Obtaining Cubatures for Rectangles and Other Planar Regions by Using Orthogonal Polynomials*

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Abstract. A. H. Stroud has recently shown the existence of cubature formulas for planar regions which use m^2 points and have polynomial precision $2m - 1$. In this paper, the author gives sufficient conditions for the existence of formulas using fewer than m^2 points, and having polynomial precision $2m - 1$. An algorithm is given for computing such formulas, and is shown to be useful in a more general setting than given in the theorem. Numerical examples are given, both in terms of previously known and new cubature formulas.

1. **Introduction.** A number of authors have attempted to use the common zeros of orthogonal polynomials in two variables as evaluation points in cubature formulas for regions in the plane. A theorem given recently by Stroud [10] shows how to construct formulas using orthogonal polynomials of degree m if they have exactly m^2 distinct common zeros. These formulas approximate the integrals of polynomials of degree $\leq 2m - 1$ exactly. The principal result of this paper is to give sufficient conditions for the existence of similar formulas which use fewer than m^2 points.

The following notation and definitions will be used in this paper.

R_2 — A region in the plane.

w — A weight function defined, and nonnegative, on R_2 such that the integrals $\int_{R_2} wx^\alpha y^\beta$ exist for all $\alpha, \beta \geq 0$.

$P_d, P_{d,i}$ — Polynomials of degree d.

$Q_d, Q_{d,i}$ — Polynomials of degree $\leq d$.

$P^{(\alpha, \beta)}$ — The orthogonal polynomials of the form $x^\alpha y^\beta + Q_{m-1}(x, y)$.

$S_{d,2}$ — A subset of pairs of nonnegative integers; $\{(\alpha, \beta): \alpha + \beta \leq d, \alpha, \beta \geq 0\}$.

S_N — A subset of $S_{d,2}$ containing N elements.

(x_i, y_i) — A common zero of two polynomials.

$X(S_N)$ — The matrix having rows $(x_1^\alpha y_1^\beta, x_2^\alpha y_2^\beta, \cdots, x_r^\alpha y_r^\beta)$, where $(\alpha, \beta) \in S_N$, and the (x_i, y_i) are specified.

$I(f)$ — $= \int_{R_2} wf$.

The cubature formulas we obtain will have the form

$$(1) \qquad\qquad I(f) \cong \sum_{k=1}^{N} A_k f(\mu_k).$$

We say that P_d is an *orthogonal polynomial*, with respect to a given region R_2 and weight function w, if $I(P_d Q_{d-1}) = 0$ for all Q_{d-1}. A cubature formula is said to have *precision d* if the approximation (1) is exact for all Q_d and is not exact for all P_{d+1}. A formula is said to be *fully symmetric* if the appearance of a point (a, b) in the formula implies the appearance of the points $(\pm a, \pm b)$, $(\pm b, \pm a)$ with the same weight. The set of points $\{(\pm a, \pm b), (\pm b, \pm a)\}$ will be denoted by $(a, b)_{FS}$.

2. Preliminary Discussion.
The following theorem was given by Stroud [10].

THEOREM 1. *Assume $P_{m,1}$ and $P_{m,2}$ are two polynomials in two variables with the following properties*:

(i) *Each $P_{m,i}$ is orthogonal to all Q_{m-1}; and*

(ii) *$P_{m,1}$ and $P_{m,2}$ have exactly m^2 common zeros, v_k, $k = 1, \cdots, m^2$, all of which are distinct and none of which are at infinity.*

Then there exist constants A_k such that

$$(2) \qquad I(Q_{2m-1}) = \sum_{k=1}^{m^2} A_k Q_{2m-1}(v_k)$$

for all Q_{2m-1}.

Numerous examples of this theorem can be given, including both previously known and new formulas. The cross product formulas for rectangles are a special case. Some particularly interesting examples are those which effectively use fewer than m^2 points by virtue of the fact that one or more of the weights are zero. The goal of this investigation is to give conditions ensuring the existence of formulas using fewer than m^2 points. These formulas do not always occur as a consequence of zero weights appearing in Theorem 1, however.

The weights of the cubature formula (2) can be found by solving the linear system of equations obtained by making (2) exact for some set of m^2 monomials of degree $\leq 2m - 1$. Let S_{m^2} be the set of pairs of integers corresponding to such a set of monomials. Then the only restriction on S_{m^2} is that the coefficient matrix $X(S_{m^2})$ be nonsingular. There is always such an S_{m^2} when the hypothesis of Theorem 1 is satisfied.

Certain information about the zeros of a polynomial in one variable will be needed. The following theorem may be found in Marden [7].

THEOREM 2. *Let $f(z) = a_0 + a_1 z + \cdots + z^m = \prod_{j=1}^{p} (z - z_j)^{m_j}$, $F(z) = (a_0 + \epsilon_0) + (a_1 + \epsilon_1)z + \cdots + z^m$ and let $0 < r_k < \min |z_k - z_j|$, $j = 1, \cdots, p$, $j \neq k$. Then there exists a positive number ϵ such that if $|\epsilon_i| \leq \epsilon$ for $i = 0, \cdots, m - 1$, then $F(z)$ has precisely m_j zeros in the circle c_k with center at z_k and radius r_k.*

Let $P(\zeta, z)$ be a polynomial in ζ and z. Consider P as a polynomial of degree m in ζ with coefficients which are polynomials in z. A function $\zeta(z)$ such that $P(\zeta(z), z) \equiv 0$ is called an algebraic function. Algebraic functions are discussed by Ahlfors [1], Bliss [3], and others. Briefly, the properties of $\zeta(z)$ we need are as follows. $\zeta(z)$ is an m-branched function, each branch of which is analytic at all points except a finite number of branch points and poles of finite order. At a branch point, $\zeta(z)$ is continuous, by Theorem 2, unless the coefficient of ζ^m in $P(\zeta, z)$ vanishes, in which case the function has a pole at the branch point. Algebraic functions form a field, hence a rational function of algebraic functions is again an algebraic function. We especially note that the only singularities that algebraic functions can have are branch points which may be poles of finite order, and poles of finite order.

