Submitted to the Annals of Applied Probability
arXiv: math.PR/0000000

OCCUPATION AND LOCAL TIMES FOR SKEW BROWNIAN MOTION WITH APPLICATIONS TO DISPERSION ACROSS AN INTERFACE*

By Thilanka Appuhamillage[†], Vrushali Bokil[†], Enrique Thomann[†], Edward Waymire[†] and Brian Wood[†]

Oregon State University[†]

Advective skew dispersion is a natural Markov process defined by a diffusion with drift across an interface of jump discontinuity in a piecewise constant diffusion coefficient. In the absence of drift this process may be represented as a function of α-skew Brownian motion for a uniquely determined value of $\alpha = \alpha^*$; see Ramirez, Thomann, Waymire, Haggerty and Wood (2006). In the present paper the analysis is extended to the case of non-zero drift. A determination of the (joint) distributions of key functionals of standard skew Brownian motion together with some associated probabilistic semigroup and local time theory is given for these purposes. An application to the dispersion of a solute concentration across an interface is provided that explains certain symmetries and asymmetries in recently reported laboratory experiments conducted at Lawrence-Livermore Berkeley Labs by Berkowitz, Cortis, Dror and Scher (2009).

1. Introduction. Skew Brownian motion was introduced by Itô and McKean (1963) to construct certain stochastic processes associated with Feller's classification of one-dimensional diffusions in terms of second order differential operators. This spawned further research leading to a number of subsequent foundational probability papers that highlight interesting and sometimes surprising special structure of skew Brownian motion, e.g., see Barlow, Burdzy, Kaspi and Mandelbaum (2001); Barlow, Pitman and Yor (1989); Burdzy and Chen (2001); Harrison and Shepp (1981); Le Gall (1984); Ouknine (1990); Ramirez (2009); Walsh (1978).

Skew Brownian motion has more recently emerged in connection with diverse applications ranging from a variant on the multi-arm bandit problem (Barlow, Burdzy, Kaspi and Mandelbaum (2000)), mathematical finance (Decamps, Goovaerts and Schoutens (2006)), Monte-Carlo simulation

*This research was partially supported by the grant GrantCMG EAR- 0724865 from the National Science Foundation.

AMS 2000 subject classifications: Primary 60K35, 60K35; secondary 60K35

Keywords and phrases: Skew Brownian motion, advection-diffusion, local time, occupation time, elastic skew Brownian motion, stochastic order, first passage time

schemes (Lejay and Martinez (2006)), and dispersion in heterogeneous media (Freidlin and Sheu (2000); Ramirez et al. (2006); Ramirez, Thomann, Waymire, Chastanet and Wood (2008)). The present paper provides new theoretical results for functionals of skew Brownian motion and its associated semi-group theory (pde), together with an application to recently reported laboratory experiments for advection-dispersion across a sharp interface by Berkowitz et al. (2009). A simple version of one of the basic issues in this regard may be posed as follows.

Question: Consider one-dimensional diffusion with two different diffusion coefficients, say $D^- < D^+$, on the left and right half-lines, respectively. Which is more likely to be removed first: a particle injected at -1 and removed at $+1$, or a particle injected at $+1$ and removed at -1 ?

We will see that the answer to this question is fundamentally tied to a corresponding effect of the interface on α-skew Brownian motion, where $\alpha = \alpha^*$ is a function of the diffusion coefficients D^- and D^+ to be determined, together with a delicate balance with its respective diffusive scalings to the left and right of the interface. In view of this basic role of skew Brownian motion, the paper is organized with an initial focus on new properties of skew Brownian motion to be followed by the more specific application to dispersion across an interface. Readers primarily interested in the application may skip from the end of this introductory section to section 5.

To set some notation and basic definitions, let $B = \{B_t : t \geq 0\}$ denote *standard Brownian motion* on a probability space $(\Omega, \mathcal{F}, P)$. Next let $J_1, J_2, \dots$ denote a fixed enumeration of the excursion intervals of the reflected process $\{|B_t| : t \geq 0\}$. For a given parameter $\alpha \in (0,1)$, let $\{A_m : m = 1, 2, \dots\}$ be an i.i.d. sequence, independent of B, of Bernoulli ± 1 valued random variables also defined on Ω with $P(A_1 = 1) = \alpha$. Define *α-skew Brownian motion* process $B^{(\alpha)} = \{B_t^{(\alpha)} : t \geq 0\}$ by

$$B_t^{(\alpha)} = \sum_{m=1}^{\infty} \mathbf{1}_{J_m}(t) A_m |B_t|, \tag{1.1}$$

where $\mathbf{1}_S$ denotes the indicator function of the set S. While skew Brownian motion is a continuous semi-martingale, Walsh (1978) showed that its local time is discontinuous. We note that throught the paper the cases $\alpha = 0$, and $\alpha = 1$ are excluded for simplicity of presentation. All the results in the paper are stated for $0 < \alpha < 1$, with obvious extensions to these other values of the parameter α.

