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Obstructions to approximating tropical curves in surfaces via intersection theory

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LIFTING REPRESENTATIONS OF FINITE REDUCTIVE GROUPS I: SEMISIMPLE CONJUGACY CLASSES

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Abstract. Suppose that $\widetilde{G}$ is a connected reductive group defined over a field k, and Γ is a finite group acting via k-automorphisms of $\widetilde{G}$ satisfying a certain quasi-semisimplicity condition. Then the identity component of the group of Γ-fixed points in $\widetilde{G}$ is reductive. We axiomatize the main features of the relationship between this fixed-point group and the pair $(\widetilde{G}, \Gamma)$, and consider any group G satisfying the axioms. If both $\widetilde{G}$ and G are k-quasisplit, then we can consider their duals $\widetilde{G}^*$ and G^*. We show the existence of and give an explicit formula for a natural map from the set of semisimple stable conjugacy classes in $G^*(k)$ to the analogous set for $\widetilde{G}^*(k)$. If k is finite, then our groups are automatically quasisplit, and our result specializes to give a map of semisimple conjugacy classes. Since such classes parametrize packets of irreducible representations of $G(k)$ and $\widetilde{G}(k)$, one obtains a mapping of such packets.

0. Introduction

Motivation. Suppose that F is a p-adic field with residue field k; E/F is a finite, tamely ramified Galois extension; H is a connected, reductive F-group; and $\widetilde{H} = R_{E/F}H$ is formed from H via restriction of scalars. Then one expects to have a *base change lifting* that takes L-packets of smooth, irreducible representations of $H(F)$ to L-packets for $H(E) = \widetilde{H}(F)$. We would like to gain an explicit understanding, in terms of compact-open data, of base change for depth-zero representations, and this problem requires us to construct a new lifting from (packets of) representations of $G(k)$ to those of $\widetilde{G}(k)$, for various connected reductive k-groups G and $\widetilde{G}$ attached to parahoric subgroups of $H(F)$ and $\widetilde{H}(F)$, respectively. Here $\Gamma = \mathrm{Gal}(E/F)$ acts on $\widetilde{G}$, and the identity component of its group of fixed points is G. (In most cases, this new lifting cannot itself be base change. For more details, see [1, 3].) Since representations of $G(k)$ can be parametrized by data associated to the dual group G^*, it is enough to construct an appropriate lifting of such data from G^* to $\widetilde{G}^*$.

This paper. In the course of creating a candidate for this new lifting, we realized that we could work in greater generality without much difficulty. Namely, let k denote an arbitrary field, $\widetilde{G}$ and G connected reductive k-groups, and Γ a finite

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group. Instead of assuming that $G = (\widetilde{G}^\Gamma)^\circ$, the identity component of the group of fixed points of Γ in $\widetilde{G}$, we make the more general assumption that G is a *parascopic group* for $(\widetilde{G}, \Gamma)$ (see Definition 7.1). There are several reasons to do this. First, it clarifies our proofs. Second, while our original motivation was to improve our explicit understanding of base change for representations of p-adic groups, we hope that our more general formulation will be applicable to the understanding of a wider collection of correspondences of representations, including endoscopic transfer. We will take this up elsewhere.

Under our hypotheses, we have the following results.

(A) Suppose $\widetilde{G}$ and G are k-quasisplit. Then we obtain a natural map $\widehat{\mathcal{N}}$ from the k-variety of semisimple geometric conjugacy classes of the dual G^* to the analogous variety for $\widetilde{G}^*$ (Proposition 10.1).

(B) By restricting and refining $\widehat{\mathcal{N}}$, one obtains a map $\widehat{\mathcal{N}}^{\mathrm{st}}$ from the set of semisimple stable conjugacy classes (in the sense of Kottwitz [9]) in $G^*(k)$ to that in $\widetilde{G}^*(k)$ (Theorem 11.1).

(C) If k is perfect of cohomological dimension ≤ 1 (e.g., k is finite), then $\widehat{\mathcal{N}}$ is a map from the set of semisimple conjugacy classes in $G^*(k)$ to that in $\widetilde{G}^*(k)$ (Corollary 11.3).

Before proving the above results, we show that the situation that motivated our notion of parascopy really is a special case of it: that in which Γ acts on $\widetilde{G}$ via k-automorphisms that all preserve a common Borel subgroup of $\widetilde{G}$ and a common maximal torus in the Borel subgroup, and $G = (\widetilde{G}^\Gamma)^\circ$. This "Borel-torus pair" need not be defined over k. Under the above hypotheses, we prove a strong form of the following:

(D) G is a reductive k-group (Proposition 3.5).

Although we assume that $\widetilde{G}$ and G are k-quasisplit in Statements (A) and (B) (it is automatic in Statement (C)), weaker hypotheses would suffice. See Remark 9.1.

Outline of this paper. After establishing some basic notation (§1), we consider in §2 how the action of a finite group on a torus $\widetilde{T}$ gives rise to a norm map $\mathcal{N}\colon \widetilde{T} \longrightarrow T$ (where T is the identity component of the group of fixed points of $\widetilde{T}$), and also corresponding maps on the modules of characters and cocharacters of these tori. In fact, we deal with the more general situation where we may replace T by any isogenous image of it (see Condition P1 of Definition 4.1). We then prove a strong version of Statement (D) above (§3). Suppose G and $\widetilde{G}$ are connected reductive k-groups, and Γ is a finite group. In §4, we say what we mean by a *parascopic datum* for the triple $(\widetilde{G}, \Gamma, G)$, and say that G is *weakly parascopic* for the pair $(\widetilde{G}, \Gamma)$ if such a datum exists. Given such a datum, we have associated maximal tori $T \subset G$ and $\widetilde{T} \subset \widetilde{G}$, an action of Γ on the Weyl group $W(\widetilde{G}, \widetilde{T})$, and a canonical embedding $W(G,T) \longrightarrow W(\widetilde{G}, \widetilde{T})^\Gamma$, which we describe explicitly in §5. Using standard cohomological arguments, we classify in §6 the set of stable conjugacy classes of maximal k-tori in a reductive k-group. In §7, we call a weakly parascopic group G *parascopic* if a compatibility condition between the maximal k-tori of G and $\widetilde{G}$ is satisfied, a condition that is automatic in many important cases (see Examples 7.2). We then define and prove some basic results on *equivalence* of parascopic data. For example, if (ϕ, j_*) is a datum for $(\widetilde{G}, \Gamma, G)$ with respect to the maximal k-tori $\widetilde{T} \subseteq \widetilde{G}$ and $T \subseteq G$, and G is parascopic with respect to this datum, then given any maximal k-torus $T' \subseteq G$, there is an equivalent datum associated to T'. This is crucial in showing that all of our constructions are independent of the choice of a maximal k-torus in G. If G is quasisplit over k, then we can form its dual group G^*, and we are then in a position to prove strong duality results (§8) between maximal k-tori in G and in G^*. In particular, up to stable conjugacy, we have a canonical one-to-one correspondence of tori, and this correspondence preserves Weyl groups and much more. If $\widetilde{G}$ is also quasisplit over k, then we can use this correspondence and our norm map $\mathcal{N} \colon \widetilde{T} \longrightarrow T$ above to define a *conorm map* $\widehat{\mathcal{N}}_{T^*} \colon T^* \longrightarrow \widetilde{T}^*$ for dual maximal k-tori $T^* \subseteq G^*$ and $\widetilde{T}^* \subseteq \widetilde{G}^*$, and can obtain explicit embeddings of Weyl groups $W(G^*, T^*) \longrightarrow W(\widetilde{G}^*, \widetilde{T}^*)$ (§9). In particular, we show (Proposition 9.4) that such embeddings have good restriction properties with respect to centralizer subgroups of G^*. We then have all of the ingredients in place to prove Statement (A) in §10. Using our cohomological results in §6, we can then prove Statement (B) in §11, and it is a simple matter to observe that Statement (C) is just a special case.

In a future work, we will address the problem of lifting other pieces of the parametrization of irreducible representations of finite reductive groups, such as unipotent conjugacy classes in dual groups.

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1. General notation and terminology

Let k denote a field, and k^{sep} denote the separable closure of k in an algebraic closure $\bar{k}$ of k. We will abbreviate $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ by $\mathrm{Gal}(k)$. Given a connected reductive k-group G and a maximal torus T of G, let $\Phi(G,T)$ (resp. $\Phi^\vee(G,T)$) denote the absolute root (resp. coroot) system of G with respect to T. Let $W(G,T)$ denote the Weyl group of G with respect to T.

For $g \in G(k^{\mathrm{sep}})$, let Int denote the natural homomorphism $G \longrightarrow \mathrm{Inn}(G)$ given by $\mathrm{Int}(g)(x) = gxg^{-1}$. If $x \in G(k^{\mathrm{sep}})$ (resp. $Y \subseteq G$), we will also denote $\mathrm{Int}(g)(x)$ (resp. $\mathrm{Int}(g)(Y)$) by ${}^g x$ (resp. ${}^g Y$). For an algebraic k-group G, let G° denote its identity component. A *geometric conjugacy class* of G is an orbit for the action of G on itself via conjugation. Following Kottwitz [9], we say that two elements $s_1, s_2 \in G(k)$ are *stably conjugate* if there is some $g \in G(k^{\mathrm{sep}})$ such that ${}^g s_1 = s_2$, and for all $\sigma \in \mathrm{Gal}(k)$, we have that $g^{-1}\sigma(g) \in C_G(s_1)^\circ(k^{\mathrm{sep}})$. Moreover, we say that two maximal k-tori $T_1, T_2 \subseteq G$ are *stably conjugate* in G if there is some element $g \in G(k^{\mathrm{sep}})$ such that $\mathrm{Int}(g)$ restricts to a k-isomorphism from T_1 to T_2. Let $\mathcal{T}_{\mathrm{st}}(G,k)$ denote the set of stable conjugacy classes of maximal k-tori in G.

If ϕ is a homomorphism from a group Γ to the group of automorphisms of some object, then we will denote the operation of taking $\phi(\Gamma)$-fixed points by $(\ \ )^{\phi(\Gamma)}$, or just by $(\ \ )^\Gamma$ when ϕ is understood.

For any k^{sep}-torus T, let $X^*(T)$ and $X_*(T)$ respectively denote the character and cocharacter modules of T. Let $\langle\ ,\ \rangle$ denote the natural bilinear pairing between $X^*(T)$ and $X_*(T)$. Let $V^*(T) = X^*(T) \otimes \mathbb{Q}$ and $V_*(T) = X_*(T) \otimes \mathbb{Q}$. Then $\langle\ ,\ \rangle$

extends to a nondegenerate pairing between the $\mathbb{Q}$-vector spaces $V^*(T)$ and $V_*(T)$. Any homomorphism $f\colon T \longrightarrow T'$ of tori determines maps $f^*\colon X^*(T') \longrightarrow X^*(T)$ and $f_*\colon X_*(T) \longrightarrow X_*(T')$, and hence maps $V^*(T') \longrightarrow V^*(T)$ and $V_*(T) \longrightarrow V_*(T')$ that we will also denote by f^* and f_*, respectively.

For $i = 1, 2$, let T_i be a maximal k-torus of G, and suppose that ${}^gT_1 = T_2$ for some $g \in G(k^{\mathrm{sep}})$. Then $\mathrm{Int}(g)$ gives an isomorphism $T_1 \to T_2$ (not necessarily defined over k). For $\chi \in X^*(T_1)$ and $\lambda \in X_*(T_1)$, define

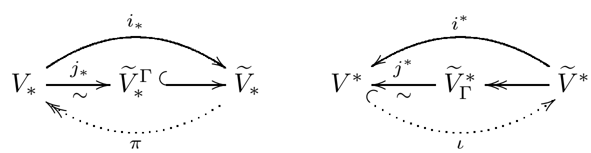
$$ {}^g\chi := \mathrm{Int}(g)^{*\,-1}\chi, \qquad {}^g\lambda := \mathrm{Int}(g)_*\lambda. $$

For a root datum $(X^*, \Phi, X_*, \Phi^\vee)$ and a root $\alpha \in \Phi$, we denote by $\alpha^\vee$ the corresponding coroot in $\Phi^\vee$.

2. Finite-group actions on character and cocharacter modules

Let $\widetilde{X}_*$ be a lattice of finite rank equipped with an action of a finite group Γ. Then Γ also acts on the dual $\widetilde{X}^*$ of $\widetilde{X}_*$, as well as on $\widetilde{V}_* = \widetilde{X}_* \otimes \mathbb{Q}$ and its dual $\widetilde{V}^* = \widetilde{X}^* \otimes \mathbb{Q}$.