For a brief discussion of the number of common zeros of two polynomials in two variables, and other algebraic geometry involved in Theorem 1, see Stroud [10].

3. **The Main Result.** For this section, let L_l denote the Legendre polynomial of degree l, with leading coefficient one. We then note that for the square whose vertices are $(\pm 1, \pm 1)$, and with unit weight function, the orthogonal polynomials $P^{(k,m-k)}$ are given by

$$(3) \qquad P^{(k,m-k)}(x, y) = L_k(x)L_{m-k}(y).$$

Let us consider the orthogonal polynomials $L_3(y)$ and $L_3(x) + \lambda L_1(x)L_2(y)$, where λ is a parameter. One can then solve for the common zeros as functions of λ, obtaining the points given in Table 1. One sees that for $\lambda \in (-9/5, 9/4)$, one can apply Theorem 1, and obtain a cubature formula of precision $2m - 1 = 5$, using those points. The weights of the formula are also given in Table 1.

TABLE 1

Point	Weight	
$(0,0)$	$\dfrac{16(4 + 5\lambda)}{9(9 + 5\lambda)}$	$a^2 = \dfrac{9 + 5\lambda}{15}$
$(\pm a, 0)$	$\dfrac{40}{9(9 + 5\lambda)}$	$b^2 = \dfrac{9 - 4\lambda}{15}$
$(\pm b, \pm c)$	$\dfrac{25}{9(9 - 4\lambda)}$	$c^2 = \dfrac{3}{5}$
$(0, \pm c)$	$\dfrac{40(1 - \lambda)}{9(9 - 4\lambda)}$	$\lambda \in \left(-\dfrac{9}{5}, \dfrac{9}{4}\right)$

For $\lambda = 0$, we obtain the cross product Gauss formula, which has positive weights. For $\lambda = 1$, we obtain Radon's formula [8]. The 7 points in Radon's formula have positive weights. For $\lambda = -4/5$, we obtain an 8-point formula, which so far as the author knows, was previously unknown, and which has positive weights. As λ approaches an endpoint of the interval $(-9/5, 9/4)$, two sets of weights become unbounded and common zeros coalesce.

This type of investigation partially motivated the following theorem.

THEOREM 3. *Let $R_2 = [a, b] \times [c, d]$ and $w(x, y) = u(x)v(y)$. Then, for $m \geq 3$, there exists a cubature formula of precision $2m - 1$ which uses fewer than m^2 points. Furthermore, the weights are all positive.*

Proof. Let $P_{l,1}(x)$ denote the orthogonal polynomial of degree l over $[a, b]$ with respect to $u(x)$. Let $P_{l,2}(y)$ denote the orthogonal polynomial of degree l over $[c, d]$ with respect to $v(y)$. We now consider the two orthogonal polynomials over R_2 with respect to $w(x, y)$,

(4) $P_{m,2}(y),$ $P_{m,1}(x) + \lambda P_{k,1}(x)P_{m-k,2}(y),$

where λ is a parameter, $0 < k < m$, and $m + k$ is an even integer. The common zeros of the polynomials (4) are of the form (x_{ij}, y_i), $i, j = 1, \cdots, m$, where the y_i are independent of λ, and the x_{ij} are continuous functions of λ, by Theorem 2.

We first establish that there is an interval J containing the origin such that $\lambda \in J$ implies the (x_{ij}, y_i) are real and distinct points. For $\lambda = 0$, the (x_{ij}, y_i) are real and distinct by the properties of orthogonal polynomials in one variable (see Jackson [6], for example). That the (x_{ij}, y_i) are real and distinct in an open neighborhood of the origin follows from continuity of the x_{ij} and the fact that complex x_{ij} appear in conjugate pairs. Thus, as λ increases, or decreases, from $\lambda = 0$, one encounters a multiple zero before one obtains complex zeros.

We must show that multiple zeros occur for a finite value of λ, i.e., that J is not the entire real axis. For multiple zeros to occur, the equations in λ and x,

(5)
$$P_{m,1}(x) + \lambda P_{k,1}(x)P_{m-k,2}(y_i) = 0,$$

$$P'_{m,1}(x) + \lambda P'_{k,1}(x)P_{m-k,2}(y_i) = 0,$$

must have a solution for one of the y_i. By the properties of orthogonal polynomials in one variable, $P_{m-k,2}(y_i) \neq 0$ for at least one of the y_i. We also know that no value of x is a zero of both $P_{k,1}(x)$ and $P'_{k,1}(x)$. Eliminating λ from the system (5), we obtain

$$P'_{m,1}(x)P_{k,1}(x) - P_{m,1}(x)P'_{k,1}(x) = 0.$$

The degree of the equation is $m + k - 1$, an odd number. Thus, the polynomial has at least one real root, x^*. Since one of $P_{k,1}(x^*)$ or $P'_{k,1}(x^*)$ is nonzero, we can solve for the corresponding value of λ, λ^* which is finite.

In practice, one might want to find the largest interval J containing zero for which the common zeros of (4) are real and distinct. We need only be concerned here that such an interval exists, and that it is not the entire real line.

Let J denote the largest interval (it is clearly open) such that $\lambda \in J$ implies the polynomials (4) have real and distinct common zeros. We note that, for any finite value of λ, the common zeros are finite. By Theorem 1, there is a cubature formula of precision $2m - 1$ which uses the points (x_{ij}, y_i) as evaluation points.

We now prove a lemma which will allow us to explicitly state the form of the weights in the formula as functions of the x_{ij}, and thus as functions of λ.

