$, which implies that $\alpha_{1,n} \ge \alpha_{2,n}$, $n \ge 1$. Consequently, $\mathrm{d}A^{\alpha_{1,n}} \ge \mathrm{d}A^{\alpha_{2,n}}$, $n \ge 1$. Hence, $\mathrm{d}A^{\alpha_1} \ge \mathrm{d}A^{\alpha_2}$, so $\mathrm{d}A^{\nu_1} \ge \mathrm{d}A^{\nu_2}$, which implies that $\mathrm{d}\nu_1 \ge \mathrm{d}\nu_2$. □

REMARK 5.6. Let $v = R^{0,T}\beta$ for some $\beta \in \mathbb{M}_T$ be such that $v \ge h$ q.e. on $E_{0,T}$. By [15, Proposition 3.4],

$$v(\mathbf{X}_t) = v(\mathbf{X}_0) - \int_0^t \mathrm{d}A_r^\beta + \int_0^t \mathrm{d}M_r, \quad t \in [0, \zeta_\upsilon]$$

for some martingale M. Let $\gamma = -(h(T, \cdot) \cdot m) \otimes \delta_{\{T\}}$. Observe that the function $\bar{v} = E.h(\mathbf{X}_{\zeta_\upsilon}) + R^{0,T}\gamma + R^{0,T}\beta$ is equal to v on $E_{0,T}$. Moreover, $\bar{v}(X)$ satisfies the same equation as $v(X)$, but on $[0, \zeta_\upsilon)$, and $\bar{v}(T, \cdot) \ge h(T, \cdot)$, from which it follows that $\bar{v}(\mathbf{X}_t) \ge h(\mathbf{X}_t)$, $t \in [0, \zeta_\upsilon]$.

We denote by $\mathcal{M}_{1,T}$ the set of all finite Borel measures on $\bar{E}_{0,T}$.

PROPOSITION 5.7. *Let the hypotheses* (H1)–(H4) *hold. Assume that* $\mu \in \mathbb{M}$ *and there exists* $\beta \in \mathbb{M}_T$ *such that* $v := R^{0,T}\beta \ge h$ *on* $E_{0,T}$*, and* $f^-(\cdot, v) \cdot m_1 \in \mathbb{M}$*. Let* (u, ν) *be a solution of* $\underline{\mathrm{O}}\mathrm{P}(\varphi, f + d\mu, h)$*. Then,* $f(\cdot, u) \cdot m_1, \nu \in \mathbb{M}$*. If we assume additionally that* $\mathcal{E}_\gamma$ *has the dual Markov property for some* $\gamma \ge 0$ *and* $\varphi \in L^1(E; m)$*,* $f(\cdot, 0)$*,* $f^-(\cdot, v) \in L^1(E_{0,T}; m_1)$*,* $\mu \in \mathcal{M}_1$*,* $\beta \in \mathcal{M}_{1,T}$*, then* $f(\cdot, u) \in L^1(E_{0,T}; m_1)$ *and* $\nu \in \mathcal{M}_1$*.*

Proof. Assume that $\mu, f^-(\cdot, v) \cdot m_1 \in \mathbb{M}$ and $\beta \in \mathbb{M}_T$. By [16, Theorem 2.13] (see also Remark 5.6), $\nu \in \mathbb{M}$. By [15, Theorem 5.4], $f(\cdot, u) \cdot m_1 \in \mathbb{M}$, which proves the first part of the proposition. Now, assume that $\mu, f^-(\cdot, v) \cdot m_1 \in \mathcal{M}_1$, $\beta \in \mathcal{M}_{1,T}$. Let $\bar{v}$ be a solution of PDE$(\varphi^+, f + f^-(\cdot, v) + \mathrm{d}\mu^+ + \mathrm{d}\beta^+)$ (it exists by [15, Theorem 5.8]). By [15, Corollary 5.9], $\bar{v} \ge v$, and consequently $\bar{v} \ge h$ q.e. on $E_{0,T}$. Observe that the pair $(\bar{v}, f^-(\cdot, v) \cdot m + \beta^+)$ is a solution to $\underline{\mathrm{O}}\mathrm{P}(\varphi^+, f + \mathrm{d}\mu^+, \bar{v})$. Hence, by Proposition 5.5, $\bar{v} \ge u$ q.e. on $E_{0,T}$. On the other hand, by [15, Corollary 5.9], $u_0 \le u$ q.e. on $E_{0,T}$, where u_0 is a solution to PDE$(\varphi, f + \mathrm{d}\mu)$. By [15, Proposition 5.10], $f(\cdot, \bar{v})$, $f(\cdot, u_0)$, $\bar{v}$, $u_0 \in L^1(E_{0,T}; m_1)$. Since $u_0 \le u \le \bar{v}$, thanks to (H2) we have $u, f(\cdot, u) \in L^1(E_{0,T}; m_1)$. Let $\gamma = f^-(\cdot, v) \cdot m + \mu^+ + \beta^+$. Observe that

$$\begin{aligned}u(z) &= E_z\varphi(\mathbf{X}_{\zeta_\upsilon}) + E_z\int_0^{\zeta_\upsilon} f(\mathbf{X}_r, u(\mathbf{X}_r))\,\mathrm{d}r + E_z\int_0^{\zeta_\upsilon} \mathrm{d}A_r^\nu + E_z\int_0^{\zeta_\upsilon} \mathrm{d}A_r^\mu \\ &\le E_z\varphi^+(\mathbf{X}_{\zeta_\upsilon}) + E_z\int_0^{\zeta_\upsilon} f(\mathbf{X}_r, \bar{v}(\mathbf{X}_r))\,\mathrm{d}r + E_z\int_0^{\zeta_\upsilon} \mathrm{d}A_r^\gamma = \bar{v}(z).\end{aligned}$$

Hence,

$$\begin{aligned}E_z\int_0^{\zeta_\upsilon} \mathrm{d}A_r^\nu &\le E_z|\varphi|(\mathbf{X}_{\tau_\upsilon}) + E_z\int_0^{\zeta_\upsilon} |f(\mathbf{X}_r, u(\mathbf{X}_r))|\,\mathrm{d}r \\ &\quad + E_z\int_0^{\zeta_\upsilon} |f(\mathbf{X}_r, \bar{v}(\mathbf{X}_r))|\,\mathrm{d}r + E_z\int_0^{\zeta_\upsilon} \mathrm{d}A_r^{|\mu+\gamma|}\end{aligned}$$

for q.e. $z \in E_{0,T}$. By the above inequality and [15, Proposition 3.13],

$$\|\nu\|_1 \le c\left(\|\varphi\|_{L^1} + \|f(\cdot,u)\|_{L^1} + \|f^-(\cdot,v)\|_{L^1} + \|\mu^+\|_1 + \|\beta^+\|_1\right),$$

which completes the proof. □

REMARK 5.8. Observe that under the assumptions of the second assertion of the above proposition, $u \in L^1(E_{0,T}; m_1)$. Moreover, by [15, Proposition 5.10],

$$\|u\|_{L^1} + \|f(\cdot,u)\|_{L^1} \le c(\|\varphi\|_{L^1} + \|f(\cdot,0)\|_{L^1} + \|\mu\|_1 + \|\nu\|_1).$$

6. PDEs with two reflecting barriers

We assume as given T, φ, f as in Sect. 5, and quasi-càdlàg functions $h_1, h_2 : \bar{E}_{0,T} \to \mathbb{R}$ such that $\hat{h}_1(T,\cdot) \le \varphi \le \hat{h}_2(T,\cdot)$.