Let X_* be another lattice with dual X^*, and let $V_* = X_* \otimes \mathbb{Q}$ and $V^* = X^* \otimes \mathbb{Q}$. We will denote by $\langle\ ,\ \rangle$ the natural bilinear pairings between V^* and V_*, and between $\widetilde{V}^*$ and $\widetilde{V}_*$. Suppose that $j_*\colon V_* \longrightarrow \widetilde{V}_*^\Gamma$ is an isomorphism such that $j_*(X_*) \supseteq \widetilde{X}_*^\Gamma$. Composing j_* with the natural inclusion $\widetilde{V}_*^\Gamma \longrightarrow \widetilde{V}_*$, and taking duals, we obtain maps i_*, j^*, and i^* as follows:



where ι and π are to be described below. Here $\widetilde{V}_\Gamma^*$ denotes the space of coinvariants for the action of Γ on $\widetilde{V}^*$.

Define $\pi\colon \widetilde{V}_* \longrightarrow V_*$ to be

$$ j_*^{-1} \circ \Bigl(\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma \Bigr). $$

We have that $\pi \circ i_* = \mathrm{id}$, so π is a projection of $\widetilde{V}_*$ onto V_*.

Let $\iota\colon V^* \longrightarrow \widetilde{V}^*$ be the transpose of π. More explicitly, one can show that if $\widetilde{v} \in \widetilde{V}^*$ is any preimage under i^* of v, then

$$ \iota(v) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma \cdot \widetilde{v}. \tag{2.1} $$

The image of ι is clearly the subspace $\widetilde{V}^{*\Gamma}$ of Γ-fixed vectors in $\widetilde{V}^*$. Moreover, ι is injective (since π is surjective) and respects the bilinear pairings $V^* \times V_* \longrightarrow \mathbb{Q}$ and $\widetilde{V}^* \times \widetilde{V}_* \longrightarrow \mathbb{Q}$ in the sense that for all $v \in V^*$ and $w \in V_*$, we have $\langle \iota(v), i_*(w) \rangle = \langle v, w \rangle$. Using ι (resp. j_*), we may therefore identify V^* (resp. V_*) with $\widetilde{V}^{*\Gamma}$ (resp. $\widetilde{V}_*^\Gamma$). We note that

$$ i^* = \iota^{-1} \circ \Bigl(\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma \Bigr). \tag{2.2} $$

Define a map $\mathcal{N}_*\colon \widetilde{V}_* \longrightarrow V_*$ by

$$\mathcal{N}_* = j_*^{-1} \circ \left(\sum_{\gamma\in\Gamma} \gamma\right) = |\Gamma|\, \pi. \tag{2.3}$$

Taking duals, we obtain the adjoint map $\mathcal{N}^*\colon V^* \longrightarrow \widetilde{V}^*$. Because of our assumption on j_*, we have that $\mathcal{N}_*$ takes $\widetilde{X}_*$ to X_*, and thus that $\mathcal{N}^*$ takes X^* to $\widetilde{X}^*$.

Suppose that $\widetilde{X}_*$ and X_* (and hence $\widetilde{X}^*$ and X^*) are equipped with $\mathrm{Gal}(k)$-actions. Further, suppose that the action of $\mathrm{Gal}(k)$ on $\widetilde{X}_*$ commutes with that of Γ. It follows that if j_* is equivariant with respect to these actions of $\mathrm{Gal}(k)$, then so are all of the other maps above.

Example 2.1. Let $\widetilde{T}$ be a k-torus and let Γ be a finite group that acts on $\widetilde{T}$ via k-automorphisms. Then Γ acts on both $X^*(\widetilde{T})$ and $X_*(\widetilde{T})$ (and thus on $\widetilde{V}^* = V^*(\widetilde{T})$ and $\widetilde{V}_* = V_*(\widetilde{T})$) via the rules

$$\gamma\cdot\chi = \gamma^{*\,-1}\chi, \qquad \gamma\cdot\lambda = \gamma_*\lambda$$

for $\chi \in X^*(\widetilde{T})$ and $\lambda \in X_*(\widetilde{T})$.

Let T be the k-torus $(\widetilde{T}^\Gamma)^\circ$. Then we have the inclusion map $i\colon T \longrightarrow \widetilde{T}$ and a norm map $\mathcal{N}_T = \mathcal{N}\colon \widetilde{T} \longrightarrow T$ given by

$$\mathcal{N}(t) = \prod_{\gamma\in\Gamma} \gamma(t), \tag{2.4}$$

both of which are defined over k. Let $V^* = V^*(T)$ and $V_* = V_*(T)$. The map $i\colon T \longrightarrow \widetilde{T}$ induces an inclusion $i_*\colon V_* \longrightarrow \widetilde{V}_*$, which gives rise to an isomorphism $j_*\colon V_* \longrightarrow \widetilde{V}_*{}^\Gamma$. Then the maps i_* and i^* (resp. $\mathcal{N}_*$ and $\mathcal{N}^*$) constructed from j_* in this section coincide with those induced by i and $\mathcal{N}$. They are all $\mathrm{Gal}(k)$-equivariant.

3. Finite-group actions on reductive groups

In this section, we establish a strong form of Statement (D) from the Introduction.

Definition 3.1. We say that an automorphism γ of a connected reductive algebraic group $\widetilde{G}$ is *quasi-semisimple* if γ preserves a Borel subgroup $\widetilde{B}_\bullet$ of $\widetilde{G}$ and a maximal torus $\widetilde{T}_\bullet$ in $\widetilde{B}_\bullet$.

Remark 3.2. If both $\widetilde{G}$ and γ are defined over k, then the groups $\widetilde{B}_\bullet$ and $\widetilde{T}_\bullet$ can be chosen to be defined over k^{sep}. We prove this in almost all cases in Lemma A.1. A general proof of Lemma A.1, as well as proofs of Lemmas 3.3 and 3.4, can be found in a preprint of Lemaire [11, §4.6]. Our proofs are different.

Lemma 3.3. *Suppose $\widetilde{G}$ is a connected reductive algebraic group, and γ is a quasi-semisimple automorphism of $\widetilde{G}$. Let $\widetilde{B}_\bullet$ be a γ-invariant Borel subgroup of $\widetilde{G}$, and $\widetilde{T}_\bullet \subseteq \widetilde{B}_\bullet$ a γ-invariant maximal torus of $\widetilde{G}$. Then the group $G := (\widetilde{G}^\gamma)^\circ$ is reductive, $T_\bullet := (\widetilde{T}_\bullet^\gamma)^\circ = G \cap \widetilde{T}_\bullet$ is a maximal torus in G, and $B_\bullet := (\widetilde{B}_\bullet^\gamma)^\circ$ is a Borel subgroup of G containing $T_\bullet$.*

Proof. Let $\widetilde{G}'$ be the simply connected cover of the derived group of $\widetilde{G}$, and let $\widetilde{Z}$ be the identity component of the center of $\widetilde{G}$. Then we have a central isogeny $\phi\colon \widetilde{Z} \times \widetilde{G}' \longrightarrow \widetilde{G}$. By [17, §9.16], the restriction of γ to the derived group of $\widetilde{G}$ lifts uniquely to an automorphism of $\widetilde{G}'$, and thus γ lifts uniquely to an automorphism of $\widetilde{Z} \times \widetilde{G}'$. Moreover, $\widetilde{G}'$ contains a maximal torus $\widetilde{T}'_\bullet$ and Borel subgroup $\widetilde{B}'_\bullet$ such that $\widetilde{Z} \times \widetilde{T}'_\bullet$ and $\widetilde{Z} \times \widetilde{B}'_\bullet$ are the inverse images under ϕ of $\widetilde{T}_\bullet$ and $\widetilde{B}_\bullet$. Note that γ preserves $\widetilde{T}'_\bullet$ and $\widetilde{B}'_\bullet$. From Steinberg [17, Theorem 8.2], we know that $G' := \widetilde{G}'^{\gamma}$ is a connected reductive group, and it is clear [*loc. cit.*, Remark 8.3(a)] that $T'_\bullet := \widetilde{T}'^{\gamma}_\bullet = G' \cap \widetilde{T}'_\bullet$ is a maximal torus in G' and $B'_\bullet := \widetilde{B}'^{\gamma}_\bullet$ is a Borel subgroup of G' containing $T'_\bullet$. We obtain our desired result by considering $\phi((\widetilde{Z}^\gamma)^\circ \times T'_\bullet)$ and $\phi((\widetilde{Z}^\gamma)^\circ \times B'_\bullet)$. □

Lemma 3.4. *Suppose $\widetilde{G}$ is a connected reductive k-group, and γ is a quasi-semisimple k-automorphism of $\widetilde{G}$. Then the group $G := (\widetilde{G}^\gamma)^\circ$ is defined over k.*

Proof. Let $\widetilde{T}_\bullet$ and $\widetilde{B}_\bullet$ be as in Definition 3.1. From Remark 3.2, we may assume that these groups are defined over k^{sep}. Since γ is defined over k, we have that $\mathrm{Gal}(k)$ preserves G. Therefore, it will be enough to show that G is defined over k^{sep} (see [16, Prop. 11.2.8(i)]). We may therefore assume that $k = k^{\mathrm{sep}}$.

Let $T_\bullet$ and $B_\bullet$ be as in Lemma 3.3. Consider the free part of the module of γ-coinvariants in the lattice $X^*(\widetilde{T}_\bullet)$. Since this is a lattice, and it is a quotient of $X^*(\widetilde{T}_\bullet)$, it is the character lattice of a k-subtorus of $\widetilde{T}_\bullet$, and this subtorus is precisely $T_\bullet$. That is, $T_\bullet$ is defined over k. Since $\widetilde{T}_\bullet$ is split over k, the root groups U_α for each $\alpha \in \Phi(\widetilde{G}, \widetilde{T}_\bullet)$ are defined over k, and there exist k-isomorphisms x_α from the additive group to each U_α. Moreover, if α is a root whose γ-orbit has size r, then since γ is defined over k, we may select the automorphisms $x_{\gamma^i\alpha}$ so that $\gamma^i \circ x_\alpha = x_{\gamma^i\alpha}$ for $i = 0, \ldots, r-1$. It follows from the description of the root groups for G with respect to $T_\bullet$ given, say, in part $(2'''')$ of the proof of [17, Theorem 8.2], that each such root group U_β ($\beta \in \Phi(G, T_\bullet)$) inherits a k-structure from that of $\widetilde{G}$. Thus, the product $Y = T_\bullet \prod U_\beta$ of $T_\bullet$ with all of the root groups U_β (in any order) is an open k-variety in G. As a result, $Y(k) \subset G(\bar{k}) \cap \widetilde{G}(k)$ is dense in Y by [16, Theorem 11.2.7], and hence dense in G. It follows [*loc. cit.*, Lemma 11.2.4(ii)] that G is defined over k. □

Note that if γ is not quasi-semisimple, then G need not be reductive. For example, suppose $\widetilde{G} = \mathrm{GL}(2)$, $g \in \widetilde{G}(k)$ is a nontrivial unipotent element of finite order, and $\gamma = \mathrm{Int}(g)$.

We now restrict our attention to automorphisms of finite order.

Proposition 3.5. *Suppose $\widetilde{G}$ is a connected reductive k-group, and Γ is a finite group that acts on $\widetilde{G}$ via k-automorphisms that preserve a common Borel subgroup $\widetilde{B}_\bullet$ of $\widetilde{G}$ and maximal torus $\widetilde{T}_\bullet$ in $\widetilde{B}_\bullet$. Let $G = (\widetilde{G}^\Gamma)^\circ$. Then:*

(i) *G is a reductive k-group.*
(ii) *For every Borel-torus pair $(\widetilde{B}, \widetilde{T})$ in $\widetilde{G}$ preserved by Γ, we have that $((\widetilde{B}^\Gamma)^\circ, (\widetilde{T}^\Gamma)^\circ)$ is a Borel-torus pair for G.*
(iii) *Let T be a maximal torus in G, and let $\widetilde{T} = C_{\widetilde{G}}(T)$. Then $\widetilde{T}$ is a maximal torus in $\widetilde{G}$.*

(iv) Let T and $\widetilde{T}$ be as in (iii). Then each root in $\Phi(G,T)$ is the restriction to T of a root in $\Phi(\widetilde{G},\widetilde{T})$.

(v) Let $\widetilde{T}$ be as in (iii). Then there is some Borel subgroup $\widetilde{B}$ of $\widetilde{G}$ containing $\widetilde{T}$ such that $(\widetilde{B},\widetilde{T})$ is a Borel-torus pair preserved by Γ.

Remarks 3.6.

(i) The reductivity of G is proved by Prasad and Yu [12] under somewhat different hypotheses. Rather than assuming that Γ fixes a Borel-torus pair, they assume that $|\Gamma|$ is not divisible by $\operatorname{char} k$. When $\operatorname{char} k > 0$, their hypotheses are neither stronger nor weaker than ours.

(ii) One can choose T in part (iii) to be defined over k, in which case the torus $\widetilde{T}$ is also defined over k.