LEMMA 4. *Let $S_{m^2} = \{(\alpha, \beta): 0 \leq \alpha, \beta < m\}$ and let (x_{ij}, y_i), $i, j = 1, \cdots, m$, be the common zeros of (4), corresponding to $\lambda \in J$. Then $X(S_{m^2})$ is nonsingular.*

Proof. Suppose there exists a $\lambda \in J$ such that $X(S_{m^2})$ is singular. This is equivalent to the existence of a nontrivial polynomial $Q_{2m-2}(x, y) = \sum_{(\alpha, \beta) \in S_{m^2}} a_{(\alpha, \beta)} x^\alpha y^\beta$ which vanishes at every point (x_{ij}, y_i). Thus, for every j, $Q_{2m-2}(x, y_i)$ is zero at x_{ij}, $i = 1, \cdots, m$. Since those x_{ij} are distinct and $Q_{2m-2}(x, y_i)$ has degree $\leq m - 1$ in x, we have $Q_{2m-2}(x, y_i) \equiv 0$ for $j = 1, \cdots, m$. But the coefficient of each power of x in Q_{2m-2} has degree $\leq m - 1$ in y. Thus $Q_{2m-2}(x, y) \equiv 0$, a contradiction. $\square$

On the basis of Lemma 4, we may solve the weights, $A_{ij}(\lambda)$, by making the approximation exact for monomials corresponding to elements of S_{m^2}. Thus, we are spared any difficulties which could arise from having to change that set in the interval J.

We note that each $A_{ij}(\lambda)$ can be expressed as a rational function of the $x_{ij}(\lambda)$. Since each x_{ij} is an algebraic function of λ, so is A_{ij}.

The proof now breaks into two cases: case (i) for some $\lambda \in J$, one or more of the $A_{ij}(\lambda) = 0$; case (ii) for all $\lambda \in J$, $A_{ij}(\lambda) \neq 0$, $i, j = 1, \cdots, m$. Assume case (i) is encountered. Let $\lambda^* \in J$ be the smallest number in absolute value such that one or more $A_{ij}(\lambda^*) = 0$. For $\lambda \in (-|\lambda^*|, |\lambda^*|)$ all $A_{ij}(\lambda) > 0$, by the continuity of the A_{ij}. Thus $A_{ij}(\lambda^*) \geqq 0$ for all $i, j = 1, \cdots, m$ and by choice of λ^*, at least one $A_{ij}(\lambda^*)$ is zero. Thus the resulting cubature formula effectively uses fewer than m^2 points.

Now assume that case (ii) is encountered. We first note that for $\lambda \in J$, $A_{ij} > 0$, for $i, j = 1, \cdots, m$, by continuity. Let $\lambda^* \in$ closure (J), $\lambda^* \notin J$, where λ^* is finite. Then for some j, i, and i', $x_{ij}(\lambda^*) = x_{i'j}(\lambda^*)$. We will show that $\lim_{\lambda \to \lambda^*; \lambda \in J} A_{ij}(\lambda)$ exists for each $i, j = 1, \cdots, m$.

Precision for constants tells us that $\sum_{i,j=1}^{m} A_{ij}(\lambda) = I(1)$, and hence each $A_{ij}(\lambda)$ is bounded for $\lambda \in J$. The only singularities possessed by algebraic functions are poles of finite order. $\lambda = \lambda^*$ is a branch point of some of the $x_{ij}(\lambda)$. Since it is a pole of at most finite order, and since each $A_{ij}(\lambda)$ is bounded as $\lambda \to \lambda^*$ through values in J, we see that $\lim_{\lambda \to \lambda^*; \lambda \in J} A_{ij}(\lambda)$ exists. Let $A_{ij}^* = \lim_{\lambda \to \lambda^*; \lambda \in J} A_{ij}(\lambda)$, and let $x_{ij}^* = x_{ij}(\lambda^*)$. Then we have

$$(6) \qquad I(Q_{2m-1}) = \sum_{i,j=1}^{m} A_{ij}^* Q_{2m-1}(x_{ij}^*, y_j)$$

for all Q_{2m-1}, since (6) is obtained by taking the limit on the right side of

$$I(Q_{2m-1}) = \sum_{i,j=1}^{m} A_{ij}(\lambda) Q_{2m-1}(x_{ij}(\lambda), y_j),$$

which holds for all Q_{2m-1}, for all $\lambda \in J$. Since some of the (x_{ij}, y_j) coincide, (6) effectively uses fewer than m^2 points. Since $A_{ij}(\lambda) > 0$ for all $\lambda \in J$, $A_{ij}^* \geqq 0$, for $i, j = 1, \cdots, m$. $\square$

4. Numerical Examples. The proof of Theorem 3 indicates an algorithm for computing formulas using fewer than m^2 points. Starting with the cross product ($\lambda = 0$), one can easily compute the common zeros of the polynomials (4), then the weights, as λ varies. For large values of m, the method may break down numerically since the matrix $X(S_{m^*})$ is ill conditioned. One can observe the behavior of the weights to determine whether case (i) or case (ii) applies. This procedure was used in the numerical examples which follow.

In the tables in this paper, the notation $(-1)0.123 \cdots$ means $0.123 \cdots \times 10^{-1}$. All calculations were done in double precision on the Univac 1108 at the University of Utah. The calculations were verified by checking the error in the approximation for those monomials which were to be approximated exactly. The errors were sufficiently small that the rounded values given in the tables should be accurate to the given number of digits.

Example 1. We consider the square with vertices at $(\pm 1, \pm 1)$ and weight function $w(x, y) = 1$. Letting L_l denote the Legendre polynomial of degree l with leading coefficient one, we consider the common zeros of

$$L_4(y), \qquad L_4(x) + \lambda L_2(x) L_2(y).$$

It is found that case (ii) applies, with

$$J = \left(-\frac{27}{490}(3(30)^{1/2} + 5), \frac{27}{490}(3(30)^{1/2} - 5)\right).$$

Two 14-point formulas of precision $2m - 1 = 7$ are obtained, corresponding to the endpoints of J. The approximate values of the points and weights are given in Table 2. Note that one formula has four points outside of the square.