DEFINITION. We say that a pair (u,ν) consisting of a quasi-càdlàg function $u : \bar{E}_{0,T} \to \mathbb{R}$ of class (D) and a smooth measure ν on $E_{0,T}$ is a solution of the obstacle problem with data φ, f, μ and barriers h_1, h_2 (we denote it by OP$(\varphi, f + \mathrm{d}\mu, h_1, h_2)$) if

- (a) (1.6) is satisfied for some local martingale M (with $M_0 = 0$),
- (b) $h_1 \le u \le h_2$ q.e. on $E_{0,T}$,
- (c) $\int(\hat{u} - \hat{h}_1)\,\mathrm{d}\nu^+ = \int(\hat{h}_2 - \hat{u})\,\mathrm{d}\nu^- = 0$.

PROPOSITION 6.1. *Let assumption* (H1) *hold. Then, there exists at most one solution of* OP$(\varphi, f + d\mu, h_1, h_2)$.

Proof. The proof is analogous to the proof of Theorem 5.3. The only difference is the estimate of the integral involving $\mathrm{d}A^\nu$. In the present situation, the estimate is as follows:

$$\begin{aligned}\int_t^\tau \mathrm{sgn}(u)(\mathbf{X}_{r-})\,\mathrm{d}A_r^\nu &\le \int_t^\tau \mathbf{1}_{\{u\neq 0\}}\frac{1}{|u|}(u_1 - h_1)(\mathbf{X}_{r-})\,\mathrm{d}A_r^{\nu_1} \\ &\quad + \int_t^\tau \mathbf{1}_{\{u\neq 0\}}\frac{1}{|u|}(h_2 - u_2)(\mathbf{X}_{r-})\,\mathrm{d}A_r^{\nu_2}\end{aligned}$$

$$= \int_t^\tau \mathbf{1}_{\{u\neq 0\}} \frac{1}{|u|} (\hat{u}_1(\mathbf{X}_r) - \hat{h}_1(\mathbf{X}_r))\, dA_r^{\nu_1} + \int_t^\tau \mathbf{1}_{\{u\neq 0\}} \frac{1}{|u|} (\hat{h}_2(\mathbf{X}_r) - \hat{u}_2(\mathbf{X}_r))\, dA_r^{\nu_2} = 0. \tag{6.1}$$

□

Consider the following hypothesis:

(H6) h_1, h_2 are quasi-càdlàg functions of class (D) such that $\hat{h}_1(T,\cdot) \le \varphi \le \hat{h}_2(T,\cdot)$ and $h_1 < h_2$, $\hat{h}_1 < \hat{h}_2$ q.e. on $E_{0,T}$, or there exists $\beta \in \mathbb{M}_T$ such that $h_1 \le R^{0,T}\beta \le h_2$ q.e. on $E_{0,T}$.

REMARK 6.2. If (H1)–(H4) (H6) are satisfied then the assertion of [16, Theorem 4.2] holds true under measure P_z for q.e. $z \in E_{0,T}$. Indeed, it is clear that (H1)–(H4) of [16] are satisfied under measure P_z for q.e. $z \in E_{0,T}$. If $h_1 < h_2$ and $\hat{h}_1 < \hat{h}_2$ q.e. on $E_{0,T}$, then (since the exceptional sets coincide with the sets of zero capacity) for q.e. $z \in E_{0,T}$

$$h_1(\mathbf{X}_t) < h_2(\mathbf{X}_t), \quad \hat{h}_1(\mathbf{X}_t) < \hat{h}_2(\mathbf{X}_t), \quad t \in (0, \zeta_\upsilon), \quad P_z\text{-a.s.} \tag{6.2}$$

Since $\mathbf{X}$ has no predictable jumps (as a Hunt process), it follows from (6.2) and Theorem 3.6 that $h_1(\mathbf{X})_{\tau-} < h_2(\mathbf{X})_{\tau-}$ for every predictable stopping time τ with values in $(0, \zeta_\upsilon]$. By this and [34] the assertion of [16, Theorem 4.2] is satisfied under measure P_z for q.e. $z \in E_{0,T}$. If $h_1 \le R^{0,T}\beta \le h_2$ on $E_{0,T}$ for some $\beta \in \mathbb{M}_T$, then hypothesis (H7) from [16] (with $L = h_1(\mathbf{X})$, $U = h_2(\mathbf{X})$, $X = v(\mathbf{X})$) is satisfied by Remark 5.6 under measure P_z for q.e. $z \in E_{0,T}$, so again the assertion of [16, Theorem 4.2] holds true.

THEOREM 6.3. *Assume that* (H1)–(H4), (H6) *are satisfied. Then, there exists a unique solution of* $\mathrm{OP}(\varphi, f + d\mu, h_1, h_2)$.

Proof. By Remark 6.2, for q.e. $z \in E_{0,T}$ there exists a solution (Y^z, M^z, R^z) of $\mathrm{RBSDE}(\varphi(\mathbf{X}_{\tau_\upsilon}), f(\mathbf{X},\cdot) + dA^\mu, h_1(\mathbf{X}), h_2(\mathbf{X}))$ under the measure P_z. To prove the existence of a solution, it suffices to show that there exists a function u and a smooth measure ν such that $Y^z = u(\mathbf{X})$, $R^z = A^\nu$ for q.e. $z \in E_{0,T}$, because then the pair (u, ν) will be a solution of $\mathrm{OP}(\varphi, f + d\mu, h_1, h_2)$. By [16, Theorem 4.1], for every $n \ge 1$ there exists a solution $(\bar{Y}^{n,z}, \bar{M}^{n,z}, \bar{A}^{n,z})$ of $\overline{\mathrm{R}}\mathrm{BSDE}(\varphi(\mathbf{X}_{\zeta_\upsilon}), f_n(\mathbf{X},\cdot) + dA^\mu, h_2(\mathbf{X}))$ with f_n defined by

$$f_n(z, y) = f(z, y) + n(y - h_1(z))^-.$$

By Theorem 5.4,

$$\bar{Y}^{n,z} = \bar{u}_n(\mathbf{X}), \quad \bar{A}^{n,z} = A^{\bar{\gamma}_n}, \tag{6.3}$$

where $(\bar{u}_n, \bar{\gamma}_n)$ is a solution of $\overline{\mathrm{OP}}(\varphi, f_n + d\mu, h_2)$. By (6.3), $\bar{M}^{n,z}$ in fact does not depend on z. By [16, Theorem 4.2],

$$\bar{Y}_t^{n,z} \nearrow Y_t^z, \quad t \in [0, \zeta_\upsilon].$$

Since the exceptional sets coincide with the sets of zero capacity, this implies that $\bar{u}_n \leq \bar{u}_{n+1}$ q.e. for $n \geq 1$, and

$$Y_t^z = u(\mathbf{X}_t), \quad t \in [0, \zeta_\upsilon]$$

for q.e. $z \in E_{0,T}$, where $u := \sup_{n\geq 1} \bar{u}_n$. By [16, Theorem 4.2], $\mathrm{d}A^{\bar{\gamma}_n} \leq \mathrm{d}A^{\bar{\gamma}^{n+1}}$, so A defined as $A_t = \lim_{n\to\infty} A_t^{\bar{\gamma}_n}$, $t \geq 0$, is a PNAF. Therefore, there exists a positive smooth measure γ such that $A = A^\gamma$. By [16, Theorem 4.2], $A^\gamma = R^{z,-}$ for q.e. $z \in E_{0,T}$. By Lemma 4.2, without loss of generality we may assume that $E_z \int_0^{\zeta_\upsilon} d|R^z|_r < \infty$ for q.e. $z \in E_{0,T}$. Observe that the triple $(Y^z, M^z, R^{z,+})$ is a solution of $\underline{\mathrm{O}}\mathrm{P}(\varphi, f + \mathrm{d}\mu + d\gamma, h_1)$. Therefore, by Theorem 5.4, there exists a positive smooth measure α such that $R^{z,+} = A^\alpha$ for q.e. $z \in E_{0,T}$, which completes the proof. $\square$

PROPOSITION 6.4. *Let the hypotheses* (H1)–(H4) *and* (H6) *with some measure* $\beta \in \mathbb{M}_T$ *hold. Assume that* $f(\cdot, v) \cdot m_1 \in \mathbb{M}$ *with* $v = R^{0,T}\beta$, *and that* $\mu \in \mathbb{M}$. *Then,* $f(\cdot, u) \cdot m_1, \nu \in \mathbb{M}$. *If, in addition,* $\mathcal{E}_\gamma$ *has the dual Markov property for some* $\gamma \geq 0$, *and* $\varphi \in L^1(E; m)$, $f(\cdot, 0)$, $f(\cdot, v) \in L^1(E_{0,T}; m_1)$, $\mu \in \mathcal{M}_1$, $\beta \in \mathcal{M}_{1,T}$, *then* $f(\cdot, u) \in L^1(E_{0,T}, m_1)$, $\nu \in \mathcal{M}_1$.