Proof. As in the proof of Lemma 3.4, if G is defined over k^{sep}, then it is defined over k. Therefore, it will be enough to prove this result under the assumption that $k = k^{\mathrm{sep}}$.

Let $\widetilde{Z}$ denote the identity component of the center of $\widetilde{G}$, and $\widetilde{G}'$ the derived subgroup of $\widetilde{G}$. Then we obtain an action of Γ on $\widetilde{Z} \times \widetilde{G}'$ via k-automorphisms. Let C be the kernel of the central k-isogeny $\widetilde{Z} \times \widetilde{G}' \longrightarrow \widetilde{G}$. Then the image of $((\widetilde{Z} \times \widetilde{G}')^\Gamma)^\circ$ under the isogeny is a normal subgroup of $\widetilde{G}^\Gamma$; that its index is finite follows from the finiteness of $H^1(\Gamma, C(\bar{k}))$ (see [18, Cor. 6.5.10]). It follows that this image is precisely G. Thus, it is enough to prove the proposition with $\widetilde{Z} \times \widetilde{G}'$ in place of $\widetilde{G}$. Since the proposition is clear for $\widetilde{Z}$, we may replace $\widetilde{G}$ by $\widetilde{G}'$. That is, we may and do assume that $\widetilde{G}$ is semisimple. Moreover, we note that the action of Γ on $\widetilde{G}$ can be lifted to an action on the simply connected cover of $\widetilde{G}$ that satisfies all of the hypotheses of this theorem. It follows from reasoning similar to that above that we may further assume that $\widetilde{G}$ is simply connected. In particular, $\widetilde{G}$ is a direct product of almost-simple groups, and Γ permutes these.

Let $(\widetilde{B}_\bullet, \widetilde{T}_\bullet)$ denote a Γ-invariant Borel-torus pair in $\widetilde{G}$.

We first prove (iii) and (v). Let $T_\bullet$ be the torus $(\widetilde{T}_\bullet^\Gamma)^\circ$. We first show that $T_\bullet$ is maximal in G and $C_{\widetilde{G}}(T_\bullet) = \widetilde{T}_\bullet$. We prove the latter by showing that the root system $\Phi(C_{\widetilde{G}}(T_\bullet), \widetilde{T}_\bullet)$ is empty. Let Φ^+ denote the positive subsystem of $\Phi(C_{\widetilde{G}}(T_\bullet), \widetilde{T}_\bullet)$ determined by $\widetilde{B}_\bullet$. Suppose for a contradiction that Φ^+ is nonempty, and choose $\alpha \in \Phi^+$. Let $\chi = \sum_{\gamma\in\Gamma} \gamma \cdot \alpha$. Since Γ preserves $\widetilde{B}_\bullet$, we have that Γ preserves Φ^+, so χ is a positive linear combination of elements of Φ^+, and is thus nonzero.

The canonical pairing $\langle\ ,\ \rangle$ between $V^*(\widetilde{T}_\bullet)$ and $V_*(\widetilde{T}_\bullet)$ is invariant under the action of Γ. Thus, for all $\lambda \in X_*(T_\bullet) = X_*(\widetilde{T}_\bullet)^\Gamma$, $\langle \chi, \lambda\rangle = \sum_\gamma \langle \gamma\cdot\alpha, \lambda\rangle = |\Gamma|\langle \alpha, \lambda\rangle$, which is 0 since α is a root of the centralizer of $T_\bullet$. Since χ is clearly Γ-invariant, we may identify it with a vector $\chi_0 \in V^*(T_\bullet)$ via the map ι^{-1} as discussed in §2. (In fact, $\chi_0 = |\Gamma|\, i^*\alpha$ by (2.2).) Then $\langle \chi_0, \lambda\rangle = \langle \chi, \lambda\rangle = 0$ for all $\lambda \in X_*(T_\bullet)$. Since $\langle\ ,\ \rangle$ is nondegenerate, it follows that $\chi_0 = 0$ and hence $\chi = 0$, a contradiction.

To see that $T_\bullet$ is maximal in G, consider a maximal torus $T'_\bullet$ containing $T_\bullet$. Then

$$T'_\bullet \subseteq C_{\widetilde{G}}(T'_\bullet) \subseteq C_{\widetilde{G}}(T_\bullet) = \widetilde{T}.$$

Taking identity components of groups of Γ-fixed points, we see that $T'_\bullet \subseteq T_\bullet$, and thus the two tori are equal.

Now let T denote an arbitary maximal torus in G. We can write $T = {}^gT_\bullet$ for some $g \in G(\bar{k})$. Thus, $C_{\widetilde{G}}(T) = {}^g\widetilde{T}_\bullet$, which is a maximal torus in $\widetilde{G}$. Also, ${}^g\widetilde{B}_\bullet$ is a Γ-invariant Borel subgroup of $\widetilde{G}$ containing $\widetilde{T}$. This proves (iii) and (v).

We now prove the other statements of the theorem simultaneously by a series of reductions.

Suppose $\widetilde{G} = \widetilde{G}_1 \times \widetilde{G}_2$, a direct product of connected reductive k-groups, each preserved by Γ. By induction on the dimension of $\widetilde{G}$, we see that each statement holds for each $\widetilde{G}_i$, and thus for $\widetilde{G}$. It thus only remains to consider the case where $\widetilde{G}$ has no such decomposition. Recall that we may assume that $\widetilde{G}$ is simply connected. In particular, we may assume that $\widetilde{G} = \prod_{i=1}^r \widetilde{G}_i$, a direct product of almost simple k-groups which are permuted transitively by Γ.

Let $\Gamma_1 = \mathrm{stab}_\Gamma(\widetilde{G}_1)$. For $1 \leq i \leq r$, choose $\gamma_i \in \Gamma$ such that $\gamma_i(\widetilde{G}_1) = \widetilde{G}_i$. Let $\widetilde{H} = \prod_{i=1}^r \widetilde{G}_1$. Then we have a k-isomorphism $\vec{\gamma} = (\gamma_1, \ldots, \gamma_r)\colon \widetilde{H} \longrightarrow \widetilde{G}$. Since Γ is generated by Γ_1 and $\{\gamma_1, \ldots, \gamma_r\}$, we have that

$$\widetilde{G}^\Gamma = \vec{\gamma}\big(\mathrm{diag}(\widetilde{G}_1^{\Gamma_1})\big) = \vec{\gamma}\big((\widetilde{H})^{S_r \times \Gamma_1}\big),$$

where $\mathrm{diag}\colon \widetilde{G}_1 \longrightarrow \widetilde{H}$ is the diagonal embedding, and S_r is the symmetric group acting on $\widetilde{H}$ by permutation of coordinates. Thus, we may replace $\widetilde{G}$ by $\widetilde{H}$ and Γ by $S_r \times \Gamma_1$.

Using induction on $|\Gamma|$ and working in stages, we see that we may always replace Γ with the successive subquotients that occur in any subnormal series for Γ. Thus, we may assume that Γ is simple. Furthermore, from the previous paragraph, we may assume either Γ either acts via permutations of coordinates, or preserves the simple factors of $\widetilde{G}$ and acts in the same way on each. In the former case, the proposition is clear. Thus, we assume that we are in the latter case, and we may assume from above that $\widetilde{G}$ is almost simple.

Since Γ is simple, it either consists of inner automorphisms (in which case it embeds in an isogenous image of $\widetilde{T}_\bullet$), or it embeds in the symmetry group of the Dynkin diagram of $\widetilde{G}$. In each case, Γ is solvable, and thus cyclic. Thus, (i) and (ii) follow from Lemma 3.3 and Lemma 3.4, and (iv) follows from the description of the root groups of G in [17, §8.2(2′′′′)]. □

Definition 3.7. If $\widetilde{G}$ and Γ are as in Proposition 3.5, we say the action of Γ on $\widetilde{G}$ is *quasi-semisimple*.

4. Parascopy: Definition

We now axiomatize the essential properties of the relationship between the group G and the pair $(\widetilde{G}, \Gamma)$ of §3.

Definition 4.1. Let $\Psi = (X^*, \Phi, X_*, \Phi^\vee)$ and $\widetilde{\Psi} = (\widetilde{X}^*, \widetilde{\Phi}, \widetilde{X}_*, \widetilde{\Phi}^\vee)$ be root data with $\mathrm{Gal}(k)$-actions, and Γ a finite group. Let $V^* = X^* \otimes \mathbb{Q}$, $V_* = X_* \otimes \mathbb{Q}$, $\widetilde{V}^* = \widetilde{X}^* \otimes \mathbb{Q}$, $\widetilde{V}_* = \widetilde{X}_* \otimes \mathbb{Q}$. A *parascopic datum* for the triple $(\widetilde{\Psi}, \Gamma, \Psi)$ is a pair (ϕ, j_*), where

- ϕ is a homomorphism from Γ to $\mathrm{Aut}(\widetilde{\Psi})$, such that $\phi(\Gamma)$ commutes with the action of $\mathrm{Gal}(k)$ and preserves some system of positive roots in $\widetilde{\Phi}$; and
- $j_*\colon V_* \longrightarrow \widetilde{V}_*^{\phi(\Gamma)}$ is a $\mathrm{Gal}(k)$-equivariant isomorphism

satisfying the following two conditions.

P1. $j_*(X_*) \supseteq \widetilde{X}_*^{\phi(\Gamma)}$.

Composing j_* with the inclusion map $\widetilde{V}_*^{\phi(\Gamma)} \longrightarrow \widetilde{V}_*$, we obtain a map $i_*\colon V_* \longrightarrow \widetilde{V}_*$ whose transpose $i^*\colon \widetilde{V}^* \longrightarrow V^*$ is assumed to satisfy:

P2. $i^*\big(\widetilde{\Phi}\big) \supseteq \Phi$.

Let G and $\widetilde{G}$ be connected reductive k-groups, and Γ a finite group. Let T (resp. $\widetilde{T}$) be a maximal k-torus of G (resp. $\widetilde{G}$). Let $\Psi(G,T)$ (resp. $\Psi(\widetilde{G},\widetilde{T})$) be the root datum of G (resp. $\widetilde{G}$) relative to T (resp. $\widetilde{T}$). These root data come equipped with an action of $\mathrm{Gal}(k)$. We will refer to a parascopic datum (ϕ, j_*) for the triple $(\Psi(\widetilde{G},\widetilde{T}),\Gamma,\Psi(G,T))$ as a *parascopic datum for* $(\widetilde{G},\Gamma,G)$ *relative to the tori* $\widetilde{T}\subseteq\widetilde{G}$ *and* $T \subseteq G$. We will say that G is a *weakly parascopic group* for the pair $(\widetilde{G},\Gamma)$ if such a parascopic datum exists, and we will feel free not to specify a particular datum if it is clear from the context.

Examples 4.2.

(a) Suppose Γ acts quasi-semisimply on $\widetilde{G}$, and $G = (\widetilde{G}^\Gamma)^\circ$, as in §3. From Proposition 3.5 and Remark 3.6(ii), we can choose maximal k-tori $T \subseteq G$ and $\widetilde{T} \subseteq \widetilde{G}$ such that Γ preserves $\widetilde{T}$, and $T = (\widetilde{T}^\Gamma)^\circ$. Then the given action $\phi\colon \Gamma \longrightarrow \mathrm{Aut}_k(\widetilde{T})$, together with the map $j_*\colon V_*(T) \longrightarrow V_*(\widetilde{T})$ induced by the inclusion $T \longrightarrow \widetilde{T}$, form a parascopic datum.

(b) If G is a Levi subgroup of $\widetilde{G}$, then G is weakly parascopic for $(\widetilde{G},1)$ with respect to an obvious parascopic datum.

(c) If G is the image under a central k-isogeny of a weakly parascopic group for $(\widetilde{G},\Gamma)$, then G is itself weakly parascopic for $(\widetilde{G},\Gamma)$.

(d) Our definition does not refer to any action of Γ on $\widetilde{G}$, but if $\widetilde{G}$ is k-quasisplit then one can indeed lift ϕ to a map $\Gamma \longrightarrow \mathrm{Aut}_k(\widetilde{G})$. Different choices of lifting can lead to groups of fixed points $\widetilde{G}^{\phi(\Gamma)}$ whose identity components are non-isomorphic. We will show elsewhere [2] that a particular lifting $\phi_0\colon \Gamma \longrightarrow \mathrm{Aut}_k(\widetilde{G})$, one that fixes a pinning, has the following property. If ϕ denotes any lifting, then $(G^{\phi(\Gamma)})^\circ$ is weakly parascopic for $((G^{\phi_0(\Gamma)})^\circ, 1)$ in a natural way. For example, considering two actions of $\mathbb{Z}/2\mathbb{Z}$ on $\mathrm{GL}(2n)$, we will see that $\mathrm{SO}(2n)$ is parascopic for $(\mathrm{Sp}(2n),1)$.

In §7, we will define and study a natural notion of equivalence between parascopic data.