<div align="center">TABLE 2</div>

x	y	Weight
$\pm$(1)0.10578 40123 71275	$\pm$ a	(-1)0.43784 15208 72291
$\pm$0.77459 66692 41483	$\pm$ b	0.36230 28638 12526
$\pm$0.46925 35221 27911	$\pm$ a	0.30407 06930 50225
0.00000 00000 00000	$\pm$ b	0.57968 45821 00041

$$\lambda = -\frac{27}{490}(3\sqrt{30} + 5)$$

$\pm$0.77459 66692 41483	$\pm$ a	0.19325 26917 43030
$\pm$0.91506 05233 80880	$\pm$ b	0.16904 99212 19002
0.00000 00000 00000	$\pm$ a	0.30920 43067 88848
$\pm$0.39619 10397 48320	$\pm$ b	0.48309 52336 43544

$$\lambda = \frac{27}{490}(3\sqrt{30} - 5)$$

$$a^2 = \frac{15 + 2\sqrt{30}}{35}, \quad a \cong 0.86113 63115 94053$$

$$b^2 = \frac{15 - 2\sqrt{30}}{35}, \quad b \cong 0.33998 10435 84856$$

Example 2. Let R_2 and w be the same as Example 1. Consider the common zeros of

$$L_5(y), \qquad L_5(x) + \lambda L_1(x)L_4(y).$$

It is found that case (i) applies, and again two formulas with fewer than $m^2 = 25$ points are obtained which have positive weights. These formulas are given in Table 3. A third formula is given; although it has a negative weight at the origin, it is interesting because it uses fewer points than either of the other two.

We also note that not all formulas obtained in case (i) have positive weights, which emphasizes the procedure used in the proof to assure positive weights.

The formulas have precision 9.

TABLE 3

x	y	Weight
$\pm$0.84592 77997 71709	$\pm$ a	(-1)0.70506 51405 64012
$\pm$0.62890 16367 32253	$\pm$ a	(-1)0.72112 15110 07611
$\pm$0.95968 14212 14621	$\pm$ b	(-1)0.97149 27360 37507
$\pm$0.43603 05962 73468	$\pm$ b	0.36854 90486 77049
$\pm$0.77459 66692 41483	0.0	0.31604 93827 16049
0.00000 00000 00000	$\pm$ a	0.18861 64397 98053
0.00000 00000 00000	$\pm$ b	(-1)0.25860 69643 71341
0.00000 00000 00000	0.0	0.50567 90123 45679

$\lambda \cong 0.80000\ 00000\ 00000;\quad 23$ points

x	y	Weight
$\pm$0.94581 37395 19925	$\pm$ a	(-1)0.49929 06230 65150
$\pm$0.46534 66248 36203	$\pm$ a	0.15844 51822 84802
$\pm$0.80425 39257 42002	$\pm$ b	0.18338 37881 51247
$\pm$0.68138 58921 63677	$\pm$ b	(-1)0.88147 65236 65422
$\pm$0.96301 84090 85396	0.0	0.11445 63755 61331
$\pm$0.42861 01432 23121	0.0	0.45443 25133 27558
0.00000 00000 00000	$\pm$ a	(-1)0.57105 28092 97435
0.00000 00000 00000	$\pm$ b	0.41419 44599 63155

$\lambda \cong 0.79012\ 34567\ 90123;\quad 24$ points

x	y	Weight
$\pm$0.94930 73500 01342	$\pm$ a	(-1)0.49452 20191 30682
$\pm$0.45817 75489 31134	$\pm$ a	0.16391 47318 81061
$\pm$0.77459 66692 41483	$\pm$ b	0.26590 48169 44092
$\pm$0.96777 69089 76724	0.0	0.11304 18390 46410
$\pm$0.41775 46715 02987	0.0	0.47992 22296 00720
0.00000 00000 00000	$\pm$ a	(-1)0.47119 90252 41204
0.00000 00000 00000	$\pm$ b	0.42544 77071 10548
0.00000 00000 00000	0.0	-(-1)0.48150 35951 64821

$\lambda \cong 0.87082\ 50331\ 67594;\quad 21$ points

$$a^2 = \frac{5}{9} + \frac{2}{63}\sqrt{70} \qquad a \cong 0.90617\ 98459\ 38664$$

$$b^2 = \frac{5}{9} - \frac{2}{63}\sqrt{70} \qquad b \cong 0.53846\ 93101\ 05683$$

Example 3. Let $R_2 = [0, \infty) \times [-1, 1]$ and $w(x, y) = e^{-x}/(1 - y^2)^{1/2}$. This example was chosen to illustrate the theorem for a nontrivial weight function on a nonsymmetric region. Let $\mathcal{L}_l$ denote the Laguerre polynomial of degree l with leading coefficient one, and T_l the Chebyshev polynomial of degree l with leading coefficient one. We consider the common zeros of