Although the excursion representation (1.1) does not extend to define a "skew Brownian motion with drift", a number of natural alternatives are available.[1] We mention two. The first describes the fundamental process intrinsic to the application to dispersion in porous media from the perspective of semi-group theory. The second is an equivalent formulation from the perspective of martingale theory. In preparation for these descriptions and through the paper, we denote by $\mathbb{R}_0 = (-\infty, 0) \cup (0, \infty) = \mathbb{R}\backslash\{0\}$.

Definition 1.1. *For* $0 < \alpha < 1, v \in (-\infty, \infty)$*, the* $\alpha-$*skew Brownian motion with drift* v $B^{(\alpha,v)}$ *is the Markov process with continuous sample paths determined by the infinitesimal generator* $\frac{1}{2}\frac{d^2}{dx^2} + v\frac{d}{dx}$ *with domain* $\mathcal{D}_{\alpha,0} = \{u \in C^2(\mathbb{R}_0) \cap C(\mathbb{R}) : \alpha u'(0^+) = (1-\alpha)u'(0^-)\}$.

The existence of unique strong solutions to stochastic differential equations of the form

$$dY_t = (2\alpha - 1)dL_t^Y(0) + v\,dt + \,dB_t, \tag{1.2}$$

where $B = \{B_t : t \geq 0\}$ is standard Brownian motion, and $L_t^Y(0)$ denotes symmetric local time of the process Y at $y = 0$, was established by Le Gall (1984). One may also check that the interface condition in Definition 1.1 implies that for $f \in \mathcal{D}_{\alpha,0}$

$$M_t = f(Y_t) - \int_0^t \{\frac{1}{2}\frac{d^2}{dx^2} + v\frac{d}{dx}\}f(Y_s)ds, \quad t \geq 0, \tag{1.3}$$

defines a martingale. The following result is generally attributed to Le Gall (1982).

Theorem 1.1. $\alpha-$*skew Brownian motion with drift* v *is the unique strong solution* $Y = B^{(\alpha,v)}$ *to (1.2).*

In the study of properties of $B^{(\alpha,v)}$, we are naturally lead to introduce a process which we denote by ${}^{(\alpha,\gamma)}B$ and refer as γ-*elastic* α-*skew Brownian motion* (without drift) in analogy to elastic Brownian motion; e.g., see (Itô and McKean (1996), p. 45). To define this process, let $\gamma > 0, 0 < \alpha < 1$. The process ${}^{(\alpha,\gamma)}B$ is the Markov process with continuous sample paths with infinitesimal generator $\frac{1}{2}\frac{d^2}{dx^2}$on the domain $\mathcal{D}_{\alpha,\gamma}$, where

$$\mathcal{D}_{\alpha,\gamma} = \{u \in C^2(\mathbb{R}_0) \cap C(\mathbb{R}) : \alpha u'(0^+) - (1-\alpha)u'(0^-) = \gamma u(0)\}.$$

[1] To the best of our knowledge this process has not been previously named in the literature. However this terminology is consistent with the usual nomenclature associated with the infinitesimal generator.

The construction of elastic skew Brownian motion defines a process with sample paths in $[\ell_t^{(\alpha)} < R_\gamma] \subset C[0,\infty)$, where R_γ denotes an exponentially distributed random variable with parameter γ, independent of $B^{(\alpha)}$. In particular, the elastic skew Brownian motion agrees with the skew Brownian motion up to the first time $\ell_t^{(\alpha)} > R_\gamma$, after which it is defined to be infinite; e.g., see Itô and McKean (1996) for the case $\alpha = 1/2$.

