Obtaining certificates for complete synchronisation of coupled oscillators

Elias August[a] and Mauricio Barahona[b]

[a]Department of Electrical Engineering, ETH Zurich, Physikerstrasse 3, 8092 Zurich, Switzerland. E-mail: august@control.ee.ethz.ch

[b]Department of Bioengineering and Institute for Mathematical Sciences, Imperial College London, South Kensington Campus, London SW7 2AZ, United Kingdom

In this paper, we provide a novel reformulation of sufficient conditions that guarantee global complete synchronisation of coupled identical oscillators to make them computationally implementable. To this end, we use semidefinite programming techniques. For the first time, we can efficiently search and obtain certificates for synchronisability and, additionally, also optimise associated cost functions. In this paper, a Lyapunov-like function (certificate) is used to certify that all trajectories of a networked system consisting of coupled dynamical systems will eventually converge towards a common one, which implies synchronisation. Moreover, we establish new conditions for complete synchronisation, which are based on the so called Bendixson's Criterion for higher dimensional systems. This leads to major improvements on the lower bound of the coupling constant that guarantees global complete synchronisation. Importantly, the certificates are obtained by analysing the connection network and the model representing an individual system only. In order to illustrate the strength of our method we apply it to a system of coupled identical Lorenz oscillators and to coupled van der Pol oscillators.

Keywords: Lyapunov theory, contraction theory, Bendixson's Criterion for higher dimensions, semidefinite programming, sum of squares decomposition, synchronisation of coupled oscillators

PACS classification codes: 87.10.Ed, 87.19.lm, 89.75.Fb

1. Introduction

While synchronised behaviour of coupled dynamical systems is fascinating when observed, it is also required for the well functioning of many manmade and natural systems. A widely used mathematical model of coupled oscillators is the Kuramoto Model of coupled phase oscillators. In his review paper on the Kuramoto model and the contributions to its analysis [1], Storgatz makes the connection between the model and different biological systems such as 'networks of pacemaker cells in the heart; circadian pacemaker cells in the suprachiasmatic nucleus [(SCN)] of the brain [. . .]; metabolic synchrony in yeast cell suspensions; congregations of synchronously flashing fireflies; and crickets that chirp in unison'. Moreover, it is well known that neurons often act in synchrony [2,3]. However, many systems are not defined by or it is not suitable to represent them through the phase of a certain property. For this reason, in this paper, we investigate the synchronisation of coupled dynamical systems which depart from the Kuramoto model [4–7].

While, on one hand, identifying the conditions that lead to synchronisation is often important for the understanding of certain biological systems (analysis), on the other hand, for many engineered systems synchronisation has to be guaranteed (synthesis) [8,9]. Necessary conditions for global synchronisation are not known and in many cases it is not trivial to check whether the known sufficient conditions are fulfilled. In this paper, we reformulate known conditions for guarantee global complete synchronisation in order to implement them computationally, which has not been done previously. Using this approach, we can determine, first, whether sufficiently strong coupling exists that will lead to synchronisation and, second, check for the weakest coupling that still guarantees it. Moreover, for all-to-all coupling of oscillators, we also provide a novel condition which is based on the so called Bendixson's Criterion for higher dimensions [10,11]. This leads to major improvements on the lower bound of the coupling constant that guarantees global complete synchronisation. Importantly, the certificates are obtained by analysing the connection network and the model representing an individual system only, which means that the analysis cost is almost independent of the size of the network (only the cost of computing the eigenvalues of the network's Laplacian increases with its size [12]). In this paper, a Lyapunov-like function (certificate) is used to certify that all trajectories of a networked system consisting of coupled dynamical systems will eventually converge towards a common one, which implies synchronisation.

First, we provide some mathematical background on the computational tools we use in this paper (Section 2). The main part of the paper consists of Section 3, where we first introduce known results on global complete synchronisation of coupled identical oscillators [4–6]. In Section 3.2, we exemplify the results through a network of coupled chaotic Lorenz oscillators. As a novelty we use semidefinite programming to make it possible to computationally implement the theoretical results of Section 3.1. We obtain numerical results that fulfill the conditions for global complete synchronisation. Moreover, our implementation allows to search for the minimal value of the coupling strength required for synchronisation. In Section 3.3, we consider systems of coupled identical oscillators with all-to-all coupling. In Section 3.3.1, we provide a result based on contraction theory [9,13,14]. In Section 3.3.3, we provide novel sufficient conditions for global complete synchronisation of coupled identical oscillators. They are based on the so called Bendixson's Criterion for higher dimensions. They signify a move away from the, at times, strict requirements derived in Sections 3.1 and 3.3.1. We also show how to computationally implement these new results in order to check for synchronisation in systems of coupled oscillators.

2. Mathematical Background

2.1. Semidefinite programming and the sum of squares decomposition

In this section, we provide some mathematical background on the computational tools we use in this paper. The main computational tool is semidefinite programming. Programmes of this type can be solved efficiently using interior-point methods. (The interested reader is referred to reference [15] and the excellent text book by the same authors [16].) In semidefinite programming, we replace the nonnegative orthant constraint of linear programming by the cone of positive semidefinite matrices and pose the following minimisation problem:

$$\begin{aligned} \text{minimise} \quad & \mathrm{c}^T x \\ \text{subject to} \quad & F(x) \geq 0, \ \text{where} \\ & F(x) = F_0 + \sum_{i=1}^{n} x_i F_i. \end{aligned} \tag{1}$$

Here, $x \in \mathbb{R}^n$ is the free variable. The so called problem data, which are given, are the vector $c \in \mathbb{R}^n$ and the matrices $F_j \in \mathbb{R}^{m \times m}$, $j = 0, \ldots, n$. Note that convexity of the set of symmetric positive semidefinite matrices in (1) implies that the minimisation problem has a global minimum.

2.1.1. Sum of squares decomposition

For problem data that consists of polynomials of any degree the requirement of positivity can be relaxed to the condition that the polynomial function is a sum of squares. On one hand, this is only a sufficient condition for positivity and it can, at times, be quite conservative; in other words, a function can be positive without being a sum of squares. On the other hand, testing positivity of a polynomial is NP-hard [17].

Consider the real-valued polynomial function $F(x)$ of degree $2d$, $x \in \mathbb{R}^n$. A sufficient condition for $F(x)$ to be nonnegative is that it can be decomposed into a sum of squares [18]:

$$F(x) = \sum_i f_i^2(x) \geq 0,$$

where f_i are polynomial functions. Now, $F(x)$ is a sum of squares if and only if there exists a positive semidefinite matrix R and

$$F(x) = \chi^{\mathrm{T}} R \chi, \ \chi = [1, \ x_1, \ x_2, \ldots, \ x_n, x_1 x_2, \ldots, \ x_n^d].$$

The length of vector χ is $\ell = \binom{n+d}{d}$. Note that R is not necessarily unique. However, $\sum_i f_i^2(x) = \chi^{\mathrm{T}} R \chi$ poses certain constraints on R of the form $\mathrm{trace}(A_j R) = c_j$, where A_j and c_j are appropriate matrices and constants respectively. (For an illustration, see Example 3.5 in [18].)