$$T_4(y), \qquad \mathcal{L}_4(x) + \lambda \mathcal{L}_2(x) T_2(y).$$

During the computations investigating the behavior of the weights, it was first thought that case (ii) applied. However, closer numerical investigation revealed that case (i)
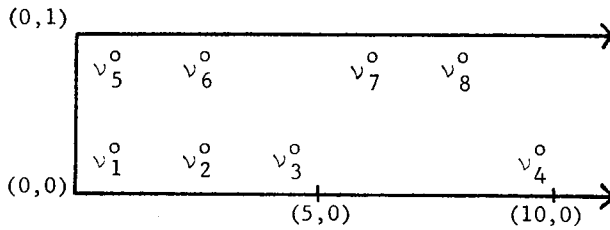
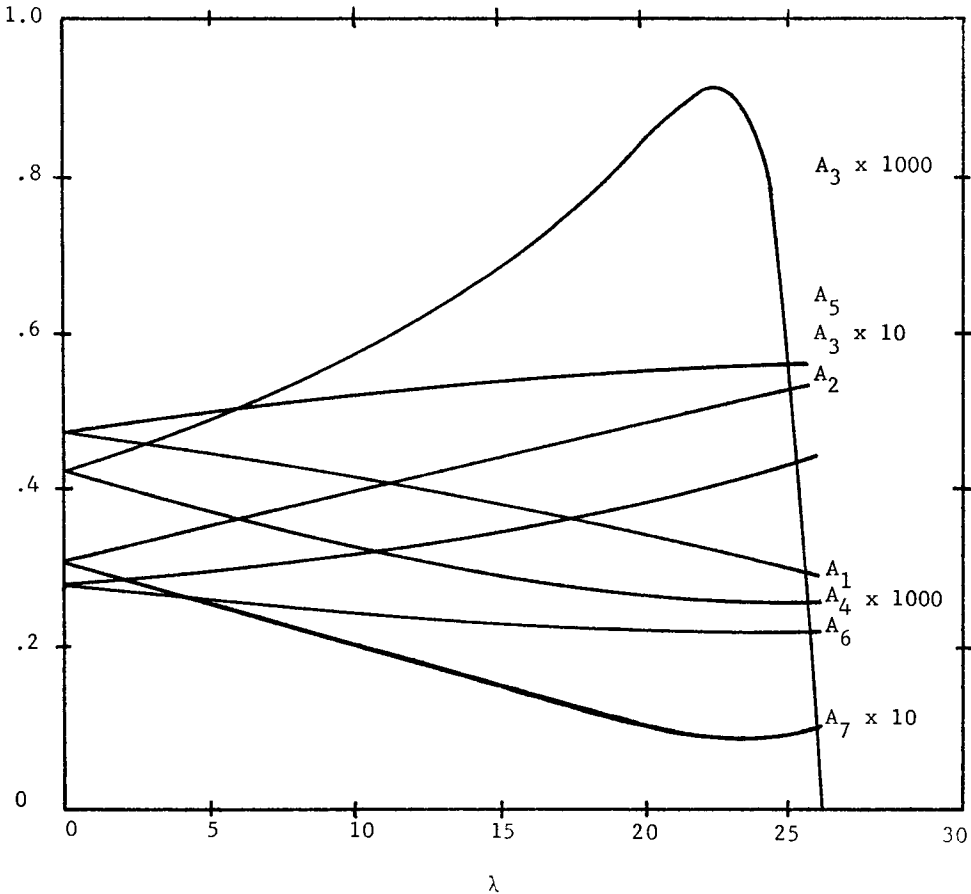General location of common zeros, $\lambda > 0$

FIGURE 1

applied. Because of the unexpected behavior of the weights, their values as functions of λ are shown in Figure 1. One of the two formulas is given in Table 4, and corresponds to $\lambda^* = (2)0.25455\ 84412\ 27157$. The second is obtained by interchanging a and b in the evaluation points, and corresponds to $\lambda = -\lambda^*$. The formulas use 14 points and have precision 7.

<div align="center">TABLE 4</div>

x	y	Weight
0.11303 28614 59250	$\pm\ a$	0.29900 62097 72373
(1)0.12290 70412 16527	$\pm\ a$	0.43310 01383 75078
(1)0.40847 50367 25518	$\pm\ a$	(-1)0.53044 24891 47755
(2)0.10573 14635 91203	$\pm\ a$	(-3)0.24756 63352 22222
0.41577 45567 83479	$\pm\ b$	0.55849 11440 03136
(1)0.22942 80360 27904	$\pm\ b$	0.21874 73164 18902
(1)0.62899 45082 93747	$\pm\ b$	(-2)0.81597 02975 41073

$$a^2 = \frac{2 - \sqrt{2}}{4} \qquad a \simeq 0.38268\ 34323\ 65090$$

$$b^2 = \frac{2 + \sqrt{2}}{4} \qquad b \simeq 0.92387\ 95325\ 11287$$

5. Generalizations. We note that in the proof of Theorem 3, we could just as well have considered the polynomials:

$$P_{m,1}(x), \qquad P_{m,2}(y) + \lambda P_{k,2}(y)P_{m-k,1}(x),$$

with the same restrictions on k. Which of the two sets of polynomials to be considered in practice might depend on the desire to preserve some symmetry.

Let us consider two situations which might lend themselves to a treatment similar to the proof of Theorem 3.

(S1) Suppose we have under consideration the square $[-a,\ a] \times [-a,\ a]$ with weight function $w(x,\ y) = u(x)u(y)$, where $u(-x) = u(x)$. Since the region and weight function are fully symmetric, it would be desirable to obtain a fully symmetric formula of precision $2m - 1$ which uses fewer than m^2 points. Accordingly, if P_l is orthogonal to all Q_{l-1} on $[-a,\ a]$ with respect to u, we could consider the orthogonal polynomials

$$(7) \qquad P_m(x) + \lambda P_k(x)P_{m-k}(y), \qquad P_m(y) + \lambda P_k(y)P_{m-k}(x),$$

where $m + k$ is even. The zeros are fully symmetric, and if they are distinct and finite, the cubature formula obtained by Theorem 1 is also fully symmetric, since it is unique.

(S2) Suppose R_2 is not a rectangle, and/or $w(x,\ y)$ is not a product of functions x and y, respectively. However, suppose that there are orthogonal polynomials $P_{m,1}(x,\ y)$ and $P_{m,2}(x,\ y)$ which satisfy the hypothesis of Theorem 1. Further, suppose the

common zeros are real, and the weights of the associated cubature formula are all positive. One could then consider the polynomials

(8) $$P_{m,1}(x, y) + \lambda P_{m,3}(x, y), \qquad P_{m,2}(x, y)$$

or the polynomials

(9) $$P_{m,1}(x, y) + \lambda P_{m,3}(x, y), \qquad P_{m,2}(x, y) + \lambda P_{m,4}(x, y),$$

where $P_{m,3}$ and $P_{m,4}$ may or may not be the same orthogonal polynomial, and neither is equal to $P_{m,1}$ or $P_{m,2}$.