In the next section we modify techniques of Itô and McKean (1963) to obtain a Feynman–Kac formula for elastic (driftless) skew Brownian motion. Using this we show that an approach of Karatzas and Shreve (1984) for Brownian motion can be extended to derive the trivariate density of position, symmetric local time at zero, and occupation time of the positive half-line, $(B_t^{(\alpha)}, \ell_t^{(\alpha)}, \Gamma_t^{(\alpha)})$, for (driftless) skew Brownian motion started at zero. Properties of the hitting time at zero for (driftless) skew Brownian motion are then shown to be sufficient to extend this to the trivariate density for the (driftless) skew Brownian motion started at arbitrary $x \in \mathbb{R}$. Some interesting new marginal distributions also follow as corollaries.

NOTE 1.1. *The special notation $\ell_t^{(\alpha)}$ will be reserved to denote the symmetric local time of α-skew Brownian motion at zero throughout this paper. Definitions of left, right, and symmetric local time used here can be found in Revuz and Yor (1991), and will be reviewed in Section 5.*

This part of the main results can be more precisely summarized as follows:

THEOREM 1.2. *Let $l > 0$, and $0 < \tau < t$. Then*

$$P_0\left(B_t^{(\alpha)} \geq y; \ell_t^{(\alpha)} \in dl, \Gamma_t^{(\alpha)} \in d\tau\right) =$$
$$\begin{cases} \dfrac{2\alpha(1-\alpha)l}{2\pi(t-\tau)^{3/2}\tau^{1/2}} \exp\left\{-\dfrac{((1-\alpha)l)^2}{2(t-\tau)} - \dfrac{(y+\alpha l)^2}{2\tau}\right\} dl d\tau & \text{if } y \geq 0, \\ \dfrac{2\alpha(1-\alpha)l}{2\pi(t-\tau)^{1/2}\tau^{3/2}} \exp\left\{-\dfrac{(\alpha l)^2}{2\tau} - \dfrac{((1-\alpha)l - y)^2}{2(t-\tau)}\right\} dl d\tau & \text{if } y \leq 0. \end{cases}$$

The proof of Theorem 1.2 is obtained from a Feynman-Kac formula for an elastic skew Brownian motion to obtain a soluble differential equation for the Laplace transform of the trivariate density that may then be inverted; an approach already known from Karatzas and Shreve (1984) to work for standard Brownian motion.

Corollary 1.1.

$$P_0\left(B_t^{(\alpha)} \in dy, \ell_t^{(\alpha)} \in dl\right) = \begin{cases} \dfrac{2\alpha(l+y)}{\sqrt{2\pi t^3}} \exp\left\{-\dfrac{(l+y)^2}{2t}\right\} dldy & \text{if } y \geq 0, l > 0 \\ \dfrac{2(1-\alpha)(l-y)}{\sqrt{2\pi t^3}} \exp\left\{-\dfrac{(l-y)^2}{2t}\right\} dldy & \text{if } y \leq 0, l > 0 \end{cases}.$$

For an initial state x one has

Corollary 1.2. *For $y \geq 0, l > 0, 0 < \tau < t$,*

$$P_x\left(B_t^{(\alpha)} \in dy, \ell_t^{(\alpha)} \in dl, \Gamma_t^{(\alpha)} \in d\tau\right)$$

$$= \begin{cases} \dfrac{\alpha\left[(1-\alpha)l\right]\left[\alpha l + y + x\right]}{\pi(t-\tau)^{3/2}\tau^{3/2}} e^{\left(-\frac{((1-\alpha)l)^2}{2(t-\tau)} - \frac{(\alpha l+y+x)^2}{2\tau}\right)} dydld\tau, \text{ if } x \geq 0, \\ \dfrac{\alpha[(1-\alpha)l-x](\alpha l+y)}{\pi(t-\tau)^{3/2}\tau^{3/2}} e^{\left(-\frac{((1-\alpha)l-x)^2}{2(t-\tau)} - \frac{(\alpha l+y)^2}{2\tau}\right)} dydld\tau, \text{ if } x \leq 0, \end{cases}$$

whereas for $y \leq 0, l > 0, 0 < \tau < t$,

$$P_x\left(B_t^{(\alpha)} \in dy, \ell_t^{(\alpha)} \in dl, \Gamma_t^{(\alpha)} \in d\tau\right)$$

$$= \begin{cases} \dfrac{(1-\alpha)\left[\alpha l + x\right]\left[(1-\alpha)l - y\right]}{\pi(t-\tau)^{3/2}\tau^{3/2}} e^{\left(-\frac{(\alpha l+x)^2}{2\tau} - \frac{((1-\alpha)l-y)^2}{2(t-\tau)}\right)} dydld\tau, \text{ if } x \geq 0, \\ \dfrac{(1-\alpha)\left(\alpha l\right)\left[(1-\alpha)l - y - x\right]}{\pi(t-\tau)^{3/2}\tau^{3/2}} e^{\left(-\frac{(\alpha l)^2}{2\tau} - \frac{((1-\alpha)l-y-x)^2}{2(t-\tau)}\right)} dydld\tau, \text{ if } x \leq 0. \end{cases}$$