In general, in order to find R, we solve the optimisation problem associated with the following semidefinite program:

$$\begin{aligned} \text{minimise} \quad & \mathrm{trace}(A_0 R) \\ \text{subject to} \quad & \mathrm{trace}(A_j R) = c_j, \ j = 1, \ldots, m \\ & R \geq 0. \end{aligned} \tag{2}$$

In this paper, to solve sum of squares programs, we used `SOSTOOLS` [19], a free, third-party `MATLAB` toolbox, which relies on the solver `SeDuMi` [20]. Finally, here are some additional remarks:

- Consider a rational function $F(x)$; that is, $F(x) = \frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomial functions. Then, $F(x) \geq 0$ if (2) is feasible with $\chi^{\mathrm{T}} R \chi = F(x) g^2(x)$ or with $\chi^{\mathrm{T}} R \chi = F(x) g(x)$ if $g(x) > 0$.
- If

$$F(x) + p(x)h(x) = \sum_i g_i^2(x) \geq 0, \ p(x) \geq 0,$$

$$h(x) = \begin{cases} \leq 0 \text{ if } a_i \leq x_i \leq b_i \ \forall i \\ > 0 \text{ otherwise} \end{cases},$$

then $F(x) \geq 0$ if $a_i \leq x_i \leq b_i$ for all i, where a_i, b_i are constants. This can be used to show that $F(x)$ is nonnegative in a specific region of the state or/and parameter space.

3. Global complete synchronisation of identical oscillators with symmetric and connected coupling configuration

In this section, we first provide results on global complete synchronisation of coupled identical oscillators with a coupling graph that is symmetric and connected but otherwise arbitrary. Thereafter, we provide respective results for the special case of all-to-all coupling. Consider two dynamical systems $f(x)$ and $f(y)$, where function $f(\cdot)$ describes the dynamics of the systems and x, y are the respective state vectors (that is, $\dot{x} = f(x)$ and $\dot{y} = f(y)$). The two systems completely synchronise if $\|x - y\| \to 0$ as $t \to \infty$. If this holds for all initial conditions, we speak of global complete synchronisation.

3.1. Sufficient conditions for global complete synchronisation of identical oscillators

A numerical approach to check whether coupled identical dynamical systems synchronise locally was introduced by Pecora and Carroll in [4]. Sufficient conditions that guarantee global complete stability of the synchronised state of a system of coupled identical oscillators and that can be checked analytically – albeit not easily – were developed independently of each other by Wu [6,21] and Belykh and colleagues [5,22]. The approaches are different but related and based on graph theory and Lyapunov stability theory. In this paper, we present and extend Wu's work.

Consider N identical n-dimensional oscillators given by $x_i \in \mathbb{R}^n$, $i = 1, \ldots, N$, and whose individual dynamics is determined through function $f(\cdot)$. If the oscillators are linearly coupled, the behaviour of the coupled system is described by:

$$\dot{x} = \tilde{f}(x) + \kappa (C \otimes D) x, \tag{3}$$

where $x = [x_1 \ldots x_N]^{\mathrm{T}}$ and $\tilde{f}(x) = [f(x_1) \ldots f(x_N)]^{\mathrm{T}}$. The second summand in the right hand side of (3) is the coupling term, where $\otimes$ denotes the Kronecker product. The positive constant κ corresponds to the coupling strength. Matrix $D \in \mathbb{R}^{n \times n}$ is the nonnegative output matrix (that is, $D_{ij} \geq 0 \ \forall i, j$) for each oscillator; in other words, it denotes the variables that are used in the coupling. Matrix $-C \in \mathbb{R}^{N \times N}$ is the Laplacian matrix of the coupling topology.[1] The proof of the following theorem can be

[1] Note that C and D are not required to be constant and can be of the form $C(x, t)$ and $D(x, t)$ respectively.

found in [6,23].

Theorem 3.1. *Given (3), where $x_i \in \mathcal{D} \subseteq \mathbb{R}^n$, $i \in \{1, \ldots, N\}$. Let $\gamma N = \lambda_{\min}(-C)$, where $\lambda_{\min}(-C)$ denotes the smallest positive eigenvalue of $-C$, and*

$$g(x_i) = f(x_i) - \gamma N \kappa D x_i. \tag{4}$$

Then, if there exists a symmetric and positive matrix $P \in \mathbb{R}^{n\times n}$ such that for all x_i and all i

$$(x_i - x_j)^{\mathrm{T}} P(g(x_i) - g(x_j)) < 0,\ x_j \neq x_i, \tag{5}$$

the network of coupled dynamical systems (3) synchronises; that is, $\|x_i - x_j\| \to 0$, $\forall i, j$, as $t \to \infty$.

We can use the *mean value theorem* to check whether inequality (5) holds. For clarity, we first state the theorem as it appears in [24]:

Theorem 3.2. (Mean value theorem) *Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable at each point x of an open set $\mathcal{S} \subset \mathbb{R}^n$. Let x and y be two points of $\mathcal{S}$ such that the line segment $L(x, y) \subset \mathcal{S}$. Then there exists a point z of $L(x, y)$ such that*

$$f(y) - f(x) = \left.\frac{\partial f}{\partial x}\right|_{x=z} (y - x). \tag{6}$$

Thus, inequality (5), whose lefthand side is a scalar-valued function, holds if

$$P\left(\frac{\partial g(y)}{\partial y}\right) < 0\ \forall y \in \overline{\mathcal{D}},\ \mathcal{D} \subseteq \overline{\mathcal{D}},\ \text{and } \overline{\mathcal{D}} \text{ is } \textit{convex}. \tag{7}$$

Finally, the following remark addresses an issue that is relevant to most results in this paper and which is not satisfactorily addressed in the literature in our opinion.

Remark 3.3. Let (7) hold for κ^*. Then it does not necessarily hold for all $\kappa > \kappa^*$. However, if matrix P is such that $PD \geq 0$ then (7) holds for all $\kappa > \kappa^*$ since then

$$P\left(\frac{\partial g(y)}{\partial y}\right) - (\kappa^* - \kappa) PD < 0.$$

3.2. Computational methods to obtain certificates for global complete synchronisation with applications

In this section, we show how to reformulate the theoretical results presented previously in order to obtain computational certificates that guarantee global complete synchronisation. First, note that condition (7) corresponds to a feasibility problem and can be implemented as:

$$\begin{array}{rl} \text{given} & \mathcal{D},\ J(\cdot),\ \gamma,\ N,\ D,\ \kappa \\ \text{search for} & P \\ \text{subject to} & P > 0,\ PD \geq 0,\ P(J(y) - \gamma N \kappa D) < 0\ \forall y \in \mathcal{D}. \end{array} \tag{8}$$

Note that we have included the requirement of Remark 3.3. In the following, we provide an application to coupled Lorenz oscillators. First, let us shortly investigate the effect of different coupling configurations.

3.2.1. Different coupling configurations

Let (7) hold for $\gamma\kappa N = 1$. Clearly, there is an inverse relationship between the values of γ and κ. It follows that since $\gamma N = \lambda_{\min}(-C)$, the coupling configuration has an influence on the value of κ. Moreover, if κ is associated with a cost function and larger values of κ are more costly then we seek a coupling that keeps κ low. For illustration, we provide examples of $\lambda_{\min}(-C)$ for different unweighted coupling configurations (Figure 1).

a) All-to-all coupling ($C = ee^{\mathrm{T}} - NI$, $e^{\mathrm{T}} = [1 \ \ldots \ 1]$): $\lambda_{\min}(-C) = N$.

b) Star-configuration: $\lambda_{\min}(-C) = 1$.

c) Ring of diffusively coupled oscillators: $\lambda_{\min}(-C) = 4\sin^2\frac{\pi}{N}$.

d) Ring of $2k$-nearest neighbor coupled oscillators: $\lambda_{\min}(-C) \simeq 2\pi^2 k(k+1)(2k+1)/3N^2$ if $k \ll N$ (see [25]).