5. Finite-group actions and Weyl groups

Suppose that $\widetilde{\Psi} = (\widetilde{X}^*, \widetilde{\Phi}, \widetilde{X}_*, \widetilde{\Phi}^\vee)$ and $\Psi = (X^*, \Phi, X_*, \Phi^\vee)$ are root data with $\mathrm{Gal}(k)$-action, and Γ is a finite group. Let (ϕ, j_*) be a parascopic datum for $(\widetilde{\Psi},\Gamma,\Psi)$. As indicated in §1, to ease notation, we will suppress reference to ϕ when considering the action of Γ. From now on, use the map j_* to to identify V_* with $\widetilde{V}_*^\Gamma$, and the map ι of §2 to identify V^* with $(\widetilde{V}^*)^\Gamma$. Since Γ preserves the root system $\widetilde{\Phi}$, it acts on the Weyl group $\widetilde{W}$ of $\widetilde{\Psi}$.

In the particular situation where Γ acts on $\widetilde{G}$ and $G = (\widetilde{G}^\Gamma)^\circ$, there is an obvious embedding $W(G,T) \longrightarrow W(\widetilde{G},\widetilde{T})^\Gamma$. Here $w \in W(G,T)$ corresponds to the unique element of $W(\widetilde{G},\widetilde{T})^\Gamma$ whose action when restricted to T coincides with that of

w. We now show that we have such an embedding in the more general setting of parascopy.

Let α be a root in Φ. Then $\alpha = i^*\widetilde{\alpha}$ for some root $\widetilde{\alpha}$ in $\widetilde{\Phi}$ by Condition P2 in the definition of parascopic datum. There are two cases to consider:

(1) The roots in $\Gamma \cdot \widetilde{\alpha}$ are mutually orthogonal.
(2) The roots in $\Gamma \cdot \widetilde{\alpha}$ are not mutually orthogonal.

In case (1), let $\Xi = \Gamma \cdot \widetilde{\alpha}$.

In case (2), [10, §1.3] implies that for each $\theta \in \Gamma \cdot \widetilde{\alpha}$, there exists a unique root $\theta' \neq \theta$ in $\Gamma \cdot \widetilde{\alpha}$ such that θ and θ' are not orthogonal. Moreover, $\theta + \theta'$ is a root in Φ and does not belong to $\Gamma \cdot \widetilde{\alpha}$. Let $\Xi = \{\theta + \theta' \mid \theta \in \Gamma \cdot \widetilde{\alpha}\}$.

Remark 5.1. Although it is assumed in [10] that Γ is cyclic, there is only one case in which the action of the stabilizer in a general group Γ of an irreducible component of Φ need not factor through a cyclic quotient (recall that Γ must preserve a positive system of roots): namely, when the component is of type D_4. One easily checks that in this situation, case (1) holds.

Remark 5.2. We note that in both cases, Ξ is an orbit of mutually orthogonal roots.

Lemma 5.3. *Suppose that $\alpha \in \Phi$. Then with $\widetilde{\alpha}$ and Ξ as above, we have*

$$\sum_{\beta \in \Xi} \beta^\vee = \frac{|\Xi|}{|\Gamma \cdot \widetilde{\alpha}|} \alpha^\vee.$$

Proof. By (2.1), we have

$$\sum_{\beta \in \Xi} \beta = \sum_{\beta \in \Gamma \cdot \widetilde{\alpha}} \beta = |\Gamma \cdot \widetilde{\alpha}| \alpha. \tag{5.1}$$

Since $\sum_\Xi \beta$ is a multiple of α, and the roots in Ξ all have the same length, it follows that $\sum_\Xi \beta^\vee$ is a multiple of $\alpha^\vee$. To determine this multiple we let $\beta_0 \in \Xi$, and compute

$$\begin{aligned}
\Big\langle \alpha, \sum_{\beta \in \Xi} \beta^\vee \Big\rangle &= \frac{1}{|\Gamma \cdot \widetilde{\alpha}|} \Big\langle \sum_{\beta' \in \Xi} \beta', \sum_{\beta \in \Xi} \beta^\vee \Big\rangle && \text{(by (5.1))} \\
&= \frac{|\Xi|}{|\Gamma \cdot \widetilde{\alpha}|} \langle \beta_0, \beta_0^\vee \rangle && \text{(by Remark 5.2)} \\
&= \frac{|\Xi|}{|\Gamma \cdot \widetilde{\alpha}|} \langle \alpha, \alpha^\vee \rangle .
\end{aligned}$$

Our result follows. □

Lemma 5.4. *The natural action of $\widetilde{W}^\Gamma$ on $(\widetilde{V}^*)^\Gamma$ is faithful.*

When Γ is cyclic, this is [17, §1.32(a)].

Proof. Let w be a nontrivial element of $\widetilde{W}^\Gamma$. Let $\widetilde{\Delta}$ be a Γ-invariant set of simple roots in $\widetilde{\Phi}$ (guaranteed to exist by Definition 4.1). Then there exists $\widetilde{\alpha} \in \widetilde{\Delta}$ such that $w(\widetilde{\alpha}) \in -\widetilde{\Delta}$. It follows that $w(\gamma \cdot \widetilde{\alpha}) = \gamma \cdot (w\widetilde{\alpha})$ belongs to $-\widetilde{\Delta}$ for every $\gamma \in \Gamma$. Let $v = \sum \gamma \cdot \widetilde{\alpha} \in \widetilde{V}^{*\Gamma}$, where the sum runs over all $\gamma \in \Gamma$. Then $w(v)$ is a linear combination of roots in $\widetilde{\Delta}$ in which all of the coefficients are nonpositive. In particular, $w(v) \neq v$. □

Let W denote the Weyl group of Ψ.

Proposition 5.5. *There is a natural* $\mathrm{Gal}(k)$*-equivariant embedding* $W \longrightarrow \widetilde{W}^{\Gamma}$*. Under this map, the image of the reflection* w_α *through the root* $\alpha \in \Phi$ *is*

$$\widetilde{w} = \prod_{\beta\in\Xi} w_\beta, \tag{5.2}$$

where Ξ *is as above and the product is taken in any order.*

Proof. We may identify $\mathrm{Aut}_{\mathbb{Q}}(V^*)$ with $\mathrm{Aut}_{\mathbb{Q}}(\widetilde{V}^{*\Gamma})$, and thus identify W with a subgroup of the latter. By Lemma 5.4, there is a natural injection

$$\widetilde{W}^{\Gamma} \longrightarrow \mathrm{Aut}_{\mathbb{Q}}(\widetilde{V}^{*\Gamma}).$$

To construct an embedding $W \longrightarrow \widetilde{W}^{\Gamma}$, it is therefore enough to show that the image of this injection contains W. Thus, given $w \in W$, we will show that there exists $\widetilde{w} \in \widetilde{W}^{\Gamma}$ whose action on $\widetilde{V}^{*\Gamma}$ coincides with that of w. It suffices to prove the existence of $\widetilde{w}$ only in the case in which w is a reflection w_α through a root $\alpha \in \Phi$. In this case, our candidate for $\widetilde{w}$ is given by (5.2).

Let $v \in V^* = \widetilde{V}^{*\Gamma}$. Since the roots in the orbit Ξ are orthogonal, we have

$$\widetilde{w}(v) = v - \sum_{\beta\in\Xi} \langle v, \beta^\vee\rangle \beta. \tag{5.3}$$

Since the pairing $\langle\ ,\ \rangle$ is Γ-invariant and v is Γ-fixed, the right-hand side of (5.3) is equal to

$$v - \langle v, \beta_0^\vee\rangle \sum_{\beta\in\Xi} \beta. \tag{5.4}$$

for any root $\beta_0 \in \Xi$. From (5.1),

$$\begin{aligned}\langle v, \beta_0^\vee\rangle \sum_{\beta\in\Xi} \beta &= |\Gamma\cdot\widetilde{\alpha}|\langle v, \beta_0^\vee\rangle\alpha \\ &= \frac{|\Gamma\cdot\widetilde{\alpha}|}{|\Xi|}\Big\langle v, \sum_{\beta\in\Xi}\beta^\vee\Big\rangle\alpha \\ &= \langle v, \alpha^\vee\rangle\alpha \qquad\qquad \text{(by Lemma 5.3).}\end{aligned}$$

Hence by (5.3) and (5.4), we have that $\widetilde{w}(v) = w_\alpha(v)$. It follows that $\widetilde{w}_\alpha = \widetilde{w}$.

The $\mathrm{Gal}(k)$-equivariance of this embedding follows from the explicit formula (5.2). □

6. Stable conjugacy classes of maximal k-tori

The statement and proof of the next result are essentially the same as those for [14, Proposition 6.1]. We include a proof here only because our hypotheses and conclusion are slightly different. Similar proofs can be found in [8, §2] and [13].

Proposition 6.1. *Let* G *denote a connected reductive group over a field* k*. Let* S *be a maximal* k*-torus of* G*. Then there is a natural injection* $\mathcal{T}_{\mathrm{st}}(G,k) \longrightarrow H^1(k, W(G,S))$*. This map is a surjection when* G *is* k*-quasisplit.*

Proof. Let T be a maximal k-torus of G. Then there exists $g \in G(k^{\mathrm{sep}})$ such that ${}^gS = T$. Let $f \in Z^1(k, N_G(S))$ be the the cocycle $\sigma \mapsto g^{-1}\sigma(g)$, and let $\bar{f}$ be the image of f in $Z^1(k, W(G,S))$. Associate to T the class of $\bar{f}$ in $H^1(k, W(G,S))$. This class is independent of the particular element g. The proof that this cohomology class uniquely determines and is determined by the stable conjugacy class of T is a standard exercise (cf. [13, §1.1]). Thus, we have the desired injection.

Suppose that G is k-quasisplit. Then G contains a maximal k-torus T_0 that is the centralizer of a maximal k-split torus. Applying the above paragraph to the case of $S = T_0$, we obtain an injection $\mathcal{T}_{\mathrm{st}}(G,k) \longrightarrow H^1(k, W(G,T_0))$, and this map is a surjection from [13, Thm. 1.1]. (Although this surjectivity result is stated only for semisimple groups, it is easily extended to the reductive case.)

Now suppose S is an arbitrary maximal k-torus of G. There is some element $h \in G(k^{\mathrm{sep}})$ such that ${}^hT_0 = S$. The map $\mathrm{Int}(h)$ then induces a natural bijection (of sets, though not of pointed sets) $H^1(k, W(G,T_0)) \longrightarrow H^1(k, W(G,S))$ (cf. [15, Ch I, Prop. 35]); one sees easily that this bijection does not depend on the choice of h. Composing the map from the preceding paragraph with this bijection yields the desired bijection. □

7. Parascopy: Basic properties

We are now equipped to state and prove the basic properties of parascopy. In this section, let $\widetilde{G}$ and G denote connected reductive k-groups, Γ a finite group, and $\widetilde{T}$ and T maximal k-tori in $\widetilde{G}$ and G. Suppose that (ϕ, j_*) is a parascopic datum for $(\widetilde{G}, \Gamma, G)$ relative to $\widetilde{T}$ and T. Thus, G is weakly parascopic for $(\widetilde{G}, \Gamma)$.

Definition 7.1. Identify $\mathcal{T}_{\mathrm{st}}(G,k)$ and $\mathcal{T}_{\mathrm{st}}(\widetilde{G},k)$ with subsets of $H^1(k, W(G,T))$ and $H^1(k, W(\widetilde{G}, \widetilde{T}))$, respectively, via the injection of Proposition 6.1. Consider the map $H^1(k, W(G,T)) \longrightarrow H^1(k, W(\widetilde{G}, \widetilde{T}))$ induced by the embedding given in Proposition 5.5. If this map restricts to give a map $\mathcal{T}_{\mathrm{st}}(G,k) \longrightarrow \mathcal{T}_{\mathrm{st}}(\widetilde{G},k)$, then we will say that G is a *parascopic group* for $(\widetilde{G}, \Gamma)$.

In other words, a weakly parascopic group G is parascopic if every maximal k-torus in G determines a unique one in $\widetilde{G}$, up to stable conjugacy.

Examples 7.2.

(a) If G is weakly parascopic for $(\widetilde{G}, \Gamma)$, and $\widetilde{G}$ is quasisplit over k, then G is parascopic by the surjectivity statement in Proposition 6.1.
(b) In the situation of Proposition 3.5, G is parascopic by part (iii) of this result.
(c) In Examples 4.2(a,b), G is parascopic for $\widetilde{G}$.