The following basic corollary identifies the role of the interface of skew Brownian motion in eventually providing an answer to the first passage time question raised at the outset. Define

$$T_y^{(\alpha)} = \inf\{t \geq 0 : B_t^{(\alpha)} = y\}. \tag{1.4}$$

Corollary 1.3. *Fix $y \geq 0$. If $1 > \alpha > 1/2$ then*

$$P_{-y}(T_y^{(\alpha)} > t) < P_y(T_{-y}^{(\alpha)} > t), \quad t > 0.$$

Theorem 1.3 below establishes a change of measure under which α-skew Brownian motion with drift parameter v is replaced by the elastic (driftless)

α-skew Brownian motion with a specific elasticity parameter $\gamma \equiv \gamma(\alpha, v)$. We refer to this as an *elastic change of measure.*

The *elastic change of measure* (via finite-dimensional distributions) is determined on a time interval prior to an elastic explosion as follows. Let

$$\Omega_t = [\ell_t^{(\alpha)} < R_\gamma] \subset C[0, \infty),$$

where R_γ denotes an exponentially distributed random variable with parameter γ, independent of $B^{(\alpha)}$. Then the elastic skew Brownian motion agrees with the skew Brownian motion up to the first time $\ell_t^{(\alpha)} > R_\gamma$. Denote the distributions of $B^{(\alpha,v)}$ and ${}^{(\alpha,\gamma)}B$ on Ω_t by $P_t^{(\alpha,v)}$ and $Q_t^{(\alpha,\gamma)}$ respectively. Also let $p^{(\alpha,v)}(t,x,y)$and $q^{(\alpha,\gamma)}(t,x,y)$ denote their corresponding transition probability densities; note from the elastic construction that $q^{(\alpha,\gamma)}(t,x,y)$ is sub-stochastic.

THEOREM 1.3. *Fix $t > 0$, and let $\gamma = |(2\alpha - 1)v|$. Then*

$$q^{(\alpha,\gamma)}(t,x,y)dy = \int_0^\infty e^{-\gamma\ell} P_x(B_t^{(\alpha)} \in dy, \ell_t^{(\alpha)} \in d\ell),$$

and

$$p^{(\alpha,v)}(t,x,y) = e^{-v(x-y)-\frac{v^2}{2}t} q^{(\alpha,\gamma)}(t,x,y).$$

In particular, on Ω_t

$$\mathbb{E}_{P_t^{(\alpha,v)}} Y = \mathbb{E}_{Q_t^{(\alpha,\gamma)}} Z_t Y$$

where $Z_t(\omega) = e^{v\omega_t - \frac{v^2}{2}t}$, $\omega \in \Omega_t$.

REMARK 1.1. *For the case $\alpha = 1/2$ the elastic change of measure exactly coincides with the Cameron-Martin-Girsanov (CMG) transformation for Brownian motion with drift, i.e., elastic standard Brownian motion with elasticity $\gamma = 0$ is a standard Brownian motion. However, in view of explosions for elastic diffusions, it is generally much more restrictive than CMG. Notice that the elasticity parameter $\gamma(\alpha, v)$ specifying the elastic change of measure is invariant under $\alpha \to 1 - \alpha, v \to -v$.*

The following formula for the distribution of α-skew Brownian motion with drift v is obtained as a consequence of the elastic change of measure in terms of the tail of the standard normal distribution $\Phi^c(y) = \frac{1}{\sqrt{2\pi}} \int_y^\infty e^{-\frac{z^2}{2}} dz$.