In this case, configuration (a) is the most desirable. Let $\gamma\kappa N = 1$. Then, the following table compares the values of κ for configurations (a) – (c) when $N = 7, \ldots, 10$:

Table 1
Cost depending on coupling configuration.

N	7			8			9			10		
config.	(a)	(b)	(c)	(a)	(b)	(c)	(a)	(b)	(c)	(a)	(b)	(c)
κ (cost)	0.143	1	0.19	0.125	1	0.213	0.111	1	0.238	0.1	1	0.262

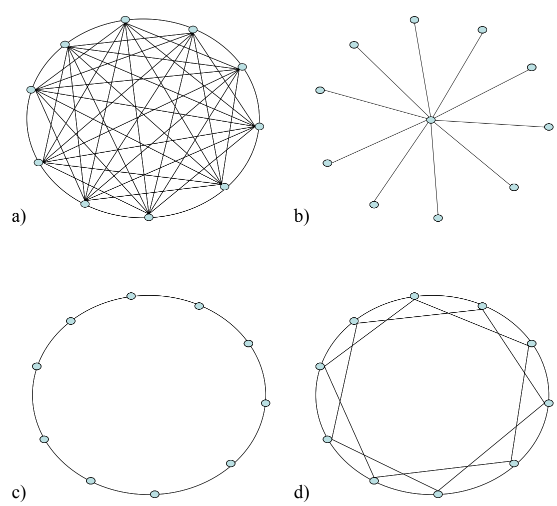

Figure 1. **Four different coupling configurations.** Here, $N = 10$ in (a), (c) and (d), $N = 11$ in (b), and $k = 2$ in (d).

3.2.2. A network of coupled identical Lorenz oscillators

The Lorenz oscillator is much referred to in the literature. It can exhibit periodic and chaotic behaviour. In several papers by Belykh et al (for instance, see [5,22]), a network of coupled identical Lorenz oscillators was analysed. We now use the results presented so far in this paper to obtain lower bounds for the coupling strength that guarantees global complete synchronisation and compare our findings with those of [5].

Consider a network of N coupled identical Lorenz oscillators. The set of equations for each individual system is given by ($i = 1, \ldots, N$):

$$\begin{aligned} \dot{x}_{i_1} &= \sigma(x_{i_2} - x_{i_1}), \\ \dot{x}_{i_2} &= r x_{i_1} - x_{i_2} - x_{i_1} x_{i_3}, \\ \dot{x}_{i_3} &= x_{i_1} x_{i_2} - b x_{i_3}. \end{aligned} \tag{9}$$

The coupled system is defined by (3). The Jacobian of each individual Lorenz oscillator is given by $J_L(x_i) = \begin{bmatrix} -\sigma & \sigma & 0 \\ r - x_{i_3} & -1 & -x_{i_1} \\ x_{i_2} & x_{i_1} & -b \end{bmatrix}$. We let $D = \mathrm{diag}([1\ 0\ 0])$, $\sigma > 1$ and $b > 2$. Similarly to [26], the following analysis will establish a positively invariant set for the system given by (9) together with (3), which describes a system of coupled Lorenz oscillators. We will then use the result to restrict our synchronisation analysis to this set.

For instance, consider

$$V(x) = \frac{1}{2}\left(\sum_{i=1}^{N} x_{i_1}^2 + \sum_{i=1}^{N} x_{i_2}^2 + \sum_{i=1}^{N} (x_{i_3} - \sigma - r)^2\right). \tag{10}$$

If follows that

$$\begin{aligned} \dot{V}(x) &= -\sigma \sum_{i=1}^{N} x_{i_1}^2 - \sum_{i=1}^{N} x_{i_2}^2 - b \sum_{i=1}^{N} (x_{i_3}^2 - (\sigma + r) x_{i_3}) + x^{\mathrm{T}} \kappa (C \otimes D) x \\ &= -\sigma \sum_{i=1}^{N} x_{i_1}^2 - \sum_{i=1}^{N} x_{i_2}^2 - b \sum_{i=1}^{N} \left(x_{i_3} - \frac{\sigma + r}{2}\right)^2 + Nb\left(\frac{\sigma + r}{2}\right)^2 + x^{\mathrm{T}} \kappa (C \otimes D) x. \end{aligned} \tag{11}$$

Let Γ be the closed set bounded by

$$\sigma \sum_{i=1}^{N} x_{i_1}^2 + \sum_{i=1}^{N} x_{i_2}^2 + b \sum_{i=1}^{N} \left(x_{i_3} - \frac{\sigma + r}{2}\right)^2 - x^{\mathrm{T}} \kappa (C \otimes D) x = Nb\left(\frac{\sigma + r}{2}\right)^2. \tag{12}$$

Then, $\dot{V}(x) < 0$ outside Γ and $\dot{V}(x) > 0$ inside Γ. It follows that, for initial conditions inside Γ, the upper bound $\mathfrak{b}$ of $V(x)$ lies on the surface given by (12), on which $\dot{V}(x) = 0$ and thus,

$$\sum_{i=1}^{N} x_{i_2}^2 = -\sigma \sum_{i=1}^{N} x_{i_1}^2 - b \sum_{i=1}^{N} (x_{i_3}^2 - (\sigma + r) x_{i_3}) + x^{\mathrm{T}} \kappa (C \otimes D) x$$

which leads to

$$V(x \in \Gamma) = \frac{1}{2}\left((1-\sigma)\sum_{i=1}^{N} x_{i_1}^2 + \sum_{i=1}^{N}(-bx_{i_3}^2 + b(\sigma+r)x_{i_3} + (x_{i_3}-\sigma-r)^2) + x^{\mathrm{T}}\kappa(C\otimes D)x\right)$$
$$\leq \tilde{V}(x \in \Gamma) = \frac{1}{2}\left((1-\sigma)\sum_{i=1}^{N} x_{i_1}^2 + \sum_{i=1}^{N}(-bx_{i_3}^2 + b(\sigma+r)x_{i_3} + (x_{i_3}-\sigma-r)^2)\right).$$

Thus, $\mathfrak{b} = V_{\max}(x \in \Gamma) \leq \tilde{V}_{\max}(x \in \Gamma)$. Now, for all i, the solutions to

$$\frac{\partial \tilde{V}(x \in \Gamma)}{\partial x_{i_1}} = (1-\sigma)x_{i_1} = 0, \quad \frac{\partial V(x \in \Gamma)}{\partial x_{i_3}} = (x_{i_3} - (\sigma + r)) - b(x_{i_3} - \frac{\sigma + r}{2}) = 0,$$

are given by $\bar{x}_{i_1} = 0$ and $\bar{x}_{i_3} = \frac{b-2}{2(b-1)}(\sigma + r) = \rho$. Note that, for all i,

$$\left.\frac{\partial^2 V(x \in \Gamma)}{\partial x_{i_1}^2}\right|_{x_{i_1}=\bar{x}_{i_1}, x_{i_3}=\bar{x}_{i_3}} < 0 \text{ and } \left.\frac{\partial^2 V(x \in \Gamma)}{\partial x_{i_3}^2}\right|_{x_{i_1}=\bar{x}_{i_1}, x_{i_3}=\bar{x}_{i_3}} < 0.$$

It follows that

$$\mathfrak{b} \leq \tilde{V}_{\max}(x \in \Gamma) = N(-b\rho^2 + b(\sigma + r)\rho + (\rho - \sigma - r)^2) = N\frac{b^2(r+\sigma)^2}{4(b-1)}.$$

Thus, for initial conditions inside Γ, the following bound holds:

$$\sum_{i=1}^{N} x_{i_1}^2 + x_{i_2}^2 + (\sigma + r - x_{i_3})^2 \leq N\frac{b^2(r+\sigma)^2}{4(b-1)}. \tag{13}$$

Note that the positively invariant set described through (13) is compact and convex. Thus, in the following, we will restrict our analysis to this set.