Proof. If $f(\cdot, v) \cdot m_1, \mu \in \mathbb{M}$, then $\nu \in \mathbb{M}$ by [16, Theorem 3.3] (see also Remark 5.6). Hence, by [15, Theorem 5.4], $f(\cdot, u) \cdot m_1 \in \mathbb{M}$. Assume that $\mathcal{E}_\gamma$ has the dual Markov property for some $\gamma \geq 0$, and that $f(\cdot, 0)$, $f(\cdot, v) \in L^1(E_{0,T}; m_1)$, $\mu \in \mathcal{M}_1$, $\beta \in \mathcal{M}_{1,T}$. Let $(\bar{v}, \bar{\nu})$ be a solution to $\overline{\mathrm{O}}\mathrm{P}(\varphi^+, f + f^-(\cdot, v) + \mathrm{d}\beta^+ + \mathrm{d}\mu^+, h_2)$. By Proposition 5.5, $\bar{v} \geq v$ q.e. on $E_{0,T}$ (since $(v, 0)$ is a solution to $\overline{\mathrm{O}}\mathrm{P}(0, f - f(\cdot, v) + \mathrm{d}\beta, v)$). Let $(\bar{u}_n, \bar{\gamma}_n)$ be a solution to $\overline{\mathrm{O}}\mathrm{P}(\varphi, f_n + \mathrm{d}\mu, h_2)$ with

$$f_n(z, y) = f(z, y) + n(y - h_1(z))^-.$$

By the proof of Theorem 6.3, $\bar{u}_n \searrow u$ and $A_t^{\bar{\gamma}_n} \nearrow A_t^{\nu^-}$, $t \in [0, \zeta_\upsilon]$, P_z-a.s. for q.e. $z \in E_{0,T}$. Since $\bar{v} \geq v$, we have $\bar{v} \geq h_1$ q.e. on $E_{0,T}$. Therefore, $f(\cdot, v) = f_n(\cdot, v)$, and in fact, $(\bar{v}, \bar{\nu})$ is a solution to $\overline{\mathrm{O}}\mathrm{P}(\varphi^+, f_n + f^-(\cdot, v) + \mathrm{d}\beta^+ + \mathrm{d}\mu^+, h_2)$. Hence, by Proposition 5.5,

$$\bar{\gamma}_n \leq \bar{\nu}.$$

By the Revuz duality, $A_t^{\bar{\gamma}_n} \leq A_t^{\bar{\nu}}$, which when combined with the convergence of $\{A^{\bar{\gamma}_n}\}$ implies that $A_t^{\nu^-} \leq A_t^{\bar{\nu}}$, $t \in [0, \zeta_\upsilon]$, P_z-a.s. for q.e. $z \in E_{0,T}$. Consequently, $\nu^- \leq \bar{\nu}$. By Proposition 5.7, $\bar{\nu} \in \mathcal{M}_1$. Hence, by [15, Proposition 3.13], $\nu^- \in \mathcal{M}_1$. Since the pair (u, ν^+) is a solution of $\underline{\mathrm{O}}\mathrm{P}(\varphi, f + \mathrm{d}\mu - \mathrm{d}\nu^-, h_1)$, applying Proposition 5.7 yields $\nu^+ \in \mathcal{M}_1$. This completes the proof. $\square$

REMARK 6.5. Under (H1)–(H4), (H6) and the assumptions of the second assertion of Proposition 6.4,

$$\|u\|_{L^1} + \|f(\cdot, u)\|_{L^1} \leq c(\|\mu\|_1 + \|\varphi\|_{L^1} + \|\nu\|_1 + \|f(\cdot, 0)\|_{L^1}).$$

This follows from Proposition 6.4 and [15, Proposition 5.10].

REMARK 6.6. Let v be a difference of potentials on $\bar{E}_{0,T}$, i.e., $v = R^{0,T}\beta$ for some $\beta \in \mathbb{M}_T$. Observe that if a pair $(\bar{u}, \bar{\nu})$ is a solution of $\mathrm{OP}(\varphi, f_v + \mathrm{d}\mu - \mathrm{d}\beta, h_1 - v, h_2 - v)$ with

$$f_v(z, y) = f(z, v + y),$$

then $(u, \nu) = (\bar{u} + v, \bar{\nu})$ is a solution of $\mathrm{OP}(\varphi, f + \mathrm{d}\mu, h_1, h_2)$. It follows that the assumption $f(\cdot, 0) \cdot m_1 \in \mathbb{M}$ in Theorems 5.4 and 6.3 may be replaced by more general assumption saying that there exists a function v which is a difference of potentials on $\bar{E}_{0,T}$ such that $f(\cdot, v) \cdot m_1 \in \mathbb{M}$. Similarly, counterparts of the results stated in Propositions 5.7, 6.4 and Remarks 5.8 and 6.5 hold true under the assumption $f(\cdot, v) \cdot m_1 \in \mathbb{M}$. To get appropriate modifications of these results, we first apply them to the pair $(\bar{u}, \bar{\nu})$, and next we use the fact that $(\bar{u}, \bar{\nu}) = (u - v, \nu)$.

In condition (b) of the definition of the obstacle problem given at the beginning of this section, we require that the solution lies q.e. between the barriers. Below we show that if we weaken (b) and require only that this property holds a.e., then our results also apply to measurable barriers h_1, h_2 satisfying the following condition: There exits v of the form $v = R^{0,T}\beta$ with $\beta \in \mathbb{M}_T$ (i.e., v is a difference of potentials) such that $h_1 \le v \le h_2$ m_1-a.e. (in case of one barrier h it is enough to assume that h^+ is of class (D)).

Before presenting our results for measurable barriers, we give a definition of a solution.

DEFINITION. We say that a pair (u, ν) consisting of a quasi-càdlàg function $u : \bar{E}_{0,T} \to \mathbb{R}$ of class (D) and a smooth measure ν on $\bar{E}_{0,T}$ is a solution of the obstacle problem with data φ, f, μ and measurable barriers $h_1, h_2 : \bar{E}_{0,T} \to \mathbb{R}$ if

(a) (1.6) is satisfied for some local martingale M (with $M_0 = 0$),

(b*) $h_1 \le u \le h_2$ m_1-a.e. on $E_{0,T}$,

(c*) $\int (\hat{u} - \hat{\eta}_1)\, \mathrm{d}\nu^+ = \int (\hat{\eta}_2 - \hat{u})\, \mathrm{d}\nu^- = 0$ for all quasi-càdlàg η_1, η_2 such that $h_1 \le \eta_1 \le u \le \eta_2 \le h_2$ m_1-a.e.

If the barriers are quasi-càdlàg, then the above definition agrees with the definition given at the beginning of Sect. 6, because for quasi-càdlàg functions condition (b*) is equivalent to (b) and in (c*) we may take $\eta_1 = h_1$ and $\eta_2 = h_2$. Therefore, the obstacle problem with data φ, f, μ and measurable barriers h_1, h_2 we still denote by $\mathrm{OP}(\varphi, f + \mathrm{d}\mu, h_1, h_2)$.