Definition 7.3. Let (ϕ', j'_*) denote another parascopic datum for $(\widetilde{G}, \Gamma, G)$, this time relative to the maximal k-tori $T' \subseteq G$ and $\widetilde{T}' \subseteq \widetilde{G}'$. We say that the parascopic data (ϕ, j_*) and (ϕ', j'_*) are *equivalent* if there exist elements $g \in G(k^{\mathrm{sep}})$ and $\widetilde{g} \in \widetilde{G}(k^{\mathrm{sep}})$ satisfying:

(a) ${}^gT = T'$ and ${}^{\widetilde{g}}\widetilde{T} = \widetilde{T}'$.
(b) For all $\gamma \in \Gamma$, $\phi'(\gamma) = \mathrm{Int}(\widetilde{g})_* \circ \phi(\gamma) \circ \mathrm{Int}(\widetilde{g})_*^{-1}$, where $\phi(\gamma)$ and $\phi'(\gamma)$ are taken to be automorphisms of $X_*(\widetilde{T})$ and $X_*(\widetilde{T}')$.
(c) $j'_* = \mathrm{Int}(\widetilde{g})_* \circ j_* \circ \mathrm{Int}(g)_*^{-1}$.

In this case, we will say that (ϕ, j_*) is equivalent to (ϕ', j'_*) *via the elements g and $\widetilde{g}$.*

It is straightforward to verify that this is indeed an equivalence relation on the set of parascopic data for $(\widetilde{G}, \Gamma, G)$.

Let g and $\widetilde{g}$ be as in Definition 7.3. For $\sigma \in \mathrm{Gal}(k)$, we have that $g^{-1}\sigma(g) \in N_G(T)(k^{\mathrm{sep}})$ and $\widetilde{g}^{-1}\sigma(\widetilde{g}) \in N_{\widetilde{G}}(\widetilde{T})(k^{\mathrm{sep}})$. In the following lemma and the remainder of the paper, we will use bars to represent the natural maps from normalizers to Weyl groups of maximal tori.

Lemma 7.4. *Let (ϕ, j_*) be a parascopic datum for $(\widetilde{G}, \Gamma, G)$ relative to $\widetilde{T}$ and T. Let $g \in G(k^{\mathrm{sep}})$ and $\widetilde{g} \in \widetilde{G}(k^{\mathrm{sep}})$, and suppose that the maximal tori $T' := {}^gT$ and $\widetilde{T}' := {}^{\widetilde{g}}\widetilde{T}$ are defined over k. Define $\phi' \colon \Gamma \longrightarrow \mathrm{Aut}(X_*(\widetilde{T}'))$ and $j'_* \colon V_*(T') \longrightarrow V_*(\widetilde{T}')$ as in Definition 7.3(b,c). Then (ϕ', j'_*) is a parascopic datum for $(\widetilde{G}, \Gamma, G)$ relative to $\widetilde{T}'$ and T' (necessarily equivalent to (ϕ, j_*)) if and only if*

$$i\big(\overline{g^{-1}\sigma(g)}\big) = \overline{\widetilde{g}^{-1}\sigma(\widetilde{g})}$$

for all $\sigma \in \mathrm{Gal}(k)$, where $i \colon W(G,T) \longrightarrow W(\widetilde{G}, \widetilde{T})^{\phi(\Gamma)}$ is the embedding given in Proposition 5.5.

Proof. Suppose (ϕ', j'_*) is a parascopic datum for $(\widetilde{G}, \Gamma, G)$. Fix $\sigma \in \mathrm{Gal}(k)$, and let $w = \overline{g^{-1}\sigma(g)} \in W(G,T)$ and $\widetilde{w} = \overline{\widetilde{g}^{-1}\sigma(\widetilde{g})} \in W(\widetilde{G}, \widetilde{T})$. Let $\gamma \in \Gamma$. Since $\phi(\gamma)$ and $\phi'(\gamma)$ are $\mathrm{Gal}(k)$-equivariant, we have

$$\mathrm{Int}(\widetilde{g})_* \circ \phi(\gamma) \circ \mathrm{Int}(\widetilde{g})_*^{-1} = \phi'(\gamma) = \mathrm{Int}(\sigma(\widetilde{g}))_* \circ \phi(\gamma) \circ \mathrm{Int}(\sigma(\widetilde{g}))_*^{-1}.$$

It follows that $\mathrm{Int}(\widetilde{g}^{-1}\sigma(\widetilde{g}))_*$ is $\phi(\Gamma)$-equivariant and hence that $\widetilde{w} \in W(\widetilde{G}, \widetilde{T})^{\phi(\Gamma)}$. By Lemma 5.4, to show that $i(w) = \widetilde{w}$, it suffices to show that the actions of w on $V_*(T)$ and of $\widetilde{w}$ on $V_*(\widetilde{T})$ correspond under the identification $j_* \colon V_*(T) \longrightarrow V_*(\widetilde{T})^{\phi(\Gamma)}$, i.e., that

$$j_* \circ w = \widetilde{w} \circ j_* \tag{7.1}$$

as maps $V_*(T) \longrightarrow V_*(\widetilde{T})^{\phi(\Gamma)}$. But since j_* and j'_* are $\mathrm{Gal}(k)$-equivariant, we have

$$\mathrm{Int}(\widetilde{g})_* \circ j_* \circ \mathrm{Int}(g)_*^{-1} = j'_* = \mathrm{Int}(\sigma(\widetilde{g}))_* \circ j_* \circ \mathrm{Int}(\sigma(g))_*^{-1},$$

and (7.1) follows immediately.

The converse is proved similarly. □

The next result addresses the question: how large is the equivalence class of the parascopic datum (ϕ, j_*)?

Proposition 7.5. *(i) If G is parascopic for $(\widetilde{G}, \Gamma)$ via the datum (ϕ, j_*), then for every maximal k-torus $T' \subseteq G$, and every element $g \in G(k^{\mathrm{sep}})$ such that $T' = {}^gT$, there exists a maximal k-torus $\widetilde{T}' \subseteq \widetilde{G}'$ and a parascopic datum (ϕ', j'_*) for $(\widetilde{G}, \Gamma, G)$ relative to $\widetilde{T}'$ and T', such that (ϕ', j'_*) is equivalent to (ϕ, j_*) via g and some $\widetilde{g} \in \widetilde{G}(k^{\mathrm{sep}})$.*

(ii) For a maximal k-torus $\widetilde{T}' \subseteq \widetilde{G}$, there exists a parascopic datum equivalent to (ϕ, j_*) relative to $\widetilde{T}'$ and T if and only if $\widetilde{T}'$ is stably conjugate to $\widetilde{T}$ in $\widetilde{G}$.

(iii) Suppose (ϕ', j'_*) is another parascopic datum for $(\widetilde{G}, \Gamma, G)$ relative to $\widetilde{T}$ and T. Then the two data are equivalent if and only if there is an element $\widetilde{w} \in W(\widetilde{G}, \widetilde{T})^{\mathrm{Gal}(k)}$ such that $\phi'(\gamma) = \widetilde{w} \circ \phi(\gamma) \circ \widetilde{w}^{-1}$ for all $\gamma \in \Gamma$, and $j'_* = \widetilde{w} \circ j_*$.

Proof. (i) Since G is parascopic for $(\widetilde{G}, \Gamma)$, we have a map $\mathcal{T}_{\mathrm{st}}(G,k) \longrightarrow \mathcal{T}_{\mathrm{st}}(\widetilde{G},k)$ induced by the map $H^1(k, W(G,T)) \longrightarrow H^1(k, W(\widetilde{G}, \widetilde{T})^{\phi(\Gamma)})$. Let $\widetilde{T}'$ be a torus in the image of the class of T'. Pick $\widetilde{g} \in \widetilde{G}(k^{\mathrm{sep}})$ such that ${}^{\widetilde{g}}\widetilde{T} = \widetilde{T}'$. The function f (resp. $\widetilde{f}$) on $\mathrm{Gal}(k)$ given by $f \colon \sigma \longmapsto \overline{g^{-1}\sigma(g)}$ (resp. $\widetilde{f} \colon \sigma \longmapsto$

$\overline{\widetilde{g}^{-1}\sigma(\widetilde{g})})$ is a cocycle in $Z^1(k, W(G,T))$ (resp. $Z^1(k, W(\widetilde{G},\widetilde{T})^{\phi(\Gamma)})$). By the definition of the map $\mathcal{T}_{\mathrm{st}}(G,k) \longrightarrow \mathcal{T}_{\mathrm{st}}(\widetilde{G},k)$, $\widetilde{f}$ is cohomologous to the image of f in $Z^1(k, W(\widetilde{G},\widetilde{T}))$. Moreover, by adjusting $\widetilde{g}$ by an appropriate element of $N_{\widetilde{G}}(\widetilde{T})(k^{\mathrm{sep}})$, one can arrange for these cocycles to coincide. In other words, $i\big(\overline{g^{-1}\sigma(g)}\big) = \overline{\widetilde{g}^{-1}\sigma(\widetilde{g})}$ for all $\sigma \in \mathrm{Gal}(k)$. But then Lemma 7.4 implies that the pair (ϕ', j'_*) is a parascopic datum for $(\widetilde{G},\Gamma,G)$ relative to $\widetilde{T}'$ and T', and that it is equivalent to (ϕ, j_*).

(ii) Suppose such a datum exists, equivalent to (ϕ, j_*) via $g \in N_G(T)(k^{\mathrm{sep}})$ and $\widetilde{g} \in \widetilde{G}(k^{\mathrm{sep}})$. We may choose $\widetilde{n} \in N_{\widetilde{G}}(\widetilde{T})(k^{\mathrm{sep}})$ such that $\overline{\widetilde{n}} = i(\bar{g})^{-1}$. Since $i\big(\overline{g^{-1}\sigma(g)}\big) = \overline{\widetilde{g}^{-1}\sigma(\widetilde{g})}$ by Lemma 7.4, it follows that $\overline{(\widetilde{g}\widetilde{n})^{-1}\sigma(\widetilde{g}\widetilde{n})} = 1$ in $W(\widetilde{G},\widetilde{T})$ for all $\sigma \in \mathrm{Gal}(k)$. Thus the map $\mathrm{Int}(\widetilde{g}\widetilde{n})\colon \widetilde{T} \longrightarrow \widetilde{T}'$ is defined over k, and so $\widetilde{T}$ and $\widetilde{T}'$ are stably conjugate. The converse is proved similarly.

(iii) Suppose that the equivalence is via the elements $g \in G(k^{\mathrm{sep}})$ and $\widetilde{g} \in \widetilde{G}(k^{\mathrm{sep}})$. Then g and $\widetilde{g}$ normalize T and $\widetilde{T}$, so we have elements $w = \overline{g}$ and $\widetilde{w}' = \overline{\widetilde{g}}$ in $W(G,T)$ and $W(\widetilde{G},\widetilde{T})$. Then Lemma 7.4 implies that $\sigma(\widetilde{w}' \cdot i(w)^{-1}) = \widetilde{w}' \cdot i(w)^{-1}$ for all $\sigma \in \mathrm{Gal}(k)$. Thus $\widetilde{w} := \widetilde{w}' \cdot i(w)^{-1} \in W(\widetilde{G},\widetilde{T})^{\mathrm{Gal}(k)}$. Moreover, since $i(w) \in W(\widetilde{G},\widetilde{T})^{\phi(\Gamma)}$, it follows that for any $\gamma \in \Gamma$,

$$\phi'(\gamma) = \widetilde{w}' \circ \phi(\gamma) \circ \widetilde{w}'^{-1} = \widetilde{w} \circ \phi(\gamma) \circ \widetilde{w}^{-1}.$$

By the definition of the embedding i,

$$j'_* = \widetilde{w}' \circ j_* \circ w^{-1} = (\widetilde{w}' \cdot i(w)^{-1}) \circ j_* = \widetilde{w} \circ j_*.$$

The converse is proved similarly. □

8. Duality

Let G be a quasisplit connected reductive k-group. Let B_0 denote a Borel k-subgroup of G, and T_0 a maximal k-torus in B_0. Suppose that (G^*, B_0^*, T_0^*) is another triple of such groups. We say that the two triples are in *k-duality* if there is a $\mathrm{Gal}(k)$-equivariant isomorphism $\delta_0\colon X^*(T_0) \longrightarrow X_*(T_0^*)$ that induces an isomorphism from the based root datum of (G, B_0, T_0) to the dual of that of (G^*, B_0^*, T_0^*); that is,

- δ_0 maps the simple roots in $\Phi(G,T_0)$ with respect to B_0 onto the simple coroots in $\Phi^\vee(G^*,T_0^*)$ with respect to B_0^*.
- The transpose $\delta_0^*\colon X^*(T_0^*) \longrightarrow X_*(T_0)$ of δ_0 maps the simple roots in $\Phi(G^*,T_0^*)$ with respect to B_0^* onto the simple coroots in $\Phi^\vee(G,T_0)$ with respect to B_0.