THEOREM 1.4. $P_0(B_t^{(\alpha,v)} \in dy) =$

$$\begin{cases} \dfrac{2\alpha}{\sqrt{2\pi t}} \exp\Big\{-\dfrac{(y-vt)^2}{2t}\Big\} \Big[1 - \gamma\sqrt{2\pi t}\ \Phi^c\Big(\dfrac{\gamma t + y}{\sqrt{t}}\Big) \exp\Big\{\dfrac{(\gamma t + y)^2}{2t}\Big\}\Big], \\ \textit{if } y > 0, l > 0, \\ \dfrac{2(1-\alpha)}{\sqrt{2\pi t}} \exp\Big\{-\dfrac{(y-vt)^2}{2t}\Big\} \Big[1 - \gamma\sqrt{2\pi t}\ \Phi^c\Big(\dfrac{\gamma t - y}{\sqrt{t}}\Big) \exp\Big\{\dfrac{(\gamma t - y)^2}{2t}\Big\}\Big], \\ \textit{if } y < 0, l > 0. \end{cases}$$

These are the essential preliminary general foundations required for the intended application, but they may also be of independent theoretical value.

The specific application is treated in the last section. Firstly, it involves explicit computation of the concentration curves for particles undergoing dispersion across an interface separating fine and coarse porous media; see Appuhamillage, Bokil, Thomann, Waymire and Wood (2009) for plots of resulting concentration curves. The relative notions of *fine* and *coarse* media are defined by their relative dispersion rates $D^- < D^+$; e.g., in a saturated fine medium, such as sand, the dispersion of solute concentrations is slower than in a saturated coarse medium, such as large gravel. For the application we adopt the convention used in experiments in which the fine medium is to the left of the interface and the coarse medium to the right. The injection and retrieval points are located at equal distances from the interface in both fine and coarse, coarse and fine regions, respectively. The flow is oriented in the direction of injection to retrieval points. Secondly, the application involves an analysis of certain empirically observed symmetries and asymmetries in the concentration curves and breakthrough times, respectively, of dispersion in symmetrically configured fine to coarse and coarse to fine injections and removal arrangements. In answer to the question raised at the outset, it was experimentally observed that fine to coarse breakthrough is faster than coarse to fine breakthrough (Berkowitz et al. (2009))! This has been interpreted as a possible breakdown of basic Fickian flux laws of transport; see Berkowitz, Cortis, Dror and Scher (2008). To the contrary, the results of this paper explain the phenomena within the framework of Fickian flux laws. The basic stochastic ordering in Corollary 1.3 will be applied to the process of physical dispersion in the final section devoted to the application; i.e, the mathematical answer to the question raised at the outset is provided by Corollary 5.2.

Finally, the next two results are formulations of individual stochastic particle properties that may serve in place of conservation properties of the concentration in the determination of the transmission parameter α^*. The first is by a variation on the martingale problem. Define the natural scale function by

$$s(x) = \sqrt{D^+}x^+ - \sqrt{D^-}x^-, \quad x \in (-\infty, \infty). \tag{1.5}$$

Martingale Problem (MP): For given $D^\pm$ determine α so that

$$f(s(B_t^{(\alpha)})) - \frac{1}{2}\int_0^t \frac{d}{dy}\left(D(y)\frac{df}{dy}\right)\Big|_{s(B_u^{(\alpha)})} du$$

is a martingale for all $f \in \mathcal{D}_{D^\pm}$ where

$$\mathcal{D}_{D^\pm} = \left\{g \in C^2(\mathbb{R}_0) \cap C(\mathbb{R}) : D^- \frac{dg}{dy}(0^-) = D^+ \frac{dg}{dy}(0^+)\right\}$$

and

$$\frac{d}{dy}\left(D(y)\frac{df}{dy}\right) = \mathbf{1}_{[B_u^{(\alpha)}>0]} D^+ \frac{d^2 f}{dy^2} + \mathbf{1}_{[B_u^{(\alpha)}\le 0]} D^- \frac{d^2 f}{dy^2}. \tag{1.6}$$

THEOREM 1.5. *The solution of (MP) for given $D^\pm$ is given by the process $Y = s(B^{(\alpha)})$ corresponding to the transmission parameter*

$$\alpha^* = \frac{\sqrt{D^+}}{\sqrt{D^+} + \sqrt{D^-}}.$$

Alternatively, α^* can be characterized as the parameter that makes a modfication of local time continuous for the skew diffusion. To be precise, for a given stochastic process $Y = \{Y_t : t \ge 0\}$, define right or left *modifed local time at a*, respectively, by

$$\tilde{A}_+^Y(t,a) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^\varepsilon \mathbf{1}([a \le Y_s < a + \varepsilon]) ds \tag{1.7}$$

and

$$\tilde{A}_-^Y(t,a) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^\varepsilon \mathbf{1}([a - \varepsilon < Y_s < a]) ds \tag{1.8}$$