The question in both situations is the same. By varying λ, can one obtain a cubature formula of precision $2m - 1$ which uses fewer than m^2 points, either through zero weights or coalescence of common zeros? A completely satisfying answer is unknown at this time. We will discuss each situation in turn, the first being the simpler. We note that in either case, complex zeros appear in conjugate pairs, since

$$\overline{Q_d(x, y)} = Q_d(\bar{x}, \bar{y}).$$

Let λ^* be a point where the common zeros of (7) are distinct. According to van der Waerden [13], the Jacobian is nonzero at each common zero. Hence, one can apply the Implicit Function Theorem (see Bochner and Martin [4]) and conclude that the common zeros are analytic functions of λ in a neighborhood of λ^*.

Assume that the leading coefficients of the P_i are all one. Then, a common zero at infinity would have to satisfy the following:

$$x^m + \lambda x^k y^{m-k} = x^k(x^{m-k} + \lambda y^{m-k}) = 0,$$
$$y^m + \lambda y^k x^{m-k} = y^k(y^{m-k} + \lambda x^{m-k}) = 0,$$

and we see that this would require $\lambda^2 = 1$. Thus, for $\lambda = \pm 1$, the polynomials (7) have common zeros on the line at infinity. Let J denote the largest interval containing the origin such that $\lambda \in J$ implies the polynomials (7) have m^2 real, distinct, and finite common zeros. Unless case (i) is obtained (zero weights), it would be necessary that J be properly contained in $(-1, 1)$. If that were the case, we would also need the common zeros to be continuous functions of λ at the point λ^* where some of the common zeros coalesce. The analogue of Theorem 2 for common zeros of two polynomials in two variables was not found in the literature. Forsythe [5, p. 212] proves a partial analogue of Theorem 2, where only the constant coefficients are allowed to vary. It is expected that the full analogue is true, but at the present we must make that assumption. We would also need to know that the limits of the weights as $\lambda \to \lambda^*$, $\lambda \in J$, exist.

In the second instance, one can tell nothing about the behavior of the common zeros of the polynomials (8) or (9) as functions of λ. Again, they are analytic in the neighborhood of any value of λ which yields m^2 distinct common zeros. Let J be defined as before, and assume continuity at a point where common zeros coalesce. Assume case (i) does not apply or we are finished. If case (ii) applies, we need J to be a proper subset of the real line; we need the common zeros of (8) or (9) to be uniformly bounded in a subinterval J' of J having a finite endpoint λ^*, which is also an endpoint of J. Again the limits of the weights $\lambda \to \lambda^*$, $\lambda \in J$, would have to exist to obtain the desired result.

We see that the first situation is contained in the second, although more can be

determined in the first situation. The above ideas can be incorporated into a theorem which is primarily useful as a guide to computations in search of cubature rules of precision $2m - 1$ using fewer than m^2 points.

THEOREM 5. *Suppose $P_{m,1}$ and $P_{m,2}$ satisfy the conditions of Theorem 1 and that $P_{m,3}$ and $P_{m,4}$ are orthogonal polynomials, neither of which is the same as $P_{m,1}$ or $P_{m,2}$. Further, suppose that the common zeros of $P_{m,1}$ and $P_{m,2}$ are all real and that the weights of the associated cubature rule are positive. Let J denote the largest interval containing the origin such that $\lambda \in J$ implies the common zeros of the polynomials (8) or (9) satisfy the hypothesis of Theorem 1. If the weights of the cubature rule obtained by Theorem 1 are all positive for all $\lambda \in J$, assume the following:*

(a) *that for $\lambda^* \in$ closure (J), $\lambda^* \notin J$, the polynomials (8) or (9) have a finite number of common zeros, none of which are at infinity;*

(b) *the common zeros are continuous functions of λ at the point $\lambda = \lambda^*$; and*

(c) *the limits of the weights as $\lambda \to \lambda^*$, $\lambda \in J$, exist.*

Then there exists a cubature formula of precision $2m - 1$ for R_2 and w which effectively uses fewer than m^2 points, and which has positive weights.

6. **Numerical Examples.** The procedure indicated by the theorem was attempted in a number of examples, with success in each instance. Both case (i) and case (ii) were encountered. Two of the examples involved numerical investigation somewhat beyond that indicated by the theorem.

Example 4. Let $R_2 = [-1, 1] \times [-1, 1]$ and $w(x, y) = 1$. Consider the polynomials

$$(10) \qquad L_4(x) + \lambda L_2(x)L_2(y), \qquad L_4(y) + \lambda L_2(y)L_2(x),$$

where L_i is the Legendre polynomial with leading coefficient one. The common zeros, and thus the cubature formulas, are fully symmetric.

For $\lambda \in (-27/35, 189/385)$, the common zeros are distinct and finite. In this range, the weights of the corresponding cubature formulas are always positive. In the limit, as $\lambda \to (189/385)-$, one obtains Tyler's formula [12], given in Table 5. For $\lambda = 189/385$, the polynomials (10) have multiple zeros on the x and y axes. If one takes the limit as $\lambda \to (-27/35)+$, one obtains a 13-point formula due to a coalescing of 4 zeros into the origin. Four of the points are outside the square, as can be seen in Table 6. The formulas in Tables 5 and 6 have precision 7.