Given a triple (G, B_0, T_0), such a triple (G^*, B_0^*, T_0^*) always exists, and is unique up to k-isomorphism. In this situation, we will say that G^* is the *k-dual* of G. We will say that a pair of maximal k-tori $T \subseteq G$ and $T^* \subseteq G^*$ are in *k-duality* if there is a $\mathrm{Gal}(k)$-equivariant isomorphism $\delta\colon X^*(T) \longrightarrow X_*(T^*)$ such that

$$\delta(\Phi(G,T)) = \Phi^\vee(G^*,T^*) \quad \text{and} \quad \delta^*(\Phi(G^*,T^*)) = \Phi^\vee(G,T), \tag{8.1}$$

where $\delta^*\colon X^*(T^*) \longrightarrow X_*(T)$ is the transpose of δ. (Note that this notion of duality of tori depends on the ambient groups G and G^*.) The isomorphism δ will be referred to as a *duality map*.

Remark 8.1. Note that a duality map δ determines a Gal(k)-equivariant isomorphism, also denoted δ, from $W(G,T)$ to $W(G^*,T^*)$ under which, for each $\alpha \in \Phi(G,T)$, the reflection w_α is sent to $w_{\delta\alpha}$. Then for every $w \in W(G,T)$ and every $\chi \in X^*(T)$, ${}^w\chi = {}^{\delta(w)}\delta(\chi)$. In other words, if we identify $W(G,T)$ and $W(G^*,T^*)$ via $w_\alpha \longleftrightarrow w_{\delta\alpha}$, then $\delta\colon X^*(T) \longrightarrow X_*(T^*)$ is a $W(G,T)$-equivariant map.

We note that when k is a finite field, it is standard (see [5–7]) to work with an *anti-action* of $W(G,T)$ on $X^*(T)$ (satisfying ${}^{w_1w_2}\chi = {}^{w_2}({}^{w_1}\chi)$). This, in turn, makes the natural map $W(G,T) \longrightarrow W(G^*,T^*)$ an *anti-isomorphism* and forces the geometric Frobenius element F to act on the Weyl group $W(G^*,T^*)$ via the *inverse* of its usual action on $W(G,T)$. However, we consider the standard action of $W(G,T)$ on $X^*(T)$, which results in the isomorphism δ of Weyl groups in the preceding paragraph. Using this map to identify $W(G,T)$ and $W(G^*,T^*)$, one sees easily that Gal(k) acts in the same way on these groups.

Proposition 8.2. *There is a canonical one-to-one correspondence $\mathcal{T}_{\mathrm{st}}(G,k) \longleftrightarrow \mathcal{T}_{\mathrm{st}}(G^*,k)$. If $T \subseteq G$ and $T^* \subseteq G^*$ correspond, then they are in k-duality, and the duality map $\delta\colon X^*(T) \longrightarrow X_*(T^*)$ is uniquely determined up to the action of $W(G,T)^{\mathrm{Gal}(k)}$.*

Proof. Fix maximal k-tori $T_0 \subseteq G$ and $T_0^* \subseteq G^*$ and a duality map $\delta_0\colon X^*(T_0) \longrightarrow X_*(T_0^*)$ of based root data as in the definition of k-duality above. As described in Proposition 6.1, we have a bijection between the set of stable conjugacy classes of maximal k-tori of G (resp. G^*) and $H^1(k, W(G,T_0))$ (resp. $H^1(k, W(G^*,T_0^*))$). Since $\delta_0\colon W(G,T_0) \xrightarrow{\sim} W(G^*,T_0^*)$ induces an isomorphism $H^1(k,W(G,T_0)) \xrightarrow{\sim} H^1(k,W(G^*,T_0^*))$ we have the desired correspondence between the sets of stable conjugacy classes of maximal k-tori in G and G^*.

Let $T \subseteq G$ and $T^* \subseteq G^*$ be maximal k-tori whose stable conjugacy classes correspond as in the preceding paragraph. Then $T = {}^gT_0$ and $T^* = {}^{g^*}T_0^*$ for some $g \in G(k^{\mathrm{sep}})$ and $g^* \in G^*(k^{\mathrm{sep}})$. The function f (resp. f^*) on Gal(k) given by $f\colon \sigma \longmapsto \overline{g^{-1}\sigma(g)}$ (resp. $f^*\colon \sigma \longmapsto \overline{\widetilde{g^*}^{-1}\sigma(\widetilde{g}^*)}$) is a cocycle in $Z^1(k,W(G,T))$ (resp. $Z^1(k,W(G^*,T^*))$). Since T and T^* correspond as above, the image of f in $Z^1(k,W(G^*,T^*))$ under the isomorphism induced by δ_0 is cohomologous to f^*. Moreover, by adjusting g^* by an appropriate element of $N_{G^*}(T_0^*)(k^{\mathrm{sep}})$, one can arrange for these cocycles to coincide.

To show that T and T^* are in k-duality, define an isomorphism $\delta\colon X^*(T) \longrightarrow X_*(T^*)$ as follows. An element of $X^*(T)$ can be written in the form ${}^g\chi$ for a unique $\chi \in X^*(T_0)$. Let

$$\delta({}^g\chi) = {}^{g^*}(\delta_0\chi). \tag{8.2}$$

It is easily verified that δ satisfies (8.1).

We need to show that δ is Gal(k)-equivariant. Suppose $\sigma \in \mathrm{Gal}(k)$. Then

$$\delta(\sigma({}^g\chi)) = \delta\big({}^{\sigma(g)}(\sigma\chi)\big) = \delta\big({}^{gg^{-1}\sigma(g)}(\sigma\chi)\big) = {}^{g^*}\big(\delta_0({}^{g^{-1}\sigma(g)}(\sigma\chi))\big) = {}^{g^*}\big(\delta_0({}^{f(\sigma)}(\sigma\chi))\big).$$

On the other hand,

$$\sigma(\delta({}^g\chi)) = \sigma({}^{g^*}(\delta_0\chi)) = {}^{\sigma(g^*)}(\sigma(\delta_0\chi)) = {}^{\sigma(g^*)}(\delta_0(\sigma\chi)).$$

It follows that δ will be Gal(k)-equivariant if and only if

$$\delta_0({}^{f(\sigma)}(\sigma\chi)) = {}^{g^{*-1}\sigma(g^*)}(\delta_0(\sigma\chi)) = {}^{f^*(\sigma)}(\delta_0(\sigma\chi))$$

for all $\sigma \in \mathrm{Gal}(k)$ and $\chi \in X^*(T_0)$. But this equality follows from the equivariance of δ_0 with respect to the action of $W(G, T_0) = W(G^*, T_0^*)$ (see Remark 8.1) and the fact that $f^*(\sigma) = \delta_0(f(\sigma))$. Thus, δ is a duality map. Finally, note that varying the choices of the above elements g and g^* has the effect of altering δ by an element of $W(G, T)^{\mathrm{Gal}(k)}$. $\square$

Remark 8.3. Given a maximal k-torus $S \subseteq G$, let $S^* \subseteq G^*$ correspond to S via Proposition 8.2. Repeating the proof with S and S^* in place of T_0 and T_0^*, we see the following. If the maximal k-torus $T \subseteq G$ corresponds to $T^* \subseteq G^*$, and we write $T = {}^g S$ with $g \in G(k^{\mathrm{sep}})$, then there exists $g^* \in G^*(k^{\mathrm{sep}})$ such that $T^* = {}^{g^*} S^*$, and the cocycle that sends $\sigma \in \mathrm{Gal}(k)$ to the image in $W(G, S)$ of $g^{-1}\sigma(g)$ corresponds to the analogous cocycle determined by g^* under the identification between $W(G, S)$ and $W(G^*, S^*)$. Thus, this latter identification leads to the same correspondence $\mathcal{T}_{\mathrm{st}}(G, k) \longleftrightarrow \mathcal{T}_{\mathrm{st}}(G^*, k)$ of Proposition 8.2. Moreover, if δ_S and δ_T are duality maps for S and T, then there is some $w \in W(G, S)^{\mathrm{Gal}(k)}$ such that for all $\chi \in X^*(S)$, we have ${}^{g^* w}\delta_S(\chi) = \delta_T({}^g\chi)$. Since $\chi \mapsto {}^w\delta_S(\chi)$ is another duality map for S, we may replace δ_S by it and then we can write ${}^{g^*}\delta_S(\chi) = \delta_T({}^g\chi)$ in analogy with (8.2).

9. The conorm map

In this section, let $\widetilde{G}$ and G be quasisplit connected reductive k-groups. Let Γ be a finite group, and suppose that (ϕ, j_*) is a parascopic datum for $(\widetilde{G}, \Gamma, G)$ relative to the maximal k-tori $\widetilde{T} \subseteq \widetilde{G}$ and $T \subseteq G$.

Remark 9.1. We assume that our groups are quasisplit only to assure that they have duals satisfying Proposition 8.2. But such duals exist for some other groups (a matter that we will take up elsewhere), and in such cases we only need to make the weaker assumption that G is parascopic for $(\widetilde{G}, \Gamma)$ via the datum (ϕ, j_*).

Remark 9.2. Given a maximal k-torus $S \subseteq G$, one has from Proposition 7.5(i,ii) a maximal k-torus $\widetilde{S} \subseteq \widetilde{G}$, uniquely determined up to stable conjugacy. By Proposition 8.2, S and $\widetilde{S}$ determine maximal k-tori $S^* \subseteq G^*$ and $\widetilde{S}^* \subseteq \widetilde{G}^*$ (up to stable conjugacy), and duality maps $\delta\colon X^*(S) \longrightarrow X_*(S^*)$ and $\widetilde{\delta}\colon X^*(\widetilde{S}) \longrightarrow X_*(\widetilde{S}^*)$ (up to conjugacy by $W(G, S)^{\mathrm{Gal}(k)}$ and $W(\widetilde{G}, \widetilde{S})^{\mathrm{Gal}(k)}$, respectively). Similarly, given a maximal k-torus $S^* \subseteq G^*$, one obtains $S \subseteq G$ and thus all of the other data above.

Using T and $\widetilde{T}$, choose T^*, $\widetilde{T}^*$, δ, and $\widetilde{\delta}$ as in the preceding Remark. From §2, the parascopic datum (ϕ, j_*) determines a $\mathrm{Gal}(k)$-equivariant map $\mathcal{N}^* = \mathcal{N}^*_T\colon X^*(T) \longrightarrow X^*(\widetilde{T})$. Define

$$\widehat{\mathcal{N}}_{T^*,*} := \widetilde{\delta} \circ \mathcal{N}^* \circ \delta^{-1}\colon X_*(T^*) \longrightarrow X_*(\widetilde{T}^*).$$

Then $\widehat{\mathcal{N}}_{T^*,*}$ determines a *conorm* homomorphism

$$\widehat{\mathcal{N}}_{T^*}\colon T^* \longrightarrow \widetilde{T}^*.$$

Since both δ and $\widetilde{\delta}$ are $\mathrm{Gal}(k)$-equivariant, so is $\widehat{\mathcal{N}}_{T^*,*}$. Hence $\widehat{\mathcal{N}}_{T^*}$ is defined over k. We also have a corresponding map $\widehat{\mathcal{N}}^*_{T^*}\colon X^*(\widetilde{T}^*) \longrightarrow X^*(T^*)$. More explicitly,

$$\widehat{\mathcal{N}}^*_{T^*} = {\delta^*}^{-1} \circ \mathcal{N}_* \circ \widetilde{\delta}^*,$$

where $\mathcal{N}_*\colon X_*(\widetilde{T}) \longrightarrow X_*(T)$ is the adjoint of $\mathcal{N}^*$.

From Remark 8.1, the duality maps δ and $\widetilde{\delta}$ determine identifications $W(G, T) \longrightarrow W(G^*, T^*)$ and $W(\widetilde{G}, \widetilde{T}) \longrightarrow W(\widetilde{G}^*, \widetilde{T}^*)$. Thus, the embedding i of $W(G, T)$ in

$W(\widetilde{G},\widetilde{T})$ (see §5) determines an embedding (which we will also denote by i) of $W(G^*,T^*)$ in $W(\widetilde{G}^*,\widetilde{T}^*)$.

Remark 9.3. If we identify $W(G^*,T^*) = W(G,T)$ with its image in $W(\widetilde{G}^*,\widetilde{T}^*) = W(\widetilde{G},\widetilde{T})$ under i, it is clear that $\mathcal{N}^*$ is $W(G,T)$-equivariant. Since δ and $\widetilde{\delta}$ are $W(G,T)$-equivariant (given the identifications in the preceding paragraph), it follows that $\widehat{\mathcal{N}}_{T^*}$ is as well.

Let $s \in T^*(\bar{k})$ and let $\widetilde{s} = \widehat{\mathcal{N}}_{T^*}(s) \in \widetilde{T}^*(\bar{k})$. Let $H^* = C_{G^*}(s)$ and $\widetilde{H}^* = C_{\widetilde{G}^*}(\widetilde{s})$.