As usual, define the symmetric local time by

$$\tilde{A}^Y = \frac{\tilde{A}_-^Y + \tilde{A}_+^Y}{2}. \tag{1.9}$$

The modification (1.7), (1.8),(1.9) to the customary definitions of one-sided and symmetric local times is that the integration is with respect to

ds and not with respect to the quadratic variation of Y. However, for the case of $Y = B^{(\alpha)}$, the modified local time coincides with usual local time. A mathematical perspective on Theorem 1.6, below, can be obtained by combining the theorem of Walsh (1978) with the celebrated theorem of Trotter (1958) for the determination of the value of α for which this local time is continuous, i.e., this yields $\alpha = 1/2$. For the present paper the extension is motivated by the physical problem as explained in the following Remark 1.2 in connection with the application.

REMARK 1.2. *One may notice that the units of local time coincide with the units of the process, namely length (L) in the case of applications of the type considered here. As a time unit this is interpreted probabilistically as the natural time scale of the diffusion. The units of the modified local time (T/L) turn out to be more appropriate for the continuity issue expressed by the following theorem and generalizing Walsh (1978). In general, the reason to consider alternative probabilistic determinations of α^* lies in their potential utility for extensions of the geometry to more complicated interfaces.*

THEOREM 1.6. *Let $Y_t = s(B_t^{(\alpha)})$. Then the process $\tilde{A}^Y$ is a.s. (spatially) continuous in a iff $\alpha = \alpha^*$.*

2. Elastic Skew Brownian Motion and a Feynman–Kac Formula. Fix parameters $\alpha \in (0,1)$ and $\gamma \geq 0$. A probability model for elastic skew Brownian motion may be defined as follows. Let R_γ be an exponentially distributed random variable with parameter $\gamma > 0$ on a probability space $(\Omega', \mathcal{F}', P')$. Define a new process $\{{}^{(\alpha,\gamma)}B_t : t \geq 0\}$ as the skew Brownian motion $B_t^{(\alpha)}$ "killed"when its local time at zero exceeds the level R_γ. More preciesly, on enlarged probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P}) = (\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', P \times P')$, define $\zeta_\gamma = \inf\{t \geq 0 : \ell_t^{(\alpha)} > R_\gamma\}$. Then we have $\widetilde{P}(\zeta_\gamma > t \mid R_\gamma) = e^{-\gamma \ell_t^{(\alpha)}}$ and the *elastic skew Brownian motion with lifetime* ζ_γ is defined by

$$
{}^{(\alpha,\gamma)}B_t = \begin{cases} B_t^{(\alpha)} & \text{if } t < \zeta_\gamma \\ \infty & \text{if } t \geq \zeta_\gamma. \end{cases} \tag{2.1}
$$

In the case $\gamma = 0$ one obtains skew Brownian motion. The transition probability densities $p^{(\alpha)}(t, x, y)$for skew Brownian motion were computed in Walsh (1978) using judicious applications of the reflection principle for Brownian motion. We record the result here for ease of reference

$$p^{(\alpha)}(t,x,y) = \begin{cases} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}} + \frac{(2\alpha-1)}{\sqrt{2\pi t}} e^{-\frac{(y+x)^2}{2t}} & \text{if } x>0, y>0 \\ \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}} - \frac{(2\alpha-1)}{\sqrt{2\pi t}} e^{-\frac{(y+x)^2}{2t}} & \text{if } x<0, y<0 \\ \frac{2\alpha}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}} \quad \text{if } x\le 0, y>0 \\ \frac{2(1-\alpha)}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}} \quad \text{if } x\ge 0, y<0. \end{cases} \tag{2.2}$$

Next we obtain a Feynman–Kac formula for this process. To simplify the presentation let $g(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$, and let $h(s;x) = \frac{|x|}{\sqrt{2\pi}} s^{-\frac{3}{2}} e^{-\frac{x^2}{2s}}$; as is very well-known $h(t,x)$ is the first passage time density to zero of standard Brownian motion starting at x, e.g., see (Bhattacharya and Waymire (2009), p. 30). Note that, by definition in terms of excursions, this coincides with the first passage time to zero for any skew Brownian motion started at x as well. We also recall the following Laplace transforms;

$$\int_0^\infty e^{-\beta t} g(x,t)dt = \frac{1}{\sqrt{2\beta}} \exp\left(-|x|\sqrt{2\beta}\right), \tag{2.3}$$

$$\int_0^\infty e^{-\lambda t} h(x,t)dt = \exp\left(-|x|\sqrt{2\lambda}\right). \tag{2.4}$$