Similarly to [5], we let $P = I$, use (13) and obtain that

$$PJ_L(x_i) + J_L(x_i)^{\mathrm{T}}P - 2\gamma N\kappa PD = J_L(x_i) + J_L(x_i)^{\mathrm{T}} - 2\gamma N\kappa D < 0 \tag{14}$$

if

$$\kappa > \frac{b(1+b)(r+\sigma)^2}{16\gamma(b-1)} - \frac{\sigma}{\gamma N}, \tag{15}$$

where γ is as in Theorem 3.1. Condition (15) follows from the Routh-Hurwitz criterion. For instance, (15) guarantees that $a_3 > 0$ and $a_1 a_2 > a_3$, where

$$\begin{aligned} a_1 &= \sigma + \gamma N\kappa + 1 + b, \\ a_2 &= \sigma + \gamma N\kappa + b + (\sigma + \gamma N\kappa)b - 0.25((\sigma + r - x_{i_3})^2 + x_{i_2}^2), \\ a_3 &= (\sigma + \gamma N\kappa)b - 0.25(b(\sigma + r - x_{i_3})^2 + x_{i_2}^2) \end{aligned}$$

are the coefficients of the characteristic equation of (14): $\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$.

Typically, the following system parameters are chosen for the Lorenz oscillator: $\sigma = 10$, $b = \frac{8}{3}$, and $r = 28$. Thus, if we assume that $P = I$ then (15) guarantees global complete synchronisation if $\kappa > 530 - \frac{10}{\gamma N}$. Figure 1 shows how γ changes according to different coupling topologies. For example, coupled Lorenz oscillators completely synchronise with all-to-all coupling if $\kappa \geq 525$ and $N = 2$, or if $\kappa \geq 529$ and $N = 10$.

By solving (8) (using `YALMIP` [27]), the bound on κ can be improved significantly. We obtain that with

$$P = \begin{bmatrix} 4.929 & 0 & 0 \\ 0 & 0.877 & 0 \\ 0 & 0 & 0.877 \end{bmatrix} \tag{16}$$

the coupled Lorenz oscillators completely synchronise if $\kappa \geq \frac{251}{\gamma}$ (thus, if $\kappa \geq 251$ for all-to-all coupling) and $N = 2$; and that with

$$P = \begin{bmatrix} 4.328 & 0 & 0 \\ 0 & 0.327 & 0 \\ 0 & 0 & 0.327 \end{bmatrix} \tag{17}$$

the coupled system synchronises if $\kappa \geq \frac{125}{\gamma}$ (if $\kappa \geq 125$ for all-to-all coupling) and $N = 10$.

Note that the bound given by (13) was implemented as $x_{i_1}^2, x_{i_2}^2, (38 - x_{i_3})^2 < 1540.3N$. In the following we require that

$$N \frac{b^2(r+\sigma)^2}{4(b-1)} - \left(\sum_{i=1}^{N} x_{i_1}^2 + x_{i_2}^2 + (\sigma + r - x_{i_3})^2 \right) \text{ is a sum of squares.}$$

However, this implies a greater computational effort and is implemented as follows:

$$\begin{array}{rl} \text{given} & J_L(\cdot),\ \gamma,\ N,\ D,\ \kappa,\ \delta > 0 \\ \text{search for} & P,\ p(v) \\ \text{subject to} & v^{\mathrm{T}}(P - \delta I)v,\ v^{\mathrm{T}} P D v,\ p(v) \text{ are SOS } \forall v \in \mathbb{R}^3 \\ & -v^{\mathrm{T}}(P(J_L(y) - \gamma N \kappa D) + \delta I)v + p(v)(y_1^2 + y_2^2 + (38 - y_3)^2 - 1540.3N) \\ & \text{is SOS } \forall y, v \in \mathbb{R}^3. \end{array} \tag{18}$$

We let $\delta = 0.01$. Searching for a lower bound on κ, we obtain that with

$$P = \begin{bmatrix} 17.077 & 0 & 0 \\ 0 & 3.777 & 0 \\ 0 & 0 & 3.782 \end{bmatrix} \tag{19}$$

the coupled Lorenz oscillators completely synchronise if $\kappa \geq \frac{222.6}{\gamma}$ (if $\kappa \geq 222.6$ for all-to-all coupling) and $N = 2$; and that with

$$P = \begin{bmatrix} 9.59 & 0 & 0 \\ 0 & 1.116 & 0 \\ 0 & 0 & 1.116 \end{bmatrix} \tag{20}$$

the coupled system synchronises if $\kappa \geq \frac{113.2}{\gamma}$ (if $\kappa \geq 113.2$ for all-to-all coupling) and $N = 10$.

Since there exist equilibrium points such that $x_1 \neq x_2$ for $N = 2$ and $\kappa < 135$, global complete synchronisation is not possible for the system consisting of coupled Lorenz oscillators in this case. If $N = 2$ and $\kappa \geq 135$ then numerics indicate that global complete synchronisation indeed occurs (Figure 2). The following table summarises the results of this example and shows how solving (18) makes it possible to guarantee global complete synchronisation with a value of the coupling strength κ that is lower than the one obtained for $P = I$.[2]

Table 2
Comparison of the values of κ^* – the minimal value of κ that guarantees global complete synchronisation – obtained with and without using programme (18) for all-to-all coupling.

	$P = I$		Solving (18)	
N	2	10	2	10
κ^*	525	529	222.6 ($\frac{222.6}{525} \approx \frac{1}{2}$)	113.2 ($\frac{113.2}{529} = 0.214$)

3.3. Global complete synchronisation of identical oscillators with all-to-all coupling

In this section, we extend the results of the previous sections for systems with all-to-all coupling. In Section 3.3.1, we provide a lemma which is similar to Theorem 3.1 but with a nonconstant matrix $P(x) > 0$. In Section 3.3.3, we provide novel sufficient conditions for global complete synchronisation of coupled identical oscillators. They are based on the so called Bendixson's Criterion for higher dimensions.

3.3.1. Sufficient conditions based on contraction theory

First, we present results on dynamical systems with a certain local contraction property. This property leads to exponential stability of equilibria. In [28], Lewis studied autonomous dynamical systems which fulfill the contraction property presented in this paper as condition (22). Slotine [13] modernised Lewis's work and made it known to the wider audience under the name of contraction theory by applying it to problems in engineering. The idea behind is that if trajectories remain in a bounded region and the distance between any two decreases with time then there exists a unique exponentially stable equilibrium point in this region. (We would like to refer the interested reader to the review paper on contraction theory by Jouffroy [29].)

The theory of dynamical systems with a certain local contraction property presents an extension of Lyapunov stability theory. Next, we provide a short overview on Lewis's and Slotine's work. Consider the following system:

$$\dot{x} = f(x), \tag{21}$$

where

[2] Note that the optimal κ is obtained by solving (18) iteratively.