TABLE 5

(*Tyler's Formula*)

Point	Weight	
$(r,0)_{FS}$	$\dfrac{98}{405}$	$r^2 = \dfrac{6}{7}$
$(s,s)_{FS}$	$\dfrac{178981 + 2769\sqrt{583}}{472230}$	$s^2 = \dfrac{114 - 3\sqrt{583}}{287}$
$(t,t)_{FS}$	$\dfrac{178981 - 2769\sqrt{583}}{472230}$	$t^2 = \dfrac{114 + 3\sqrt{583}}{287}$

TABLE 6

Point	Weight	
$(0,0)$	$\dfrac{392}{405}$	$a^2 = \dfrac{3}{2}$
$(a,a)_{FS}$	$\dfrac{16}{2025}$	$b^2 = \dfrac{3}{7}\left(1 + \sqrt{\dfrac{10}{31}}\ \right)$
$(b,c)_{FS}$	$\dfrac{1519}{4050}$	$c^2 = \dfrac{3}{7}\left(1 - \sqrt{\dfrac{10}{31}}\ \right)$

Example 5. Let $R_2 = [-1, 1] \times [-1, 1]$ and $w(x, y) = 1$. Consider the polynomials

(11) $L_5(x) + \lambda L_3(x)L_2(y) + \mu L_1(x)L_4(y)$, $L_5(y) + \lambda L_3(y)L_2(x) + \mu L_1(y)L_4(x)$.

The formulas obtained are fully symmetric. It was found that case (i) was obtained for a range of values for μ.

For $\mu \cong -0.53880\ 66531\ 67134$, $\lambda = 0$, a 21-point formula of precision 9 is obtained by virtue of a set of four weights (say A_1) being zero. We note that this formula has a negative weight at the origin. Now letting μ decrease, and finding a corresponding λ such that $A_1 = 0$, we eventually find that corresponding to $\mu \cong -0.75540\ 45432\ 80930$, $\lambda \cong 0.29165\ 21592\ 17893$, the weight at the origin has increased to zero, thus yielding a 20-point formula. The formula is given in Table 7, and was previously given by Rabinowitz and Richter [9], who computed it by a different method.

TABLE 7
(Rabinowitz-Richter Formula)

Point	Weight	
$(a,0)_{FS}$	$0.45409\ 03525\ 51545$	$a \cong 0.48888\ 63428\ 42372$
$(b,0)_{FS}$	$(-1)0.71613\ 42470\ 98110$	$b \cong 0.98453\ 98119\ 42252$
$(c.c)_{FS}$	$(-1)0.42784\ 61546\ 67780$	$c \cong 0.93956\ 72874\ 21522$
$(r,s)_{FS}$	$0.21575\ 58036\ 35933$	$r \cong 0.50737\ 67736\ 74613$
		$s \cong 0.83671\ 03250\ 23989$

For $\lambda = 0$, $\mu \cong -0.39506\ 17283\ 95062$, a 24-point formula is obtained through the weight at the origin being zero. By decreasing μ and selecting λ to maintain a zero weight at the origin, we expected that the formula of Table 7 would again be obtained. This was not the case. Another 20-point formula of precision 9 was found, corresponding to $\mu \cong -0.72380\ 71335\ 86019$, $\lambda \cong 0.26938\ 24624\ 64281$. Note that the values of μ and λ are near the values obtained above, however a different set of 4

weights are zero, which yields a formula with a different disposition of points than that obtained by Rabinowitz and Richter. The formula is given in Table 8.

TABLE 8

Point	Weight	
$(a,0)_{FS}$	0.45416 39606 86749	a $\cong$ 0.48892 68569 74369
$(b,b)_{FS}$	0.21420 03609 26862	b $\cong$ 0.69088 05504 86344
$(c,c)_{FS}$	(-1)0.42731 23186 57758	c $\cong$ 0.93956 52580 96838
		r $\cong$ 0.91862 04410 56722
$(r,s)_{FS}$	0.14445 22232 60307	s $\cong$ 0.34487 20253 64404

$\lambda \cong 0.26938\ 24624\ 64281,\ \mu \cong -0.72380\ 71335\ 86019$

Example 6. Let R_2 be the triangle with vertices at $(0, 0)$, $(1, 0)$, and $(0, 1)$, with weight function $w(x, y) = 1$. The following orthogonal polynomials were given by Appell and Kampé de Feriet [2], and may be found in Stroud [11].

$$P_{4,1}(x, y) = 70x^4 + 140x^3y + 90x^2y^2 + 20xy^3 + y^4$$
$$- 140x^3 - 180x^2y - 60xy^2 - 4y^3$$
$$+ 90x^2 + 60xy + 6y^2 - 20x - 4y + 1,$$

$$P_{4,2}(x, y) = 35x^4 + 160x^3y + 180x^2y^2 + 64xy^3 + 5y^4$$
$$- 80x^3 - 240x^2y - 144xy^2 - 16y^3$$
$$+ 60x^2 + 96xy + 18y^2 - 16x - 8y + 1,$$

$$P_{4,3}(x, y) = 15x^4 + 120x^3y + 216x^2y^2 + 120xy^3 + 15y^4$$
$$- 40x^3 - 216x^2y - 216xy^2 - 40y^3$$
$$+ 36x^2 + 108xy + 36y^2 - 12x - 12y + 1,$$

$$P_{4,4}(x, y) = P_{4,2}(y, x),$$

$$P_{4,5}(x, y) = P_{4,1}(y, x).$$

We consider the common zeros of two polynomials of the form

$$P_{4,6}(x, y) = \tau P_{4,1}(x, y) + \lambda P_{4,2}(x, y) + \mu P_{4,3}(x, y) + \gamma P_{4,4}(x, y),$$

$$(12) \qquad P_{4,7}(x, y) = \tau P_{4,5}(x, y) + \lambda P_{4,4}(x, y) + \mu P_{4,3}(x, y) + \gamma P_{4,2}(x, y)$$

$$= P_{4,6}(y, x)$$

in an attempt to find a formula of precision 7 which uses fewer than 16 points. The form of the polynomials (12) was chosen to obtain a formula with the same symmetry as the region, i.e., if $(a, b) \in R_2$, then $(b, a) \in R_2$, and if (a, b) is a common zero of (12), so is (b, a). Including $P_{4,5}(x, y)$ in the first and $P_{4,1}(x, y)$ in the second polynomial leads to no more generality than the above.