In case of measurable barriers the proof of uniqueness of solutions to the problem $\mathrm{OP}(\varphi, f + \mathrm{d}\mu, h_1, h_2)$ goes as in the case of quasi-càdlàg barriers, the only difference being in the fact that in (6.1) we replace h_1 by $u_1 \wedge u_2$ and h_2 by $u_1 \vee u_2$, and then we apply (c*).

The problem of existence of a solution is more delicate. Observe that if (u, ν) is a solution to $\mathrm{OP}(\varphi, f + \mathrm{d}\mu, h_1, h_2)$ with measurable barriers h_1 and h_2, then it is a solution to $\mathrm{OP}(\varphi, f + \mathrm{d}\mu, \eta_1, \eta_2)$ with η_1, η_2 as in condition (c*). It appears that under the additional assumption on barriers (mentioned before the last definition) one can construct quasi-càdlàg η_1, η_2 such that if (u, ν) is a solution to $\mathrm{OP}(\varphi, f + \mathrm{d}\mu, \eta_1, \eta_2)$, then it is a solution to $\mathrm{OP}(\varphi, f + \mathrm{d}\mu, h_1, h_2)$. This shows that as long as we only require (b*), the study of the obstacle problem with measurable barriers can be reduced to the study of the obstacle problem with quasi-càdlàg barriers. Finally, note that, unfortunately, there is no construction of η_1, η_2 depending only on h_1 and h_2 (the barriers η_1, η_2 depend also on φ, f, μ). The reason is that the class of càdlàg functions is neither inf-stable nor sup-stable.

Let L, U be measurable adapted processes ($L \le U$). Following [16] we say that a triple of processes (Y, M, R) is a solution of the reflected BSDE with terminal condition ξ, right-hand side $f + \mathrm{d}V$, lower barrier L and upper barrier U if

(a) Y is an $\mathbb{F}$-adapted càdlàg process of class (D), $M \in \mathcal{M}_{0,loc}$, $R \in {}^p\mathcal{V}$,
(b*) $L_t \le Y_t \le U_t$ P-a.s. for a.e. $t \in [0, T]$,
(c*) $\int_0^T (Y_{t-} - H^1_{t-})\,\mathrm{d}R^+_t = \int_0^T (H^2_{t-} - Y_{t-})\,\mathrm{d}R^-_t = 0$ P-a.s. for all càdlàg processes H^1, H^2 such that $L_t \le H^1_t \le Y_t \le H^2_t \le U_t$ P-a.s. for a.e. $t \in [0, T]$,
(d) $[0, T] \ni t \mapsto f(t, Y_t) \in L^1(0, T)$ and

$$Y_t = \xi + \int_t^T f(r, Y_r)\,\mathrm{d}r + \int_t^T \mathrm{d}V_r + \int_t^T \mathrm{d}R_r - \int_t^T \mathrm{d}M_r, \quad t \in [0, T], \quad P\text{-a.s.}$$

It is obvious that for càdlàg barriers the above definition agrees with the definition given in Sect. 4. We see that (u, ν) is a solution to $\mathrm{OP}(\varphi, f + \mathrm{d}\mu, h_1, h_2)$ if and only if $(u(\mathbf{X}), A^\nu, M)$ is a solution to $\mathrm{RBSDE}(\varphi(\mathbf{X}_{\zeta_\upsilon}), f(\mathbf{X}, \cdot) + \mathrm{d}A^\mu, h_1(\mathbf{X}), h_2(\mathbf{X}))$ under the measure P_z for q.e. $z \in E_{0,T}$ (because $m(A) = 0$ if and only if $E_z \int_0^{\zeta_\upsilon} \mathbf{1}_A(\mathbf{X}) = 0$ for q.e. $z \in E_{0,T}$, q.e., and the last condition is satisfied if and only if for q.e. $z \in E_{0,T}$ we have $\mathbf{1}_A(\mathbf{X}_t) = 0$ P_z-a.s. for a.e. $t \in [0, \zeta_\upsilon]$).

PROPOSITION 6.7. *Assume that* (H1)–(H4) *are satisfied and* h_1, h_2 *are measurable functions such that* $h_1 \le v \le h_2$ m_1*-a.e. for some function* v *being a difference of potentials on* $\bar{E}_{0,T}$*. Then, there exist quasi-càdlàg functions* η_1, η_2 *such that* $h_1 \le \eta_1 \le v \le \eta_2 \le h_2$ m_1*-a.e., and moreover, having the property that if* (u, ν) *is a solution to* $\mathrm{OP}(\varphi, f + d\mu, \eta_1, \eta_2)$*, then* (u, ν) *is a solution to* $\mathrm{OP}(\varphi, f + d\mu, h_1, h_2)$*.*

Proof. By Remark 6.2, for q.e. $z \in E_{0,T}$ there exists a unique solution (Y^z, M^z, R^z) to $\mathrm{RBSDE}(\varphi(\mathbf{X}_{\zeta_\upsilon}), f(\mathbf{X}, \cdot) + \mathrm{d}A^\mu, h_1(\mathbf{X}), h_2(\mathbf{X}))$ under the measure P_z. By [16, Theorem 4.2],

$$Y^{z,n}_t \to Y^z_t, \quad t \in [0, \zeta_\upsilon], \quad P_z\text{-a.s.}$$

for q.e. $z \in E_{0,T}$, where $(Y^{z,n}, M^{z,n})$ is a solution to $\mathrm{BSDE}(\varphi(\mathbf{X}_{\zeta_\upsilon}), f_n(\mathbf{X}, \cdot) + \mathrm{d}A^\mu)$ with

$$f_n(z, y) = f(z, y) + n(y - h_1(z))^- - n(y - h_2(z))^+.$$

By [15, Theorem 5.8], $Y^{z,n}_t = u_n(\mathbf{X}_t)$, $t \in [0, \zeta_\upsilon]$, P_z-a.s. for q.e. $z \in E_{0,T}$, where u_n is a solution to $\mathrm{PDE}(\varphi, f_n + \mathrm{d}\mu)$. Let $w := \sup_{n \ge 1} u_n$. It is clear that

$$w(\mathbf{X}_t) = Y^z_t, \quad t \in [0, \zeta_\upsilon], \quad P_z\text{-a.s.}$$

for q.e. $z \in E_{0,T}$. Write $\eta_1 = w \wedge v$, $\eta_2 = w \vee v$ and denote by (u, ν) a solution to $\mathrm{OP}(\varphi, f+\mathrm{d}\mu, \eta_1, \eta_2)$. Then, $(u(\mathbf{X}), M, A^\nu)$ is a solution to $\mathrm{RBSDE}(\varphi(\mathbf{X}_{\zeta_\upsilon}), f(\mathbf{X}, \cdot)+\mathrm{d}A^\mu, \eta_1(\mathbf{X}), \eta_2(\mathbf{X}))$. Since (Y^z, M^z, R^z) is a solution to the same equation (since $\eta_1(\mathbf{X}_t) \le Y^z_t \le \eta_2(\mathbf{X}_t)$, $t \in (0, \zeta_\upsilon)$), we have $(Y^z, M^z, R^z) = (u(\mathbf{X}), M, A^\nu)$ for q.e. $z \in E_{0,T}$, which implies that (u, ν) is a solution to $\mathrm{OP}(\varphi, f + \mathrm{d}\mu, h_1, h_2)$. □

COROLLARY 6.8. *Under the assumptions of Proposition 6.7, there exists a unique solution* (u, ν) *to* $\mathrm{OP}(\varphi, f + \mathrm{d}\mu, h_1, h_2)$.