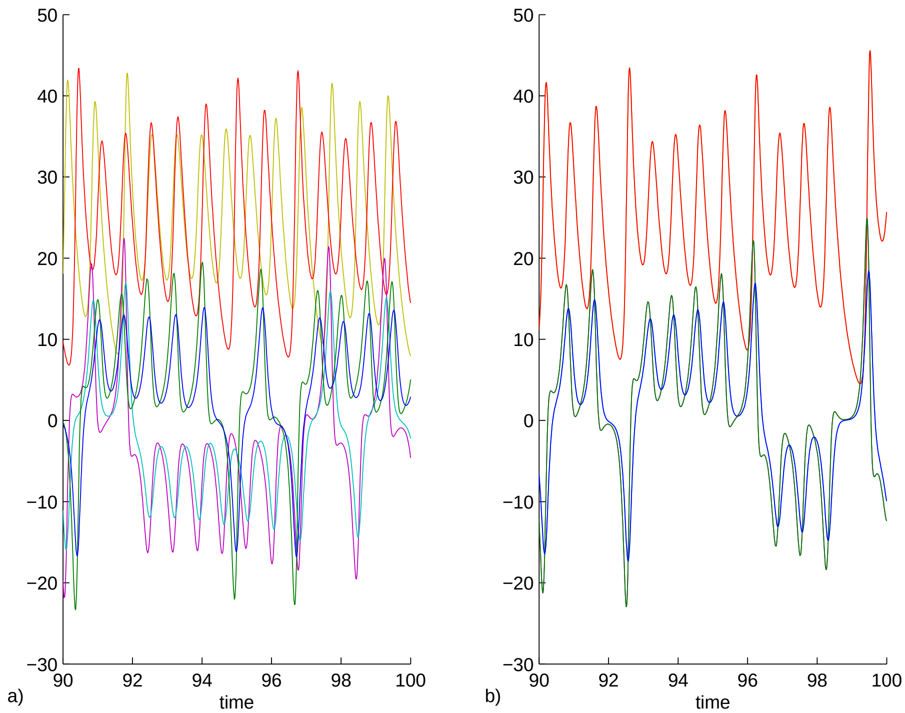

Figure 2. **Two coupled identical Lorenz oscillators.** Let $D = \text{diag}([1\ 0\ 0])$ and consider all-to-all coupling. (a) For $\kappa = 1$, two coupled identical Lorenz oscillators do not seem to synchronise in numerical simulations (the six trajectories clealry differ). (b) For $\kappa = 135$, numerical simulations seem to indicate that they globally completely synchronise as the three trajectories of the second system are superimposed on the ones of the first (note that for $\kappa < 135$, asynchron states are possible solutions of the system such as the following for $\kappa = 134$: $x_{11} = 0.1385$, $x_{12} = 3.8505$, $x_{13} = 0.2$, $x_{21} = -0.1385$, $x_{22} = -3.8505$, $x_{23} = 0.2$).

- $x \in \mathcal{B} \subseteq \mathbb{R}^n$ and $\mathcal{B}$ is a compact, connected and positively invariant set of (21).

The following theorem is a reformulation of Lewis's Theorem 9 from [28]. It establishes conditions under which asymptotic stability of solutions is guaranteed.

Theorem 3.4. (Lewis, Theorem 9 of [28]) *If there exits a matrix $M(x) > 0$, $x \in \mathcal{B}$, such that*

$$M(x)\frac{\partial f(x)}{\partial x} + \frac{1}{2}M'(x) < 0,\ \forall x \in \mathcal{B},\ \text{with}\ M'_{ij}(x) = \sum_{k=1}^{n} \frac{\partial M_{ij}(x)}{\partial x_k} f_k \qquad (22)$$

holds, then any two solutions x_1 and x_2 of (21) must approach each other asymptotically.

Now, the next theorem is an immediate consequence of Theorem 3.4. The proof is based on Lyapunov stability theory and constitutes an extension of the standard proof of Krasovskii's Theorem [13,24].

Theorem 3.5. *If $\mathcal{B} \subset \overline{\mathcal{B}}$, $\overline{\mathcal{B}}$ is convex and there exists a matrix $M(x) > 0$ such that (22) holds for all $x \in \overline{\mathcal{B}}$ then (21) has a unique asymptotically stable equilibrium point in $\mathcal{B}$. Additionally, if $M(x)$ is bounded in $\overline{\mathcal{B}}$ then the equilibrium point is exponentially stable.*

Proof. First, since $\mathcal{B}$ is a compact, connected and positively invariant set of (21), by the Brouwer fixed point theorem, there exists at least one equilibrium point in $\mathcal{B}$. We denote it by x_{eq}. Without loss of generality, let $x_{\text{eq}} = 0$, for otherwise the equilibrium point can be shifted to the origin through a change of a variables. Now, let there exists a matrix $M(x)$ such that (22) holds. Then, consider the Lyapunov function $V(x) = \frac{1}{2}x^{\mathrm{T}}M(x)x$.

Since $\overline{\mathcal{B}}$ is convex, a straight path that connects a pair of points $\{x, y\}$ in $\overline{\mathcal{B}}$ also lies in $\overline{\mathcal{B}}$. Thus, it follows from Theorem 3.2 that there exists a z such that $z_j \in [x_j, y_j]$ for all j, $j = 1, \ldots, n$, and

$$\begin{aligned}\dot{V}(x) &= x^{\mathrm{T}}M(x)f(x) + \frac{1}{2}x^{\mathrm{T}}M'(x)x = x^{\mathrm{T}}M(x)\left(f(x) - f(x_{\text{eq}})\right) + \frac{1}{2}x^{\mathrm{T}}M'(x)x \\ &= x^{\mathrm{T}}\left(M(x)\left.\frac{\partial f(x)}{\partial x}\right|_{x=z} + \frac{1}{2}M'(x)\right)x < 0.\end{aligned}$$

This implies that x_{eq} is unique and asymptotically stable. Exponential stability follows from Theorem 4.10 in [24] if $M(x)$ is bounded, because there exist positive constants k_1, k_2, k_3 such that $k_1x^{\mathrm{T}}x < V(x) < k_2x^{\mathrm{T}}x$ and $\dot{V}(x) < -k_3x^{\mathrm{T}}x$. □

Based on Theorem 3.5, the following lemma provides an extension to Theorem 3.1. A similar result is given in [30] with a different proof. In the following,

$$P'_{ij}(z) = \sum_{k=1}^{n} \frac{\partial P_{ij}(z)}{\partial z_k}(f_k(z) - (\kappa N D z)_k).$$

Lemma 3.6. *Consider the coupled system given by (3) with all-to-all coupling. Let $x_i \in \mathcal{D}$, $\mathcal{D}$ be a connected, compact and positively invariant set of (3), $\mathcal{D} \subset \overline{\mathcal{D}}$ and $\overline{\mathcal{D}}$ be convex. If there exists a nonconstant symmetric matrix $P(z) > 0$ such that*

$$P(z)J(x_i)|_{x_i=z} - \kappa N P(z) D + \frac{1}{2}P'(z) < 0 \tag{23}$$

for all $z \in \overline{\mathcal{D}}$ then (3) completely synchronies as $t \to \infty$.

Proof. First, for all i, consider the difference $x_i - x_1 \equiv X_i$ ($X_1 = 0$). Then,

$$X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_N \end{bmatrix} = \left(\begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ -1 & 1 & 0 & \ldots & 0 \\ -1 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \ldots & 1 \end{bmatrix} \otimes I_n\right) x. \tag{24}$$

As shown in [5], without coupling, the following holds

$$\dot{X}_i = \int_0^1 J(\beta x_i - (1 - \beta x_j))\mathrm{d}\beta X_i, \ i = 1, \ldots, N$$

where $\frac{\partial f(x_i)}{\partial x_i} \equiv J(x_i)$. It follows that if $J(\beta x_i - (1-\beta x_j))\mathrm{d}\beta < \alpha < 0$ for all β, where $0 \leq \beta \leq 1$ and α is a constant, then

$$\int_0^1 J(\beta x_i - (1-\beta x_j))\mathrm{d}\beta \leq \alpha \tag{25}$$

This implies that if $\overline{\mathcal{D}}$ is a convex set, $z, x_i, x_j \in \overline{\mathcal{D}}$, and $J(z) < \alpha < 0$ then (25) holds.