We found that the common zeros of (12) for $\tau = 1$, $\lambda = \mu = \gamma = 0$ are all real, distinct, and inside the triangle. However, the cubature rule obtained by using these points has two negative weights. By decreasing λ, decreasing μ, or increasing γ, while maintaining $\tau = 1$, and the other two of λ, μ, and γ at zero, it was found that the two negative weights increased, and became zero for some value of each of the three parameters. Thus, three different 14-point rules can be obtained in this manner, each of which has positive weights and one of which has all its points interior to the triangle. The latter formula is given in Table 9. Using the same procedure as in Example 5, an attempt was made to obtain a formula using fewer than 14 points. One approach which was used, but without success, was to start with the formula given in Table 9, change λ slightly, and obtain a new 14-point formula by proper choice of μ. Because this is simply a trial and error method, a complete investigation was impractical. However, it appeared that one of two things would happen: (1) a common zero would tend to infinity; or (2) a common zero would coalesce with the common zero which had a zero weight. Thus, the effort failed to improve on the results indicated by Theorem 5.

The same procedure was tried, starting with a 14-point formula occurring for $\lambda = -0.08$, $\mu \cong (-2)0.71524\ 25347\ 51567$. This formula is given in Table 10. While the disposition of the points is similar to that of Table 9, the zero weight occurs for a "different" point, as determined by its position relative to the other points. The points in the formula are interior to the triangle, although the two common zeros with zero weights are outside the triangle. Again no formula with fewer than 14 points was obtained, for the same reasons as in the other attempt.

A very large number of 16-point formulas were computed for various values of λ, μ,

TABLE 9

Point	Weight	
(a_1, a_1)	$(-1)0.26332\ 15013\ 60460$	$a_1 \cong (-1)0.64634\ 10980\ 16171$
(a_2, a_2)	$(-1)0.66675\ 06099\ 02085$	$a_2 \cong \ \ \ \ \ 0.25047\ 87642\ 60821$
(a_3, a_3)	$(-1)0.59839\ 84722\ 97514$	$a_3 \cong \ \ \ \ \ 0.40528\ 81131\ 34598$
(a_4, a_4)	$(-1)0.30224\ 43080\ 27287$	$a_4 \cong \ \ \ \ \ 0.48342\ 85070\ 60240$
(b_1, c_1)	$(-1)0.38713\ 91024\ 62897$	$b_1 \cong (-1)0.49024\ 15490\ 57468$
(c_1, b_1)		$c_1 \cong \ \ \ \ \ 0.31241\ 81290\ 02285$
(b_2, c_2)	$(-1)0.22310\ 31308\ 16147$	$b_2 \cong (-1)0.27265\ 49172\ 25016$
(c_2, b_2)		$c_2 \cong \ \ \ \ \ 0.64982\ 99188\ 30148$
(b_3, c_3)	$(-2)0.93095\ 64046\ 94027$	$b_3 \cong (-2)0.74809\ 20050\ 42521$
(c_3, b_3)		$c_3 \cong \ \ \ \ \ 0.92292\ 92246\ 98637$
(b_4, c_4)	$(-1)0.36538\ 29270\ 09296$	$b_4 \cong \ \ \ \ \ 0.16671\ 86876\ 51425$
(c_4, b_4)		$c_4 \cong \ \ \ \ \ 0.77579\ 68804\ 94268$
(b_5, c_5)	$(-1)0.51592\ 17534\ 48585$	$b_5 \cong \ \ \ \ \ 0.15196\ 95753\ 82297$
(c_5, b_5)		$c_5 \cong \ \ \ \ \ 0.56910\ 13418\ 00312$

$$\tau = 1, \ \lambda \cong -(-1)0.44573\ 47683\ 19604, \ \mu = \gamma = 0$$

TABLE 10

Point	Weight		
(a_1,a_1)	(-1)0.26081 61608 68233	$a_1 \cong$	(-1)0.64321 15701 15959
(a_2,a_2)	(-1)0.66360 23579 26664	$a_2 \cong$	0.25299 83313 85515
(a_3,a_3)	(-1)0.56173 11713 92644	$a_3 \cong$	0.40974 73142 94030
(a_4,a_4)	(-1)0.23952 62212 75731	$a_4 \cong$	0.48546 22879 28209
(b_1,c_1)	(-1)0.39457 61269 86614	$b_1 \cong$	(-1)0.50408 59335 70127
(c_1,b_1)		$c_1 \cong$	0.31055 48435 59296
(b_2,c_2)	(-1)0.23550 12185 40342	$b_2 \cong$	(-1)0.27912 25784 37840
(c_2,b_2)		$c_2 \cong$	0.64767 37979 23676
(b_3,c_3)	(-1)0.16265 84769 36259	$b_3 \cong$	(-1)0.36407 93788 27516
(c_3,b_3)		$c_3 \cong$	0.89578 58440 39319
(b_4,c_4)	(-1)0.31823 23669 04684	$b_4 \cong$	0.20234 89156 94331
(c_4,b_4)		$c_4 \cong$	0.74337 52526 60890
(b_5,c_5)	(-1)0.52619 38549 00464	$b_5 \cong$	0.16004 02557 10345
(c_5,b_5)		$c_5 \cong$	0.56485 12168 76248

$\tau = 1$, $\lambda = 0.08$, $\mu \cong$ (-2)0.71524 25347 51567, $\gamma = 0$

and γ in (12) in an effort to determine the behavior of the weights as functions of those parameters. Although the investigation was far from complete, none of the computations gave any indication of the existence of a formula using fewer than 14 points.

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