7. Switching problem

We first describe informally the switching problem. Precise definitions will be given later on. Consider a factory in which we can change a mode of production. Let $c_{j,i}(X)$ be the cost of the change from mode $j \in \{1, \dots, N\}$ to mode i from some set $A_j \subset \{1, \dots, j-1, j+1, \dots, N\}$, and let $f^i(X) + \mathrm{d}A^{\mu^i}$ be the payoff rate in mode i. Then a management strategy $\mathcal{S} = (\{\tau_n\}, \{\xi_n\})$ consists of a pair of two sequences of random variables. The variable τ_n is the moment when we decide to switch the mode of production, and ξ_n is the mode to which we switch at time τ_n. If $\xi_0 = j$, then we start the production at mode j. If $\xi_0 = j$, then under strategy $\mathcal{S}$ the expected profit on the interval $[0, \zeta_\upsilon]$ is given by the formula

$$J(z, \mathcal{S}, j) = E_z\Big(\int_0^{\zeta_\upsilon} f^{w^j_r}(\mathbf{X}_r)\,\mathrm{d}r + \int_0^{\zeta_\upsilon} \mathrm{d}A_r^{\mu^{w^j_r}} - \sum_{n\ge1} c_{w^j_{\tau_{n-1}}, w^j_{\tau_n}}(\mathbf{X}_{\tau_n})\mathbf{1}_{\{\tau_n<\zeta_\upsilon\}} + \varphi^{w^j_{\zeta_\upsilon}}(\mathbf{X}_{\zeta_\upsilon})\Big), \tag{7.1}$$

where

$$w^j_t = j\mathbf{1}_{[0,\tau_1)}(t) + \sum_{n\ge1} \xi_n \mathbf{1}_{[\tau_n,\tau_{n+1})}(t).$$

The main problem is to find an optimal strategy, i.e., the strategy $\mathcal{S}^*$ such that

$$J(x, \mathcal{S}^*, j) = \sup_{\mathcal{S}} J(x, \mathcal{S}, j).$$

In this section, we show that $\mathcal{S}^*$ exists and $\mathcal{S}^*$, $J(z, \mathcal{S}^*, j)$ are determined by a solution of the system (1.14)–(1.16).

7.1. Systems of BSDEs with oblique reflection

In what follows, $N \in \mathbb{N}$, $\xi = (\xi^1, \dots, \xi^N)$ is an $\mathcal{F}_T$-measurable random vector, $V = (V^1, \dots, V^N)$ is an $\mathbb{F}$-adapted process such that $V_0 = 0$ and each component V^j is a càdlàg process of finite variation, $f : \Omega \times [0, T] \times \mathbb{R}^N \to \mathbb{R}^N$ is a measurable function such that for every $y \in \mathbb{R}^N$ the process $f(\cdot, y)$ is $\mathbb{F}$-progressively measurable. Consider a family $\{h_{j,i}; i, j = 1, \dots, N\}$ of measurable functions $h_{j,i} : \Omega \times [0, T] \times$

$\mathbb{R} \to \mathbb{R}$ such that $h_{j,i}(\cdot, y^i)$ is progressively measurable for every $y \in \mathbb{R}$. For given sets $A_j \subset \{1, \dots, j-1, j+1, \dots, N\}$, $j = 1, \dots, N$, we set

$$H^j(t, y) = \max_{i \in A_j} h_{j,i}(t, y_i),$$
$$H(t, y) = (H^1(t, y), \dots, H^N(t, y)), \quad t \in [0, T],\ y \in \mathbb{R}^N.$$

We consider the following system of BSDEs with oblique reflection:

$$\begin{cases} Y_t^j = \xi^j + \int_t^T f^j(r, Y_r)\,\mathrm{d}r + \int_t^T \mathrm{d}V_r + \int_t^T \mathrm{d}K_r^j - \int_t^T \mathrm{d}M_r^j, \quad t \in [0, T], \\ Y_t^j \geq H^j(t, Y_t), \quad t \in [0, T], \\ \int_0^T \left(Y_{t-}^j - [H^j(\cdot, Y)]_{t-} \right) \mathrm{d}K_t^j = 0, \ j = 1, \dots, N. \end{cases} \tag{7.2}$$

DEFINITION. We say that a triple (Y, M, K) of adapted càdlàg processes is a solution of BSDE with oblique reflection (7.2) if Y is of class (D), M is a local martingale with $M_0 = 0$, K is an increasing process with $K_0 = 0$ and (7.2) is satisfied.

If $A_j = \emptyset$, then by convention, $H^j = -\infty$, so Y^j has no lower barrier. We then take $K^j = 0$ in the above definition.

7.2. Systems of quasi-variational inequalities

Fix $N \geq 1$. Let $\mu^j, j = 1, \dots, N$, be smooth measures on $E_{0,T}$, and let $\varphi^j : E \to \mathbb{R}$, $f^j : \bar{E}_{0,T} \times \mathbb{R}^N \to \mathbb{R}$, $h_{j,i} : \bar{E}_{0,T} \times \mathbb{R} \to \mathbb{R}$, $i, j = 1, \dots, N$, be measurable functions. We set

$$f^j(z, y; a) = f^j\left(z, y_1, \dots, y_{j-1}, a, y_{j+1}, \dots, y_N\right), \quad y \in \mathbb{R}^N,\ a \in \mathbb{R},$$

and for given sets $A_j \subset \{1, \dots, j-1, j+1, \dots, N\}$, $j = 1, \dots, N$, we set

$$H^j(z, y) = \max_{i \in A_j} h_{j,i}(z, y_i), \quad H(z, y) = \left(H^1(z, y), \dots, H^N(z, y)\right),$$
$$z \in \bar{E}_{0,T},\ y \in \mathbb{R}^N.$$

We adopt the convention that the maximum over the empty set equals $-\infty$. Consequently, if $A_j = \emptyset$ for some j, then $H^j(z, y) = -\infty$.

Set

$$\varphi = (\varphi^1, \dots, \varphi^N), \quad f = (f^1, \dots, f^N), \quad \mu = (\mu^1, \dots, \mu^N),$$

and consider the following system of equations:

$$-\frac{\partial u^j}{\partial t} - L_t u^j = f^j(t, x, u) + \mu^j \quad \text{in } E_{0,T}, \quad u^j(T, \cdot) = \varphi^j \quad \text{on } E \tag{7.3}$$

for $j = 1\dots, N$. In the sequel, we denote (7.3) by $\mathrm{PDE}(\varphi, f + \mathrm{d}\mu)$.

DEFINITION. We say that measurable function $u = (u^1, \dots, u^N) : \bar{E}_{0,T} \to \mathbb{R}^N$ is a subsolution (resp. supersolution) of PDE(φ, $f + \mathrm{d}\mu$) if there exist positive measures $\beta^j \in \mathbb{M}$ and $\underline{\varphi} \le \varphi$, $\underline{\varphi} \cdot m \otimes \delta_{\{T\}} \in \mathbb{M}_T$ (resp. $\overline{\varphi} \ge \varphi$, $\overline{\varphi} \cdot m \otimes \delta_{\{T\}} \in \mathbb{M}_T$) such that u^j is a solution of PDE($\underline{\varphi}^j$, $f^j + \mathrm{d}\mu^j - \mathrm{d}\beta^j$) (resp. PDE($\overline{\varphi}^j$, $f^j + \mathrm{d}\mu^j + \mathrm{d}\beta^j$)), $j = 1, \dots, N$.

DEFINITION. We say that a quasi-càdlàg function $u = (u^1, \dots, u^N) : \bar{E}_{0,T} \to \mathbb{R}^N$ is a solution of (1.14)–(1.16) if there exist positive smooth measures ν^j on $E_{0,T}$ such that (u^j, ν^j) is a solution of $\underline{\text{O}}$P(φ^j, $f^j(\cdot, u; \cdot) + \mathrm{d}\mu^j$, $H^j(\cdot, u)$), $j = 1, \dots, N$.