Note that the following holds for all-to-all coupling

$$\dot{X} = (\tilde{J}(\tilde{z}) - \kappa N D)X,\ \ \tilde{J}(\tilde{z}) = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & J(x_2)|_{x_2=z_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J(x_N)|_{x_N=z_N} \end{bmatrix}. \tag{26}$$

Moreover, consider the Lyapunov function given by

$$V(X) = \frac{1}{2}X^{\mathrm{T}}\tilde{P}(\tilde{z})X > 0,\ \ X \neq 0,\ \ \tilde{P}(\tilde{z}) = \begin{bmatrix} P(z_1) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & P(z_N) \end{bmatrix},\ \ z_1, \dots, z_n \in \overline{\mathcal{D}}.$$

It follows that

$$\dot{V}(X) = X^{\mathrm{T}}\left(\tilde{P}(\tilde{z})\tilde{J}(\tilde{z}) - \kappa N\tilde{P}(\tilde{z})D + \frac{1}{2}\tilde{P}'(\tilde{z})\right)X.$$

Hence, $\dot{V}(X) < 0$ since (23) holds, which implies that (3) completely synchronises as $t \to \infty$ and concludes the proof. □

Now, condition (23) compromises a feasibility problem that we implement together with the requirement of Remark 3.3 as follows using the sum of squares decomposition:

$$\begin{array}{rl} \text{given} & J(\cdot),\ N,\ D,\ \kappa,\ \delta > 0 \\ \text{search for} & P(z) \\ \text{subject to} & v^{\mathrm{T}}(P(z) - \delta I)v,\ v^{\mathrm{T}}\left(P(z)D + \frac{1}{2}\bar{P}'(z)\right)v \text{ are SOS } \forall z, v \in \mathbb{R}^n \\ & -v^{\mathrm{T}}\left(P(z)(J(z) - N\kappa D) + \frac{1}{2}P'(z) + \delta I\right)v \text{ is SOS } \forall z, v \in \mathbb{R}^n, \end{array} \tag{27}$$

where

$$P'_{j\ell}(z) = \sum_{k=1}^{n} \frac{\partial P_{j\ell}(z)}{\partial z_k}(f_k(z) - (\kappa N D z)_k) \text{ and } \bar{P}'_{j\ell}(z) = \sum_{k=1}^{n} \frac{\partial P_{j\ell}(z)}{\partial z_k}\ (Dz)_k.$$

3.3.2. A network of coupled van der Pol oscillators

In [31], it was shown that in a network of coupled van der Pol oscillators representing individual heart cells, one group of coupled oscillators could represent the right atrium called sino-atrial node. This cell aggregate generates the normal cardiac rhythm. In addition, there is another pacemaker, the atrio-ventricular node, which takes over when the former fails to perform well. As discussed in [31], this node could be represented by the other group and both groups interact with each other. Investigating conditions for synchrony of the two heart cell aggregates is of major importance as unsynchronised behaviour is associated with cardiac dysrhythmia, a life threatening heart disease. The following analysis provides valuable information about conditions for which synchronised behaviour between heart cells is guaranteed.

Consider an all-to-all coupling scheme for a network of coupled identical van der Pol oscillators. The individual oscillator with $k = -1$ is described by:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= (1 - x_1^2)x_2 - x_1, \end{aligned} \tag{28}$$

The Jacobian of an individual oscillator is given by

$$J_v = \begin{bmatrix} 0 & 1 \\ -(2x_1x_2 + 1) & 1 - x_1^2 \end{bmatrix}.$$

The equations for the coupled system are given by (28) together with (3), where $D = \text{diag}([0\ 1])$.

Let us assume that the coupled system has a compact, convex and positively invariant set. Note that the origin is an unstable equilibrium point of the coupled system. Thus, the unique synchronised state corresponds to the limit cycle that is the union of the limit cycles of the individual uncoupled oscillators. In order to apply the result of Lemma 3.6, we search for a matrix $P(x) > 0$ that guarantees complete synchronisation of the coupled system. Moreover, we are interested to see whether the value of κ decreases as we increase the order of the polynomial functions which are entries to $P(x)$. The results are as follows (note that $P(x)$ is a sum of squares only if the polynomials are of even order [14]):

- For polynomials of order 0, a constant matrix $P(x) = P > 0$ and a constant $\kappa > 0$ such that (23) holds was not found.
- For polynomials of order 2, a matrix $P(x) > 0$ and a constant $\kappa > 0$ such that (23) holds was not found.
- For polynomials of order 4, inequality (23) holds if $\kappa N > 1$. Note that this case corresponds to the van der Pol oscillator with $\dot{x}_2 = (-k^* - x_1^2)x_2 - x_1$, $1 - \kappa \equiv k^* > 0$.

For $\kappa = 1.1$:

$$\begin{aligned}
P_{(1,1)}(x) &= 1.748 + 1.102x_1^2 - 1.379x_1x_2 + 1.473x_2^2 + 1.321x_1^4 \\
&\quad +0.452x_1^2x_2^2 - 0.037x_1x_2^3 + 0.002x_2^4, \\
P_{(1,2)}(x) &= 0.056 + 1.506x_1^2 - 0.935x_1x_2 + 0.145x_2^2 - 0.101x_1^3x_2 \\
&\quad +0.028x_1^2x_2^2 - 0.001x_1x_2^3, \\
P_{(2,2)}(x) &= 1.761 + 1.211x_1^2 - 0.460x_1x_2 + 0.064x_2^2 + 0.085x_1^4 - 0.019x_1^3x_2 \\
&\quad +0.002x_1^2x_2^2.
\end{aligned}$$

- Additional increases in polynomial order fail to lower the value of κ for which (23) holds. This was expected since $\kappa N < 1$ corresponds to the case when the fixed point of the van der Pol oscillator with $\dot{x}_2 = (-k^* - x_1^2)x_2 - x_1$, $k^* < 0$. Therefore, a $P(x)$ such that (23) holds cannot exist for $\kappa N < 1$.

For $N = 2$, numerical simulations indcate that complete synchronisation occurs if $\kappa \geq 0.017$ (see, Figure 3). Unsurprisingly, our condition is conservative (the numerical value seems to be about 30 times lower) but it provides guaranteed complete synchronisation.