Proposition 9.4. *The embedding $i : W(G^*,T^*) \longrightarrow W(\widetilde{G}^*,\widetilde{T}^*)$ restricts to give embeddings of $W(H^*,T^*)$ in $W(\widetilde{H}^*,\widetilde{T}^*)$ and of $W(H^{*\circ},T^*)$ in $W(\widetilde{H}^{*\circ},\widetilde{T}^*)$.*

Proof. Suppose that $w \in W(H^*,T^*)$. Then by Remark 9.3,

$$(i(w))(\widetilde{s}) = (i(w))(\widehat{\mathcal{N}}_{T^*}(s)) = \widehat{\mathcal{N}}_{T^*}(w(s)) = \widehat{\mathcal{N}}_{T^*}(s) = \widetilde{s}.$$

Thus i gives an embedding of $W(H^*,T^*)$ in $W(\widetilde{H}^*,\widetilde{T}^*)$.

Now suppose $w \in W(H^{*\circ},T^*)$. According to [5, Theorem 3.5.3], w is a product of reflections through roots $\alpha^* \in \Phi(G^*,T^*)$ such that $\alpha^*(s) = 1$. For such a root α^*, let α be the corresponding root in $\Phi(G,T)$: $\alpha = \delta^{-1}(\alpha^{*\vee})$. Then

$$i(w_{\alpha^*}) = i(w_\alpha) = \prod_{\beta\in\Xi} w_\beta$$

in the notation of Proposition 5.5. For each $\beta \in \Xi$, let β^* be the corresponding root in $\Phi(\widetilde{G}^*,\widetilde{T}^*)$: $\beta^* = \widetilde{\delta}^{*\,-1}(\beta^\vee)$. Then $i(w_{\alpha^*}) = \prod_\Xi w_{\beta^*}$, and by *loc. cit.*, to show that $i(w)$ lies in $W(\widetilde{H}^{*\circ},\widetilde{T}^*)$, it suffices to show that $\beta^*(\widetilde{s}) = 1$ for all $\beta \in \Xi$. For such a root β^*,

$$\beta^*(\widetilde{s}) = \beta^*(\widehat{\mathcal{N}}_{T^*}(s)) = (\widehat{\mathcal{N}}^*_{T^*}\beta^*)(s),$$

so it is enough to show that $\widehat{\mathcal{N}}^*_{T^*}\beta^*$ is an integer multiple of α^*.

Recall the identifications of $V_*(T)$ with $V_*(\widetilde{T})^\Gamma$ and $V^*(T)$ with $V^*(\widetilde{T})^\Gamma$ described at the beginning of §5. We have

$$\begin{aligned}\mathcal{N}_{T,*}\beta^\vee &= \sum_{\gamma\in\Gamma}\gamma\cdot\beta^\vee && \text{(by (2.3))}\\ &= |\operatorname{stab}_\Gamma\beta| \sum_{\beta'\in\Xi}(\beta')^\vee && \text{(by Remark 5.2)}\\ &= \frac{|\Xi||\operatorname{stab}_\Gamma\beta|}{|\Gamma\cdot\widetilde{\alpha}|}\alpha^\vee && \text{(by Lemma 5.3).}\end{aligned}$$

But the constant $|\Xi||\operatorname{stab}_\Gamma\beta|/|\Gamma\cdot\widetilde{\alpha}|$ is always integral. Indeed, (in the terminology of §5) in case (1), we have $\Xi = \Gamma\cdot\widetilde{\alpha}$, while in case (2), $|\Gamma\cdot\widetilde{\alpha}| = 2|\Xi|$ and $|\operatorname{stab}_\Gamma\beta|$ is even. Translating this to the dual setting, we have that $\widehat{\mathcal{N}}^*\beta^*$ is an integer multiple of α^*, and the proposition follows. □

10. Lifting of semisimple geometric conjugacy classes

We now prove Statement (A) from the Introduction. Let $\widetilde{G}$, G, Γ, $\widetilde{T}$, T, and (ϕ, j_*) be as in §9.

Given a maximal k-torus S^* of G^*, choose corresponding maximal k-tori $\widetilde{S}$, S^*, and $\widetilde{S}^*$, and and duality maps δ_S and $\delta_{\widetilde{S}}$ as in Remark 9.2. From Proposition 7.5(i), there exists a parascopic datum (ϕ', j'_*) for $(\widetilde{G}, \Gamma, G)$ with respect to $\widetilde{S}$ and S that is equivalent to (ϕ, j_*). Thus from §9, corresponding to these arbitrary, implicit choices, we have a $W(G^*, S^*)$-equivariant k-morphism $\widehat{\mathcal{N}}_{S^*}\colon S^* \longrightarrow \widetilde{S}^*$.

Proposition 10.1. *There is a canonical k-morphism $\widehat{\mathcal{N}}$ from the k-variety of geometric semisimple conjugacy classes in G^* to the analogous variety for $\widetilde{G}^*$. Moreover, if S^* is a maximal k-torus of G^* and $s \in S^*(\bar{k})$, then*

$$\widehat{\mathcal{N}}_{S^*}(s) \in \widehat{\mathcal{N}}([s])(\bar{k}),$$

where $[s]$ is the geometric conjugacy class of s in G^*. That is, $\widehat{\mathcal{N}}$ is compatible with the conorms $\widehat{\mathcal{N}}_{S^*}$ on all maximal k-tori in G^*.

Proof. Let S^* be a maximal k-torus in G^*. As noted above, implicit in the construction of $\widehat{\mathcal{N}}_{S^*}$ are k-tori S, $\widetilde{S}$, and $\widetilde{S}^*$, and duality maps δ_S and $\delta_{\widetilde{S}}$ (as in Remark 9.2), as well as a parascopic datum (ϕ', j'_*) for $(\widetilde{G}, \Gamma, G)$ with respect to $\widetilde{S}$ and S that is equivalent to (ϕ, j_*). Since $\widehat{\mathcal{N}}_{S^*}\colon S^* \longrightarrow \widetilde{S}^*$ is a $W(G^*, S^*)$-equivariant k-morphism, we obtain a k-morphism

$$\widehat{\mathcal{N}}\colon S^*/W(G^*, S^*) \longrightarrow \widetilde{S}^*/W(\widetilde{G}^*, \widetilde{S}^*).$$

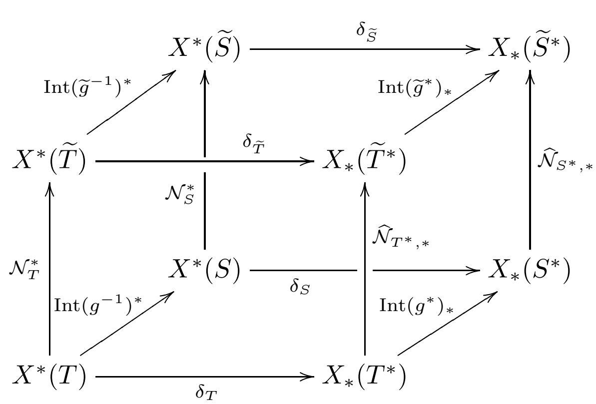
But these latter two varieties are k-isomorphic to the varieties of geometric semisimple conjugacy classes in G^* and $\widetilde{G}^*$, respectively.

From the maximal k-tori $T \subseteq G$ and $\widetilde{T} \subseteq \widetilde{G}$, Proposition 8.2 gives us maximal k-tori $T^* \subseteq G^*$, and $\widetilde{T}^* \subseteq \widetilde{G}^*$, and duality maps δ_T and $\delta_{\widetilde{T}}$. We now show that $\widehat{\mathcal{N}}$ is independent of the choice of the torus S^* by showing that we would have obtained the same map had we chosen $S^* = T^*$ and $(\phi', j'_*) = (\phi, j_*)$.

Suppose (ϕ', j'_*) is equivalent to (ϕ, j_*) via $g \in G(k^{\mathrm{sep}})$ and $\widetilde{g} \in \widetilde{G}(k^{\mathrm{sep}})$. Then ${}^gT = S$, ${}^{\widetilde{g}}\widetilde{T} = \widetilde{S}$, $\phi'(\gamma) = \mathrm{Int}(\widetilde{g})_* \circ \phi(\gamma) \circ \mathrm{Int}(\widetilde{g})_*^{-1}$ for all $\gamma \in \Gamma$, and $j'_* = \mathrm{Int}(\widetilde{g})_* \circ j_* \circ \mathrm{Int}(g)_*^{-1}$. It follows from this and the definition of $\mathcal{N}^*$ in §2 that

$$\mathcal{N}^*_S = \mathrm{Int}(\widetilde{g}^{-1})^* \circ \mathcal{N}^*_T \circ \mathrm{Int}(g)^*. \tag{10.1}$$

Choose $g^* \in G^*(k^{\mathrm{sep}})$ and $\widetilde{g}^* \in \widetilde{G}^*(k^{\mathrm{sep}})$ such that ${}^{g^*}T^* = S^*$ and ${}^{\widetilde{g}^*}\widetilde{T}^* = \widetilde{S}^*$ and such that g^* and $\widetilde{g}^*$ are compatible with g and $\widetilde{g}$ (respectively) as in Remark 8.3. Then there exist duality maps δ_T and $\delta_{\widetilde{T}}$ such that the top and bottom faces of the following diagram commute:



The front and back faces also commute by the definitions of $\widehat{\mathcal{N}}_{T^*,*}$ and $\widehat{\mathcal{N}}_{S^*,*}$. The left face commutes by (10.1). Since all of the horizontal maps are isomorphisms, the right face must also commute. That is, we have the equality

$$\widehat{\mathcal{N}}_{S^*,*} = \operatorname{Int}(\widetilde{g}^*)_* \circ \widehat{\mathcal{N}}_{T^*,*} \circ \operatorname{Int}(g^*)_*^{-1}$$

of maps $X_*(S^*) \longrightarrow X_*(\widetilde{S}^*)$. Therefore, the corresponding homomorphisms $S^* \longrightarrow \widetilde{S}^*$ must be equal, i.e.,

$$\widehat{\mathcal{N}}_{S^*} = \operatorname{Int}(\widetilde{g}^*) \circ \widehat{\mathcal{N}}_{T^*} \circ \operatorname{Int}(g^*)^{-1}. \tag{10.2}$$

It follows immediately that the definition of $\widehat{\mathcal{N}}$ is independent of the particular choice of torus S^*. □

11. Main Theorem

Let $\widetilde{G}$, G, Γ, $\widetilde{T}$, T, and (ϕ, j_*) be as in §9 and §10.

Theorem 11.1. *Let $s_1, s_2 \in G^*(k)$ be semisimple elements that are stably conjugate, and let T_i^* be maximal k-tori in G^* containing s_i. Then the elements $\widehat{\mathcal{N}}_{T_i^*}(s_i) \in \widetilde{G}^*(k)$ are stably conjugate.*

Remark 11.2. That is, there is a well-defined map $\widehat{\mathcal{N}}^{\mathrm{st}}$ from the set of semisimple stable conjugacy classes in $G^*(k)$ to the set of such classes in $\widetilde{G}^*(k)$, such that for every semisimple $s \in G^*(k)$ and every maximal k-torus T^* in G containing s, we have that $\widehat{\mathcal{N}}_{T^*}(s) \in \widehat{\mathcal{N}}^{\mathrm{st}}([s]_{\mathrm{st}})$, where $[s]_{\mathrm{st}}$ is the stable conjugacy class of s.

Proof of Theorem 11.1. From T_i^*, choose T_i, δ_i, $\widetilde{T}_i$, $\widetilde{T}_i^*$, and $\widetilde{\delta}_i$ as in Remark 9.2, and a parascopic datum $(\phi_i, j_{i,*})$ for $(\widetilde{G}, \Gamma, G)$ with respect to $\widetilde{T}_i$ and T_i that is equivalent to (ϕ, j_*). (These choices are implicit in the definition of $\widehat{\mathcal{N}}_{T_i^*}$.) Let $\widetilde{s}_i = \widehat{\mathcal{N}}_{T_i^*}(s_i) \in \widetilde{G}^*(k)$. According to Proposition 10.1, $\widetilde{s}_1$ is geometrically conjugate to $\widetilde{s}_2$ in $\widetilde{G}^*$. We want to show that the stable conjugacy classes of $\widetilde{s}_1$ and $\widetilde{s}_2$ in $\widetilde{G}^*(k)$ coincide.

Let $H_1^* = C_{G^*}(s_1)$ and $\widetilde{H}_1^* = C_{\widetilde{G}^*}(\widetilde{s}_1)$. We have that ${}^{g^*}s_1 = s_2$ for some $g^* \in G^*(k^{\mathrm{sep}})$ such that ${g^*}^{-1}\sigma(g^*) \in H_1^{*\circ}(k^{\mathrm{sep}})$ for all $\sigma \in \operatorname{Gal}(k)$. Moreover, by replacing g^* by g^*h^* for an appropriate element $h^* \in H_1^{*\circ}(k^{\mathrm{sep}})$, we may assume, in addition, that ${}^{g^*}T_1^* = T_2^*$. It follows that ${g^*}^{-1}\sigma(g^*) \in N_{H_1^{*\circ}}(T_1^*)(k^{\mathrm{sep}})$ for all $\sigma \in \operatorname{Gal}(k)$.