Let us consider the following hypotheses:

(A1) $\mu, \varphi \cdot m \otimes \delta_{\{T\}} \in \mathbb{M}_T$,

(A2) for $j = 1, \dots, N$ the function is $a \mapsto f^j(z, y; a)$ is nonincreasing for all $z \in E_{0,T}$, $y \in \mathbb{R}^N$,

(A3) f is off-diagonal nondecreasing, i.e., for $j = 1, \dots, N$ we have $f^j(z, y) \le f^j(z, \bar{y})$ for all $y, \bar{y} \in \mathbb{R}^N$ such that $y \le \bar{y}$ and $y^j = \bar{y}^j$,

(A4) $y \mapsto f(z, y)$ is continuous for every $z \in E_{0,T}$,

(A5) $f^j(\cdot, y) \in qL^1(E_{0,T}; m)$ for all $y \in \mathbb{R}^N$, $j = 1, \dots, N$,

(A6) there exists a subsolution $\underline{u}$ and a supersolution $\overline{u}$ of PDE(φ, $f + \mathrm{d}\mu$) such that

$$\underline{u} \le \overline{u}, \quad H(\cdot, \overline{u}) \le \overline{u}, \quad \sum_{j=1}^{N} (|f^j(\cdot, \overline{u})| + |f^j(\cdot, \underline{u}; \overline{u}^j)|) \cdot m_1 \in \mathbb{M},$$

(A7) H^j is continuous on $\bar{E}_{0,T} \times \mathbb{R}^N$ with the product topology consisting of quasi-topology on $\bar{E}_{0,T}$ and Euclidean topology on $\mathbb{R}^N$, and $h_{j,i}$, $i, j = 1, \dots, N$, are nondecreasing with respect to the second variable.

THEOREM 7.1. *Let the assumptions* (A1)–(A7) *hold. Then, there exists a minimal solution of* (1.14)–(1.16) *such that* $\underline{u} \le u \le \overline{u}$.

Proof. First observe that the data $f(\mathbf{X}, \cdot)$, $H^j(\mathbf{X}, \cdot)$, $\xi := \varphi(\mathbf{X}_{\zeta_\upsilon})$, $\underline{Y} := \underline{u}(\mathbf{X})$, $\overline{Y} := \overline{u}(\mathbf{X})$, $V = A^\mu$ satisfy the assumptions of [18, Theorem 3.11] under the measure P_z for q.e. $z \in E_{0,T}$. Set $u_0 = \underline{u}$ and $Y^0 = \underline{Y}$. By Theorem 5.4 (see also Remark 6.6), for every $n \ge 1$,

$$u_n^j(X_t) = Y_t^{n,j}, \quad A_t^{\nu_n} = K_t^{n,j},$$

where (u_n^j, ν_n^j) is a solution of $\underline{\text{O}}$P(φ^j, $f^j(\cdot, u_{n-1}; \cdot) + \mu^j$, $H^j(\cdot, u_{n-1})$) and the triple $(Y^{n,j}, K^{n,j}, M^{n,j})$ is a solution of $\underline{\text{R}}$BSDE(ξ^j, $f^j(\mathbf{X}, Y^{n-1}; \cdot) + \mathrm{d}V^j$, $H^j(\cdot, Y^{n-1})$). By Proposition 5.5, $u_n \le u_{n+1}$ q.e. Set $u := \sup_{n \ge 0} u_n$. By [18, Theorem 3.11],

$$Y_t^{n,j} \nearrow Y_t, \quad t \in [0, \zeta_\upsilon], \quad P_z\text{-a.s.}$$

for q.e. $z \in E_{0,T}$, where (Y, M, K) is the minimal solution of (7.2) such that $\underline{Y} \le Y \le \overline{Y}$. Hence, since the exceptional sets coincide with the sets of zero capacity, we have

$$u^j(\mathbf{X}_t) = Y_t^j, \quad t \in [0, \zeta_\upsilon], \quad P_z \text{a.s.}$$

for q.e. $z \in E_{0,T}$. We see that the triple (Y^j, M^j, K^j) is a solution to the problem $\underline{\mathrm{R}}\mathrm{BSDE}(\varphi^j(\mathbf{X}_{\zeta_\upsilon}), f^j(\mathbf{X}, u(\mathbf{X}); \cdot)+\mathrm{d}A^{\mu^j}, H^j(\cdot, u(\mathbf{X})))$. By Theorem 5.4, $K^j = A^{\nu^j}$, where (u^j, ν^j) is a solution to $\underline{\mathrm{O}}\mathrm{P}(\varphi^j, f^j(\cdot, u; \cdot) + \mathrm{d}\mu^j, H^j(\cdot, u))$, $j = 1, \dots, N$, which implies that the pair (u, ν) is a solution of (1.14)–(1.16). Minimality of u follows from the minimality of Y. □

REMARK 7.2. Let u be the minimal solution of Theorem 7.1. Observe that under assumptions of Theorem 7.1 $f^j(\cdot, u) \cdot m_1 \in \mathbb{M}$ and $\nu^j \in \mathbb{M}$, $j = 1, \dots, N$. Indeed, first observe that $\bar{u}^j \geq H^j(\cdot, u)$ and

$$f^j(\cdot, \bar{u}) \leq f^j(\cdot, u; \bar{u}) \leq f^j(\cdot, \underline{u}; \bar{u}). \tag{7.4}$$

Since u^j is a solution of $\underline{\mathrm{O}}\mathrm{P}(\varphi^j, f^j(\cdot, u; \cdot) + \mathrm{d}\mu^j, H^j(\cdot, u))$, $j = 1, \dots, N$ we get the result by Remark 6.6. Moreover, if we assume that $\mathcal{E}_\gamma$ has the dual Markov property for some $\gamma \geq 0$, assumptions (A1), (A6) are satisfied with $\mathbb{M}$ replaced by $\mathcal{M}_1$ and the measures β^j appearing in the definition of the supersolution $\overline{u}$ belong to $\mathcal{M}_1$. Then by Remark 6.6 under assumptions of Theorem 7.1 $\nu^j \in \mathcal{M}_1$, and $f^j(\cdot, u) \in L^1(E_{0,T}; m_1)$, $j = 1, \dots, N$.

Let us consider the following hypothesis:

(A8) there exists a subsolution $\underline{u}$ and a supersolution $\overline{u}$ of $\mathrm{PDE}(\varphi, f + \mathrm{d}\mu)$ and a function $v = (v^1, \dots, v^N)$ which is a difference of potentials on $\bar{E}_{0,T}$ such that

$$\sum_{j=1}^N (|f^j(\cdot, \overline{u}; v^j)| + |f^j(\cdot, \underline{u}; v^j)|) \cdot m_1 \in \mathbb{M}.$$

PROPOSITION 7.3. *Let assumptions* (A1)–(A5), (A8) *hold. Then, there exists minimal solution u of* $\mathrm{PDE}(\varphi, f + d\mu)$ *such that $\underline{u} \leq u \leq \overline{u}$ q.e.*