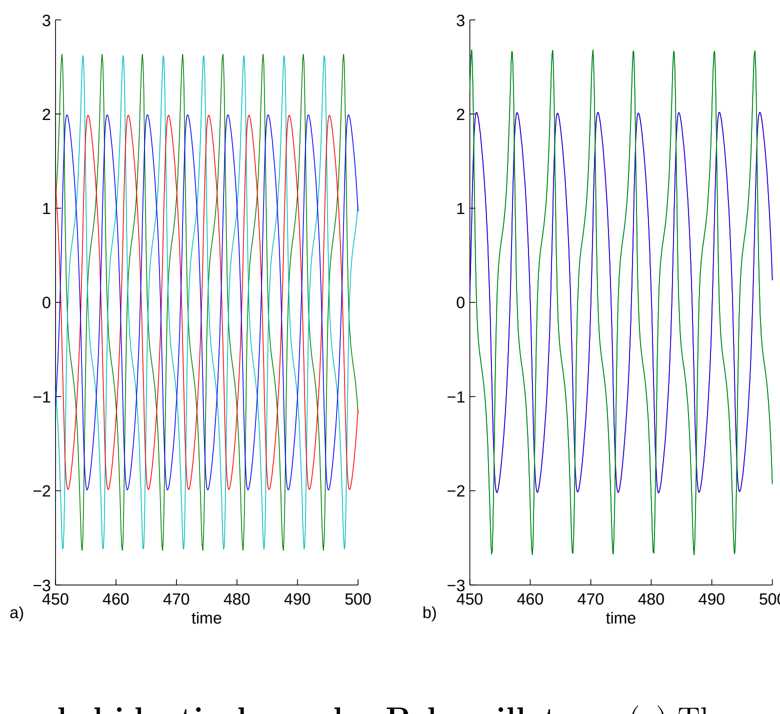

Figure 3. **Two coupled identical van der Pol oscillators.** (a) The coupling strength is $\kappa = 0.01$ and the two coupled van der Pol oscillators do not seem to synchronise, as their two solution trajectories remain distinct. (b) The two coupled identical van der Pol oscillators seem to completely synchronise for $\kappa = 0.02$ (the solution trajectories of the two systems are indistinguishable from each other).

3.3.3. Sufficient conditions based on the Bendixson's Criterion for higher dimensions

In this section, we provide novel sufficient conditions for global complete synchronisation of coupled identical oscillators. They are based on the so called Bendixson's Criterion for higher dimensions [10,11]. Before presenting them, we require the following definition from [11]. The *second additive compound* $A^{[2]}$ of matrix $A \in \mathbb{R}^{n\times n}$ is the $\binom{n}{2} \times \binom{n}{2}$ matrix defined as follows:

For any integer $i = 1, \ldots, \binom{n}{2}$, let $(i) = (i_1, i_2)$ be the ith member in the lexicographic ordering of integer pairs (i_1, i_2) such that $1 \le i_1 < i_2 \le n$. Then, the element of the ith row and jth column of $A^{[2]}$ is

$$\begin{cases} A_{i_1 i_1} + A_{i_2 i_2} & \text{if } (j) = (i), \\ (-1)^{r+s} A_{i_r j_s} & \text{if exactly one entry } i_r \text{ of } (i) \text{ does not} \\ & \text{occur in } (j) \text{ and } j_s \text{ does not occur in } (i), \\ 0 & \text{if neither entry from } (i) \text{ occurs in } (j). \end{cases}$$

For example, if $n = 3$ then $(1) = (1, 2)$, $(2) = (1, 3)$, $(3) = (2, 3)$ and

$$A^{[2]} = \begin{bmatrix} A_{11} + A_{22} & A_{23} & -A_{13} \\ A_{32} & A_{11} + A_{33} & A_{12} \\ -A_{31} & A_{21} & A_{22} + A_{33} \end{bmatrix}.$$

Note that $(A + B)^{[2]} = A^{[2]} + B^{[2]}$ and that the eigenvalues of $\frac{1}{2}(A + A^{\mathrm{T}})^{[2]}$ are given by $\lambda_i + \lambda_j$, where λ_i, λ_j are the eigenvalues of $\frac{1}{2}(A + A^{\mathrm{T}})$, $1 \le i < j \le n$ [32].

Let $\mathcal{B} \subseteq \mathbb{R}^n$ be a compact, simply connected and positively invariant set of $\dot{x} = f(x)$, and $x \in \mathcal{B}$. Then, the following theorem by Li and Muldowney proves global asymptotic stability (Theorem 2.5 in [33] with equality (2.6) and inequality (2.7) from [11]).

Theorem 3.7. (Li's and Muldowney's theorem on global asymptotic stability) *Let the origin be the unique equilibrium point of*

$$\dot{x} = f(x), \ x \in \mathcal{B} \subset \mathbb{R}^n. \tag{29}$$

If there exists a $\binom{n}{2} \times \binom{n}{2}$ *matrix* $P(x)$ *and*

$$\frac{1}{2}P'(x) + P(x)\left(\frac{\partial f(x)}{\partial x}\right)^{[2]} < 0, \ \forall x \in \mathcal{B}, \tag{30}$$

then the origin is globally asymptotically stable. Here, $P'_{ij}(x) = \sum_{k=1}^{n} \frac{\partial P_{ij}(x)}{\partial x_k} f_k(x)$.

Using the results of Theorem 3.7,[3] we obtain the following theorem:

[3]Note that Theorem 3.7 implies that if (30) holds then periodic solutions cannot exist. Moreover, this is also true if $\frac{1}{2}P'(x) + P(x)\left(\frac{\partial f(x)}{\partial x}\right)^{[2]} > 0, \ \forall x \in \mathcal{B}$ (see [10,11]). Thus, if any of the two inequalities holds then periodic solutions cannot exist, which means that this is a version of Bendixson's Criterion (which is for $n = 2$) for systems of higher dimensions ($n > 2$).

Theorem 3.8. *Consider the coupled system given by (3). Let $x_i \in \mathcal{D}$, $\mathcal{D}$ be a compact, connected and positively invariant set of (3), $\mathcal{D} \subset \overline{\mathcal{D}}$ and $\overline{\mathcal{D}}$ be convex. If the origin is the unique equilibrium point of (26) and there exists a matrix $P(z) > 0$ and a coupling strength κ^* such that*

$$\frac{1}{2}P'(z) + P(z)(J(z) - N\kappa^* D)^{[2]} < 0 \tag{31}$$

for all $z \in \overline{\mathcal{D}}$ then (3) completely synchronies as $t \to \infty$.

Proof. The proof uses Theorem 3.7 and is otherwise similar to the proof of Lemma 3.6 and hence, is omitted. □

Remark 3.9. Let λ_i denote an eigenvalue of $J(z) + J(z)^{\mathrm{T}} - 2N\kappa^* D,\ i = 1, \ldots, n$. Then condition (31) holds for a constant P if $\sup\{\lambda_i + \lambda_j\} < 0$, which clearly holds if (23) holds for a constant P but also if there exists a $\lambda_i > 0$ such that $\lambda_i < -\lambda_j$ for all j, $j \neq i$. Thus, inequality (31) connotes a more relaxed requirement than inequality (23).

A network of coupled Lorenz oscillators

For two coupled identical Lorenz oscillators with all-to-all coupling, we first show that the origin is the unique equilibrium point of (26) by using the sum of squares decomposition to show that for all $y \in \mathrm{R}^n$ and all $z \in \overline{\mathcal{D}}$, where $\overline{\mathcal{D}}$ is given by (13), the following holds:

$$\|(J(z) - N\kappa^* D)y\|_2^2 > 0 \tag{32}$$

for $N = 2$ and different $\kappa^* \geq 160$. When applying Theorem 3.8, we obtain that global complete synchronisation can be guaranteed for any $\kappa^* \geq 160$ if (32) holds. The following table summarises our results for coupled identical Lorenz oscillators and also provides an excellent opportunity to display some of the achievements of the research work presented in this paper (considering that, so far, only the first result could be obtained from the literature):

Table 3
Comparing the different approach present in the paper

$N = 2$	reference [5]	Theorem 3.1/Lemma 3.6	Theorem 3.8	numerics
κ^*	525	222.6	160	135

Comparing the different approaches to obtain the minimal value of κ ($= \kappa^*$) that guarantees global complete synchronisation of all-to-all coupled identical Lorenz oscillators.

In summary, we could improve the result from the literature and obtain now a value of κ^* that guarantees global complete synchronisation which is almost the value we observe in numerical trials.