Choose $g \in G(k^{\mathrm{sep}})$ compatible with g^* as in Remark 8.3 such that ${}^{g}T_1 = T_2$. By Proposition 7.5(i) and Lemma 7.4, there exists $\widetilde{g} \in \widetilde{G}(k^{\mathrm{sep}})$ such that ${}^{\widetilde{g}}\widetilde{T}_1 = \widetilde{T}_2$ and $i\big(\overline{g^{-1}\sigma(g)}\big) = \overline{\widetilde{g}^{-1}\sigma(\widetilde{g})}$. Choose $\widetilde{g}^* \in G(k^{\mathrm{sep}})$ compatible with $\widetilde{g}$ as in Remark 8.3 such that ${}^{\widetilde{g}^*}\widetilde{T}_1^* = \widetilde{T}_2^*$.

Since $(\phi_1, j_{1,*})$ and $(\phi_2, j_{2,*})$ are equivalent to (ϕ, j_*), they are equivalent to each other. By Lemma 7.4, g and $\widetilde{g}$ implement an equivalence of $(\phi_1, j_{1,*})$ with a parascopic datum for $(\widetilde{G}, \Gamma, G)$ with respect to $\widetilde{T}_2$ and T_2. The latter datum must therefore be equivalent to $(\phi_2, j_{2,*})$, and hence is related to $(\phi_2, j_{2,*})$ as in Proposition 7.5(iii). It follows that the conorms $T_2^* \longrightarrow \widetilde{T}_2^*$ corresponding to these data differ by the action of an element of $W(\widetilde{G}^*, \widetilde{T}_2^*)^{\operatorname{Gal}(k)}$. Thus the stable conjugacy classes of the images of s_2 under these conorms coincide. Hence we may assume that $(\phi_1, j_{1,*})$ and $(\phi_2, j_{2,*})$ are equivalent via the particular elements g and $\widetilde{g}$.

As in (10.2), we have $\widehat{\mathcal{N}}_{T_2^*} = \operatorname{Int}(\widetilde{g}^*) \circ \widehat{\mathcal{N}}_{T_1^*} \circ \operatorname{Int}(g^*)^{-1}$. Thus

$$\widetilde{s}_2 = \widehat{\mathcal{N}}_{T_2^*}(s_2) = {}^{\widetilde{g}^*}\big(\widehat{\mathcal{N}}_{T_1^*}({}^{{g^*}^{-1}}s_2)\big) = {}^{\widetilde{g}^*}\big(\widehat{\mathcal{N}}_{T_1^*}(s_1)\big) = {}^{\widetilde{g}^*}\widetilde{s}_1. \tag{11.1}$$

It remains to show that $\widetilde{g}^{*-1}\sigma(\widetilde{g}^*) \in \widetilde{H}_1^{*\circ}(k^{\rm sep})$ for all $\sigma \in \mathrm{Gal}(k)$. But $\widetilde{g}^*$ was chosen so that the image of $\widetilde{g}^{*-1}\sigma(\widetilde{g}^*)$ in $W(G^*, T_1^*)$ is $i\big(\overline{g^{*-1}\sigma(g^*)}\big)$. Since $\overline{g^{*-1}\sigma(g^*)} \in W(H_1^{*\circ}, T_1^*)$, it follows that $i\big(\overline{g^{*-1}\sigma(g^*)}\big)$ lies in $W(\widetilde{H}_1^{*\circ}, \widetilde{T}_1^*)$ by Proposition 9.4. Therefore, $\widetilde{g}^{*-1}\sigma(\widetilde{g}^*) \in N_{\widetilde{H}_1^{*\circ}}(\widetilde{T}_1^*)(k^{\rm sep}) \subseteq \widetilde{H}_1^{*\circ}(k^{\rm sep})$, so $\widetilde{s}_1$ and $\widetilde{s}_2$ are stably conjugate. □

Corollary 11.3. *In the situation of Theorem 11.1 and Remark 11.2, suppose that k is perfect and has cohomologial dimension ≤ 1. Then $\widehat{\mathcal{N}}$ refines to a map $\widehat{\mathcal{N}}^{st}$ from the set of semisimple, $G^*(k)$-conjugacy classes in $G^*(k)$ to the set of semisimple, $\widetilde{G}^*(k)$-conjugacy classes in $\widetilde{G}^*(k)$.*

Proof. From the Lang-Steinberg Theorem (see [15, §III.2.3]), $H^1(k, M)$ is trivial for every connected reductive k-group M. Thus, from the first paragraph of [9, §3], we see that semisimple stable conjugacy classes and semisimple rational conjugacy classes coincide in the group of k-points of a connected reductive k-group, so our result now follows from Theorem 11.1. □

Appendix A. When must a quasi-semisimple automorphism preserve a Borel-torus pair defined over $k^{\rm sep}$?

We now offer a proof promised in Remark 3.2.

Lemma A.1. *Suppose $\widetilde{G}$ is a connected reductive k-group, and γ is a quasi-semisimple k-automorphism of $\widetilde{G}$. Suppose that at least one of the following holds:*

(i) the characteristic of k is not two;
(ii) k is perfect;
(iii) no power of γ acts via a non-inner automorphism on any factor of $\widetilde{G}$ of type A_{2n}.

Then γ preserves a Borel-torus pair defined over $k^{\rm sep}$.

Proof. Note that this statement is obvious when k is perfect.

We may assume that $k = k^{\rm sep}$ is separably closed. It is enough to show that $\widetilde{G}$ has a γ-stable maximal k-torus $\widetilde{T}_\bullet$ contained in a γ-stable Borel subgroup $\widetilde{B}_\bullet$. For then $\widetilde{T}_\bullet$ must be split over k, so the root groups corresponding to the roots in $\Phi(\widetilde{G}, \widetilde{T}_\bullet)$ must be defined over k, and hence so must $\widetilde{B}_\bullet$.

Fix a Borel $\bar{k}$-subgroup $\widetilde{B}_*$ of $\widetilde{G}$ and a maximal $\bar{k}$-torus $\widetilde{T}_* \subseteq \widetilde{B}_*$. The homogeneous space $\widetilde{G}/\widetilde{T}_*$ can be viewed, via the map $g\widetilde{T}_* \mapsto ({}^g\widetilde{B}_*, {}^g\widetilde{T}_*)$, as the $\bar{k}$-variety X of all pairs $(\widetilde{B}, \widetilde{T})$, where $\widetilde{B}$ is a Borel $\bar{k}$-subgroup of $\widetilde{G}$, and $\widetilde{T}$ is a maximal $\bar{k}$-torus of $\widetilde{B}$. Taking $\widetilde{T}_*$ to be defined over k shows that X can be given the structure of a k-variety; this structure is easily seen to be independent of the particular choice of $(\widetilde{B}_*, \widetilde{T}_*)$.

Assume for now that $\widetilde{T}_*$ is defined over k. There is an obvious action of γ on X. Let X^γ be the (nonempty) $\bar{k}$-variety of γ-fixed points in X. A point $x \in X^\gamma(\bar{k})$ corresponds to a pair $(\widetilde{B}, \widetilde{T})$ as above such that $\widetilde{B}$ and $\widetilde{T}$ are γ-stable. Moreover, if $x \in X(k) \cap X^\gamma(\bar{k})$ is represented by $g \in \widetilde{G}(\bar{k})$, then $\mathrm{Int}(g) : \widetilde{T}_* \longrightarrow \widetilde{T} := {}^g\widetilde{T}_*$ is a k-isomorphism. Thus $\widetilde{T}$ (and hence $\widetilde{B}$) is defined over k [4, Cor. 14.5]. It follows that the desired Borel-torus pair exists provided that $X(k) \cap X^\gamma(\bar{k}) \neq \emptyset$. To prove this nonemptiness, it suffices to show that X^γ is defined over k, for then

$X^\gamma(k) = X(k) \cap X^\gamma(\bar{k})$ must be dense in X^γ [4, Cor. 13.3]. By [16, Thm. 11.2.13], this will follow in turn if we can prove the equality of tangent spaces

$$\text{(A.1)} \qquad \mathrm{T}_x(X^\gamma) = \mathrm{T}_x(X)^\gamma \quad \text{for all } x \in X^\gamma(\bar{k}).$$

Let $\widetilde{T}_\bullet$ and $\widetilde{B}_\bullet$ be as in Definition 3.1. The Bruhat decomposition can be used to express $\widetilde{G}$ as a disjoint union

$$\text{(A.2)} \qquad \coprod_{w \in W(\widetilde{G}, \widetilde{T}_\bullet)} \widetilde{U}\widetilde{U}_w n_w \widetilde{T}_\bullet\,,$$

where n_w is a representative of w in $N_{\widetilde{G}}(\widetilde{T}_\bullet)(\bar{k})$, $\widetilde{U}$ is the unipotent radical of $\widetilde{B}_\bullet$, and $\widetilde{U}_w$ is the group generated by the root groups corresponding to $\alpha \in \Phi^-(\widetilde{G}, \widetilde{T}_\bullet) \cap w\Phi^+(\widetilde{G}, \widetilde{T}_\bullet)$, where positivity is with respect to $\widetilde{B}_\bullet$.

Let $z \in X(\bar{k}) = (\widetilde{G}/\widetilde{T}_\bullet)(\bar{k})$ be the point corresponding to the coset $\widetilde{T}_\bullet$. Since the obvious map $\widetilde{U} \times \widetilde{U}_w \times \widetilde{T}_\bullet \longrightarrow \widetilde{U}\widetilde{U}_w n_w \widetilde{T}_\bullet$ is an isomorphism of varieties for each $w \in W(\widetilde{G}, \widetilde{T}_\bullet)$, it follows from (A.2) that

$$\text{(A.3)} \qquad (\widetilde{U}\widetilde{U}_w n_w \widetilde{T}_\bullet \cdot z)^\gamma = \begin{cases} \widetilde{U}^\gamma \widetilde{U}_w^\gamma n_w \cdot z & \text{if } w \in W(\widetilde{G}, \widetilde{T}_\bullet)^\gamma, \\ \emptyset & \text{otherwise.} \end{cases}$$

From Lemma 3.3, $G = (\widetilde{G}^\gamma)^\circ$ is a reductive group, $T_\bullet := (\widetilde{T}_\bullet^\gamma)^\circ = G \cap \widetilde{T}_\bullet$ is a maximal torus in G, and $B_\bullet := (\widetilde{B}_\bullet^\gamma)^\circ$ is a Borel subgroup of G containing $T_\bullet$.

Observe that G acts on X^γ via left translation. It follows from (A.3) that X^γ is the union of a finite number of G-orbits represented by some subset of $\{n_w\}$. (In fact, we show in Proposition 5.5 that $W(G, T_\bullet)$ embeds in $W(\widetilde{G}, \widetilde{T}_\bullet)^\gamma$, implying that one can take as representatives for $G\backslash X^\gamma$ any subset of $\{n_w\}$ corresponding to representatives of $W(G, T_\bullet)\backslash W(\widetilde{G}, \widetilde{T}_\bullet)^\gamma$ in $W(\widetilde{G}, \widetilde{T}_\bullet)^\gamma$.) Moreover, the stabilizer in G of $n_w \cdot z$ is precisely $T_\bullet$. In particular, each G-orbit has dimension $\dim G - \dim T_\bullet$. Since these orbits are irreducible and of the same dimension as X^γ, it is straightforward to show that they are precisely the irreducible components of X^γ. Since they are disjoint and finite in number, they must also be the connected components of X^γ. It therefore suffices to verify (A.1) for $x = n_w \cdot z$.

Since the dimension of any G-orbit in X^γ is $\dim G - \dim T_\bullet$, we have $\dim \mathrm{T}_x(X^\gamma) = \dim \mathrm{T}_x(G \cdot x) \geq \dim G - \dim T_\bullet$. On the other hand, there is a natural identification $\mathrm{T}_x(X) = L(\widetilde{G})/L(\widetilde{T}_\bullet)$, where L denotes the Lie algebra functor, so we have

$$\mathrm{T}_x(X^\gamma) \subseteq \mathrm{T}_x(X)^\gamma = \Big(L(\widetilde{G})/L(\widetilde{T}_\bullet)\Big)^\gamma = \Bigg(\bigoplus_{\alpha \in \Phi(\widetilde{G}, \widetilde{T}_\bullet)} L(\widetilde{U}_\alpha)\Bigg)^\gamma.$$

It follows from [17, §8.2(2'''')] that if either assumption (i) or (iii) holds, the last space is equal to

$$\bigoplus_{\beta \in \Phi(G, T_\bullet)} L(U_\beta),$$

which has dimension $|\Phi(G, T_\bullet)| = \dim G - \dim T_\bullet$. Thus $\mathrm{T}_x(X^\gamma) = \mathrm{T}_x(X)^\gamma$, as desired. □

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