Proof. Observe that $f(\mathbf{X}, \cdot)$, $\underline{Y} := \underline{u}(\mathbf{X})$, $\overline{Y} := \overline{u}(\mathbf{X})$, $S = v(\mathbf{X})$ (see Remark 5.1), $V = A^\mu$, $\xi := \varphi(\mathbf{X}_{\zeta_\upsilon})$ satisfy the assumptions of [18, Theorem 2.12] under the measure P_z for q.e. $z \in E_{0,T}$. Set $u_0 = \underline{u}$. By [15, Theorem 5.8], $Y^{n,j} = u_n^j(\mathbf{X})$, where u_n^j is a solution of $\mathrm{PDE}(\varphi, f^j(\cdot, u_{n-1}; \cdot) + \mathrm{d}\mu^j)$ and $(Y^{n,j}, M^{n,j})$ is a solution of $\mathrm{BSDE}(\xi^j, f^j(\mathbf{X}, Y^{n-1}; \cdot) + \mathrm{d}V^j)$, $j = 1, \dots, N$. By [15, Corollary 5.9] $u_n \leq u_{n+1}$ q.e. Set $u = \sup_{n\geq 1} u_n$. By [18, Theorem 2.12], it follows that $Y_t^n \nearrow Y_t$, $t \in [0, \zeta_\upsilon]$, P_z-a.s. for q.e. $z \in E_{0,T}$, where (Y, M) is a minimal solution of $\mathrm{BSDE}(\varphi(\mathbf{X}_{\zeta_\upsilon}), f(\mathbf{X}, \cdot) + \mathrm{d}A^\mu)$ such that $\underline{u}(\mathbf{X}) \leq Y \leq \overline{u}(\mathbf{X})$. Hence, (since the exceptional sets coincide with the sets of zero capacity) $u(\mathbf{X}_t) = Y_t$, $t \in [0, \zeta_\upsilon]$, P_z-a.s. for q.e. $z \in E_{0,T}$, which implies that u is the minimal solution of $\mathrm{PDE}(\varphi, f + \mathrm{d}\mu)$ such that $\underline{u} \leq u \leq \overline{u}$ q.e. □

THEOREM 7.4. *Assume* (A1)–(A7). *Then, there exists minimal solution u_n of the system*

$$-\frac{\partial u_n}{\partial t} - L_t u_n^j = f^j(\cdot, u_n) + n(u_n^j - H^j(\cdot, u_n))^- + \mu \tag{7.5}$$

such that $\underline{u} \le u_n \le \overline{u}$. *Moreover,* $u_n \nearrow u$ *q.e., where* u *is minimal solution of* (1.14)–(1.16) *such that* $\underline{u} \le u \le \overline{u}$.

Proof. Observe that $\overline{u}$ is a supersolution of (7.5), and $\underline{u}$ is a subsolution of (7.5). Moreover, (A8) for (7.5) is satisfied with $v = \overline{u}$. By Proposition 7.3, there exists a minimal solution u_n of (7.5). By the definition and construction of minimal solution to (7.5) (see Proposition 7.3) and minimal solution to BSDE($\varphi(\mathbf{X}_{\zeta_\upsilon}), f_n(\mathbf{X}, \cdot) + dA^\mu$) (see [18, Theorem 2.12]), $u_n(\mathbf{X})$ is the first component of minimal solution of BSDE($\varphi(\mathbf{X}_{\zeta_\upsilon}), f_n(\mathbf{X}, \cdot) + dA^\mu$) with $f_n^j(z, y) = f^j(z, y) + n(y^j - H^j(z, y))^-$. By [18, Theorem 3.15], the sequence $\{u_n(\mathbf{X})\}$ is nondecreasing and $u_n(\mathbf{X})_t \nearrow Y_t$, $t \in [0, \zeta_\upsilon]$, P_z-a.s. for q.e. $z \in E_{0,T}$, where Y is the first component of the minimal solution (Y, M, K) of (7.2) such that $\underline{u}(\mathbf{X}) \le Y \le \overline{u}(\mathbf{X})$. Since the sets coincide with the sets of zero capacity $u_n \le u_{n+1}$, q.e. on $E_{0,T}$. Let $u := \sup_{n\ge1} u_n$. It is clear that $Y_t = u(\mathbf{X}_t)$, $t \in [0, \zeta_\upsilon]$, P_z-a.s. for q.e. $z \in E_{0,T}$. Now we see that (Y^j, M^j, K^j) is a solution to $\underline{\text{R}}$BSDE($\varphi^j(\mathbf{X}_{\zeta_\upsilon}), f^j(\mathbf{X}, u(\mathbf{X}); \cdot) + dA^{\mu^j}, H^j(\cdot, u(\mathbf{X}))$). By Theorem 5.4 $K^j = A^{\nu^j}$, where (u^j, ν^j) is a solution to $\underline{\text{O}}$P($\varphi^j, f^j(\cdot, u; \cdot) + d\mu^j, H^j(\cdot, u)$), $j = 1, \dots, N$. This implies that (u, ν) is the minimal solution of (1.14)–(1.16) such that $\underline{u} \le u \le \overline{u}$. Of course, $u_n \nearrow u$ q.e. $\square$

7.3. Value function for the switching problem

In what follows, we assume that H^j are of the form

$$H^j(z, y) = \max_{i \in A_j} \left(-c_{j,i}(z) + y^i \right), \tag{7.6}$$

where $c_{j,i}$ are quasi-continuous functions on $E_{0,T}$ such that for some constant $c > 0$,

$$c_{j,i}(z) \ge c, \quad z \in E_{0,T}, \quad i \in A_j, \quad j = 1, \dots, N.$$

By a strategy, we call a pair $\mathcal{S} = (\{\xi_n\}, \{\tau_n\})$ consisting of a sequence $\{\tau_n, n \ge 1\}$ of increasing $\mathbb{F}$-stopping times such that

$$P_z(\tau_n < \zeta_\upsilon, \forall\, n \ge 1) = 0$$

for q.e. $z \in E_{0,T}$, and a sequence $\{\xi_n, n \ge 1\}$ of random variables taking values in $\{1, \dots, N\}$ such that ξ_n is $\mathcal{F}_{\tau_n}$-measurable for each $n \ge 1$. The set of all strategies we denote by $\mathbf{A}$. For $\mathcal{S} \in \mathbf{A}$, we set

$$w_t^j = j\mathbf{1}_{[0,\tau_1)}(t) + \sum_{n\ge1} \xi_n \mathbf{1}_{[\tau_n, \tau_{n+1})}(t).$$

REMARK 7.5. In Theorem 7.3, assume additionally that $\mu \in \mathbb{M}_c$, $h_{j,i}$ are strictly increasing with respect to y, and that the following condition considered in [11] is satisfied:

(A9) there are no $(y_1, \dots, y_k) \in \mathbb{R}^k$ and $j_2 \in A_{j_1}, \dots, j_k \in A_{j_{k-1}}, j_1 \in A_{j_k}$ such that

$$y_1 = h_{j_1,j_2}(z, y_2), y_2 = h_{j_2,j_3}(z, y_3), \dots, y_{k-1} = h_{j_{k-1},j_k}(z, y_k), y_k = h_{j_k,j_1}(z, y_1).$$

Then, $\nu \in \mathbb{M}_c$ and u is quasi-continuous. This follows from [18, Remark 3.14]. Observe that (A9) is satisfied for $h_{j,i}$ defined by (7.6).

THEOREM 7.6. *Assume that f does not depend on y, the functions H^j are of the form (7.6) and $f^j \cdot m_1, \mu^j \in \mathbb{M}$, $j = 1, \dots, N$. Then, there exists a unique solution u of (1.14)–(1.16). Moreover,*

$$u^j(z) = \sup_{\mathcal{S} \in \mathbf{A}} J(z, \mathcal{S}, j)$$

and

$$u^j(z) = E_z\Big(\int_0^{\zeta_\upsilon} f^{w_r^{j,*}}(\mathbf{X}_r)\,\mathrm{d}r + \int_0^{\zeta_\upsilon} \mathrm{d}A_r^{\mu^{w_r^{j,*}}} \\ - \sum_{n\ge 1} c_{w^{j,*}_{\tau_{n-1}}, w^{j,*}_{\tau_n}}(\mathbf{X}_{\tau_n})\mathbf{1}_{\{\tau_n < \zeta_\upsilon\}} + \varphi^{w^{j,*}_{\zeta_\upsilon}}(\mathbf{X}_{\zeta_\upsilon})\Big),$$

where

$$w_t^{j,*} = j\mathbf{1}_{[0,\tau_1^{j,*})}(t) + \sum_{n\ge 1} \xi_n^{j,*}\mathbf{1}_{[\tau_n^{j,*}, \tau_{n+1}^{j,*})}(t)$$