4. Conclusions

In Section 3, we presented sufficient conditions for global complete synchronisation. As a novelty, we reformulated them to obtain certificates that guarantee synchronisation computationally. Using state of the art computational tools, we improved previous results on the lower bound of the coupling strength required for the synchronisation of coupled Lorenz oscillators. The example in Section 3.3.2 highlights the necessity of advanced computational techniques in order to guarantee synchronisation of, for example, coupled van der Pol oscillators. In Section 3.3.3, we provided new results that guarantee global complete synchronisation for all-to-all coupling and showed how to implement them computationally. The required conditions on the system are more relaxed than previously published ones and, thus, lead to lower values of the coupling strength that guarantees global complete synchronisation. For instance, we lowered the previously known value of this coupling strength for coupled Lorenz oscillators and obtain now a value which is almost the value we observe in numerical trials. Importantly, the certificates were obtained by analysing general properties of the connection network and the model representing an individual system only, which means that the analysis cost is almost independent of the size of the network (only the cost of computing the eigenvalues of the network's Laplacian increases with its size).

We would like to emphasise that we used a system of coupled Lorenz oscillators for illustration because this is a widely used system in the field. However, our method is general and can be applied to many systems. Particularly, for applications in engineering that require synchronised behaviour, consensus or so called stable flocking behaviour [34], it is of outmost importance to be able to provide certificates for such behaviour. In many cases, this is impossible without computational tools such as the ones presented in this paper. Finally, the results for non-identical oscillators presented in [5] can also be reformulated along the lines of the approaches presented in this paper in order to implement them computationally. To this end, in Section 2.1.1 we described how to include uncertainties in parameters using the sum of squares decomposition.

Acknowledgments Research funded by EPSRC and SNF. We thanks the anonymous reviewers for their valuable comments.

REFERENCES

1. S. H. Strogatz. From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled oscillators. *Physica D*, 143:1–20, 2000.
2. S. Coombes. Neuronal networks with gap junctions: A study of piecewise linear planar neuron models. *SIAM J. Appl. Dyn. Syst.*, 7(3):1101–1129, 2008.
3. E. Steur, I. Tyukin, and H. Nijmeijer. Semi-passivity and synchronization of diffusively coupled neuronal oscillators. *Physica D: Nonlinear Phenomena*, 238(21):2119–2128, 2009.
4. L. M. Pecora and T. L. Carroll. Master Stability Functions for Synchronized Coupled Systems. *Physical Review Letters*, 80:2109–2112, 1998.
5. V. N. Belykh, I. V. Belykh, and M. Hasler. Connection graph stability method for synchronized coupled chaotic systems. *Physica D*, 195:159–187, 2004.

6. C. W. Wu. Synchronization in networks of nonlinear dynamical systems coupled via a directed graph. *Nonlinearity*, 18:1057–1064, 2005.
7. D. Gonze, S. Bernard, C. Waltermann, A. Kramer, and H. Herzel. Spontaneous Synchronization of Coupled Circadian Oscillators. *Biophysical Journal*, 89:120–129, 2005.
8. C. W. Wu. On two approaches to analyzing consensus in complex networks. *IEEE International Symposium on Circuits and Systems*, pages 2638–2641, 2007.
9. S. J. Chung and J.-J. E. Slotine. Cooperative Robot Control and Concurrent Synchronization of Lagrangian Systems. *IEEE Transactions on Robotics*, 25(3):686–700, 2009.
10. R. A. Smith. Some applications of Hausdorff dimension inequalities for ordinary differential equations. *Proc. Roy. Soc. Edinburgh Sect. A*, 104:235–259, 1986.
11. Y. Li and J. S. Muldowney. On Bendixson's criterion. *Journal of Differential Equations*, 106:27–93, 1993.
12. C. W. Wu. *On a matrix inequality and its application to the synchronization in coupled chaotic systems* in *Complex Computing-Networks: Brain-like and Wave-oriented Electrodynamic Algorithms*, volume 104, pages 279–288. Proceedings in Physics, 2006.
13. W. Lohmiller and J.-J. E. Slotine. On contraction analysis for nonlinear systems. *Automatica*, 34(6):683–696, 1998.
14. E. M. Aylward, P. A. Parrilo, and J.-J. E. Slotine. Stability and Robustness Analysis of Nonlinear Systems via Contraction Metrics and SOS Programming. *Automatica*, 44(8):2163–2170, 2008.
15. L. Vandenberghe and S. Boyd. Semidefinite programming. *SIAM Review*, 38(1):49–95, 1996.
16. S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, Cambridge, UK, 2004.
17. K. G. Murty and S. N. Kabadi. Some NP-complete problems in quadratic and nonlinear programming. *Math. Program.*, 39:117–129, 1987.
18. P. A. Parrilo. Semidefinite programming relaxations for semialgebraic problems. *Math. Program., Ser. B*, 96:293–320, 2003.
19. S. Prajna, A. Papachristodoulou, and P. A. Parrilo. SOSTOOLS – Sum of Squares Optimization Toolbox, User's Guide. Available at `http://www.cds.caltech.edu/sostools`. 2002.
20. J. F. Sturm. Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. *Optimization Methods and Software*, 11–12:625–653, 1999.
21. C. W. Wu. Synchronization in an array of chaotic systems coupled via a directed graph. *IEEE International Symposium on Circuits and Systems*, 6:6046–6049, 2005.
22. I. Belykh, V. Belykh, K. Nevidin, and M. Hasler. Persistent clusters in lattices of coupled nonidentical chaotic systems. *Chaos*, 13(1):165–178, 2003.
23. C. W. Wu. Perturbation of coupling matrices and its effect on the synchronizability in arrays of coupled chaotic systems. *Physics Letters A*, 319:495–503, 2003.
24. H. K. Khalil. *Nonlinear Systems*. Prentice-Hall, Upper Saddle River, New Jersey, 3^{rd} edition, 2000.
25. M. Barahona and L. M. Pecora. Synchronization in Small-World Systems. *Physical Review Letters*, 89(5):1–4, 2002.

26. D. Li, J. Lu, X. Wu, and G. Chen. Estimating the ultimate bound and positively invariant set for the Lorenz system and a unified chaotic system.
27. J. Löfberg. YALMIP: A Toolbox for Modeling and Optimization in MATLAB. In *Proceedings of the CACSD Conference*, Taipei, Taiwan, 2004. Available from `http://control.ee.ethz.ch/~joloef/yalmip.php`.
28. D. C. Lewis. Metric properties of differential equations. *American Journal of Mathematics*, 71:294–312, 1949.
29. J. Jouffroy. Some ancestors of contraction analysis. *Proc. IEEE Conf. Dec. Contr.*, 2005.
30. W. Wang and J.-J. E. Slotine. On partial contraction analysis for coupled nonlinear oscillators. *Biological Cybernetics*, 92:38–53, 2005.
31. A. M. dos Santos, S. R. Lopes, and R. L. Viana. Rhythm synchronization and chaotic modulation of coupled Van der Pol oscillators in a model for the heartbeat. *Physica A*, 338:335–355, 2004.
32. J. S. Muldowney. Compound matrices and ordinary differential equations. *Rocky Mountain J. Math.*, 120:857–872, 1990.
33. M. Y. Li and J. S. Muldowney. A Geometric Approach to Global-Stability Problems. *SIAM J. Math. Anal.*, 27(4):1070–1083, 1996.
34. U. Münz, A. Papachristodoulou, and F. Allgöwer. Delay-Dependent Rendezvous and Flocking of Large Scale Multi-Agent Systems with Communication Delays. In *Proceedings of the 47^{th} IEEE Conference on Decision and Control*, pages 2038–2043, Cancun, Mexico, 2008.