and

$$\begin{aligned} \tau_0^{j,*} &= 0, \qquad \xi_0^{j,*} = j, \\ \tau_k^{j,*} &= \inf\Big\{t \ge \tau_{k-1}^{j,*} : u^{\xi_{k-1}^{j,*}}(\mathbf{X}_t) = H^{\xi_{k-1}^{j,*}}(\mathbf{X}_t, u(\mathbf{X}_t))\Big\} \wedge \zeta_\upsilon, \quad k \ge 1, \\ \xi_k^{j,*} &= \max\Big\{i \in A_{\xi_{k-1}^{j,*}};\ H^{\xi_{k-1}^{j,*}}\Big(\mathbf{X}_{\tau_k^{j,*}}, u\Big(\mathbf{X}_{\tau_k^{j,*}}\Big)\Big) = -c_{\xi_{k-1}^{j,*},i}\Big(\mathbf{X}_{\tau_k^{j,*}}\Big) + u^i(\mathbf{X}_{\tau_k^{j,*}})\Big\}, \\ &\quad k \ge 1. \end{aligned}$$

Proof. Follows from Proposition 7.3, Remark 7.5 and [18, Theorem 4.3]. □

REFERENCES

[1] Bensoussan, A., Lions, J.-L.: Applications of variational inequalities in stochastic vontrol. North-Holland, Amsterdam, (1982)

[2] Bögelein, V., Duzaar, F., Scheven, C.: The obstacle problem for parabolic minimizers, J. Evol. Equ. (2017). https://doi.org/10.1007/s00028-017-0384-4

[3] Chung, K.L., Walsh, J.B.: Meyer's theorem on predictability. Z. Wahrscheinlichkeitstheorie verw. Gebiete **24**, 85–93 (1972)
[4] Dal Maso, G., Murat, F., Orsina, L., Prignet, A.: Renormalized solutions of elliptic equations with general measure data. Ann. Scuola Norm. Sup. Pisa Cl. Sci. **28** (4), 741–808 (1999)
[5] Dellacherie, C., Meyer, P.-A.: Probabilities and Potential. North-Holland Publishing Co., Amsterdam (1978)
[6] Djehiche B., Hamadène S., Popier, A.: The finite horizon optimal multiple switching problem. SIAM J. Control Optim. **48** (4), 2751–2770 (2009)
[7] El Asri, B., Hamadène, S.: The finite horizon optimal multi-modes switching problem: The viscosity solution approach. Appl. Math. Optim. **60**, 213–235 (2009)
[8] Getoor, R.K., Sharpe, M.J.: Naturality, standardness, and weak duality for Markov processes. Z. Wahrsch. verw. Gebiete **67**, 1–62 (1984)
[9] Hamadène, S., Jeanblanc, M.: On the starting and stopping problem: application in reversible investments Math. Oper. Res. **32**, 182–192 (2007)
[10] Hamadène, S., Morlais, M.A.: Viscosity solutions of systems of PDEs with interconnected obstacles and multi-modes switching problem. arXiv:1104.2689v2
[11] Hamadène, S., Zhang, J.: Switching problem and related system of reflected backward SDEs. Stochastic Process. Appl. **120**, 403–426 (2010)
[12] Hu, Y., Tang, S.: Multi-dimensional BSDE with oblique reflection and optimal switching. Probab. Theory Related Fields **147**, 89–121 (2010)
[13] Klimsiak, T.: Reflected BSDEs and the obstacle problem for semilinear PDEs in divergence form. Stochastic Process. Appl. **122**, 134–169 (2012)
[14] Klimsiak, T.: Cauchy problem for semilinear parabolic equation with time-dependent obstacles: a BSDEs approach. Potential Anal. **39**, 99–140 (2013)
[15] Klimsiak, T.: Semi-Dirichlet forms, Feynman-Kac functionals and the Cauchy problem for semilinear parabolic equations. J. Funct. Anal. **268**, 1205–1240 (2015)
[16] Klimsiak, T.: Reflected BSDEs on filtered probability spaces. Stochastic Process. Appl. **125**, 4204–4241 (2015)
[17] Klimsiak, T.: Reduced measures for semilinear elliptic equations involving Dirichlet operators. Calc. Var. Partial Differential Equations **55**, no. 4, Paper No. 78, 27 pp. (2016)
[18] Klimsiak, T.: Systems of quasi-variational inequalities related to the switching problem. arXiv:1609.02221v2 (2016)
[19] Klimsiak, T., Rozkosz, A.: Dirichlet forms and semilinear elliptic equations with measure data. J. Funct. Anal. **265**, 890–925 (2013)
[20] Klimsiak, T., Rozkosz, A.: Obstacle problem for semilinear parabolic equations with measure data. J. Evol. Equ. **15**, 457–491 (2015)
[21] Klimsiak, T., Rozkosz, A.: Renormalized solutions of semilinear equations involving measure data and operator corresponding to Dirichlet form. NoDEA Nonlinear Differential Equations Appl. **22**, 1911–1934 (2015)
[22] Lenglart, E.; Lépingle, D.; Pratelli, M. Présentation unifiée de certaines inégalités de la théorie des martingales. (French) With an appendix by Lenglart. Lecture Notes in Math. **784**, Seminar on Probability, XIV (Paris, 1978/1979) pp. 26–52, Springer, Berlin, 1980
[23] Lundström, N., Nyström, K., Olofsson, M.: Systems of variational inequalities for non-local operators related to optimal switching problems: existence and uniqueness. Manuscripta Math. **145**, 407–432 (2014)
[24] Lundström, N. Nyström, K. Olofsson, M.: Systems of variational inequalities in the context of optimal switching problems and operators of Kolmogorov type. Ann. Mat. Pura Appl. (4) **193**, 1213–1247 (2014)
[25] Mignot, F., Puel, J.P.: Inéquations d'évolution paraboliques avec convexes dépendant du temps. Applications aux inéquations quasi-variationnelles d'évolution. Arch. Ration. Mech. Anal. **64**, 59–91 (1977)
[26] Oshima Y.: Semi-Dirichlet forms and Markov processes. Walter de Gruyter, Berlin (2013)
[27] Oshima, Y.: Some properties of Markov processes associated with time dependent Dirichlet forms. Osaka J. Math. **29**, 103–127 (1992)

[28] Peng, S.: Monotonic Limit Theorem of BSDE and Nonlinear Decomposition Theorem of Doob-Meyers Type. Probab. Theory Related Fields **113**, 473–499 (1999)

[29] Petitta, F., Ponce, A.C., Porretta, A.: Diffuse measures and nonlinear parabolic equations. J. Evol. Equ. **11**, 861–905 (2011)

[30] Pierre, M.: Problèmes d'évolution avec contraintes unilatérales et potentiel paraboliques. Comm. Partial Differential Equations **4**, 1149–1197 (1979)

[31] Pierre, M.: Représentant précis d'un potentiel parabolique. Séminaire de Théorie du Potentiel, Paris, No. 5, Lecture Notes in Math. **814**, 186–228 (1980)

[32] Pierre, M.: Parabolic capacity and Sobolev spaces, Siam J. Math. Anal. **14**, 522–533 (1983)

[33] Stannat, W.: The theory of generalized Dirichlet forms and its applications in analysis and stochastics. Mem. Amer. Math. Soc. **142**, no. 678, viii+101 pp. (1999)

[34] M. Topolewski, Reflected BSDEs with general filtration and two completely separated barriers. arXiv:1611.06745 (2016)

Tomasz Klimsiak
Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8
00-956 Warsaw
Poland
E-mail: tomas@mat.umk.pl

and

Faculty of Mathematics and
Computer Science
Nicolaus Copernicus University
Chopina 12/18
87-100 Toruń
Poland