THEOREM 2.1. *Let $x\in\mathbb{R}$, $y>0$, and $\lambda>0$. Suppose that f is a bounded continuous function on $\mathbb{R}\backslash\{y\}$. Define a function u by*

$$u(x) = \mathbb{E}_x \int_0^\infty e^{-\lambda t} f({}^{(\alpha,\gamma)}B_t)dt = \mathbb{E}_x \int_0^\infty e^{-\lambda t} e^{-\gamma \ell_t^{(\alpha)}} f(B_t^{(\alpha)})dt.$$

Then u is bounded and continuous on $\mathbb{R}$, C^1 on $\mathbb{R}_0$, C^2 on $\mathbb{R}_0\backslash\{y\}$, and

$$(\lambda - \frac{1}{2}\frac{d^2}{dx^2})u = f, \quad \alpha u'(0^+) - (1-\alpha)u'(0^-) = \gamma u(0).$$

PROOF. To simplify notation let $\tau_0 = T_{\cdot,0}^{(\alpha)} \equiv T_{\cdot,0}^{(\frac{1}{2})}$. We first make the claim that for f satisfying the hypothesis of the theorem and $\gamma\ge 0$, one has

$$\mathbb{E}_x \int_{\tau_0}^\infty e^{-\lambda t} e^{-\gamma \ell_t^{(\alpha)}} f(B_t^{(\alpha)})dt = \mathbb{E}_x e^{-\lambda\tau_0}\, \mathbb{E}_0 \int_0^\infty e^{-\lambda t} e^{-\gamma \ell_t^{(\alpha)}} f(B_t^{(\alpha)})dt \tag{2.5}$$

Indeed, by a simple change of variables and conditioning on the $\sigma-$field $\mathcal{F}_{\tau_0}$ to use the strong Markov property of $B^{(\alpha)}$, the left hand side can be written as

$$\begin{aligned} & \mathbb{E}_x \int_0^\infty e^{-\lambda(t+\tau_0)} e^{-\gamma \ell^{(\alpha)}_{t+\tau_0}} f(B^{(\alpha)}_{t+\tau_0}) dt \\ = & \mathbb{E}_x \left[\mathbb{E}_x \left[e^{-\lambda\tau_0} \int_0^\infty e^{-\lambda t} e^{-\gamma \ell^{(\alpha)}_{t+\tau_0}} f(B^{(\alpha)}_{t+\tau_0}) dt | \, \mathcal{F}_{\tau_0} \right]\right] \\ = & \mathbb{E}_x e^{-\lambda\tau_0} \, \mathbb{E}_0 \int_0^\infty e^{-\lambda t} e^{-\gamma \ell^{(\alpha)}_t} f(B^{(\alpha)}_t) dt \end{aligned}$$

as claimed.

Then, from the definition of u, noting that $l^{(\alpha)}_t = 0$ for $t < \tau_0$, and using (2.5), one has

$$\begin{aligned} u(x) &= \mathbb{E}_x \int_0^{\tau_0} e^{-\lambda t} f(B^{(\alpha)}_t) dt + \mathbb{E}_x \int_{\tau_0}^\infty e^{-\lambda t} e^{-\gamma \ell^{(\alpha)}_t} f(B^{(\alpha)}_t) dt \\ &= \mathbb{E}_x \int_0^{\tau_0} e^{-\lambda t} f(B^{(\alpha)}_t) dt + \mathbb{E}_x e^{-\lambda\tau_0} \, \mathbb{E}_0 \int_0^\infty e^{-\lambda t} e^{-\gamma \ell^{(\alpha)}_t} f(B^{(\alpha)}_t) dt \\ &= \mathbb{E}_x \int_0^\infty e^{-\lambda t} f(B^{(\alpha)}_t) dt - \mathbb{E}_x \int_{\tau_0}^\infty e^{-\lambda t} f(B^{(\alpha)}_t) dt + u(0) \mathbb{E}_x e^{-\lambda\tau_0} \\ &= \mathbb{E}_x \int_0^\infty e^{-\lambda t} f(B^{(\alpha)}_t) dt \\ &\quad + \mathbb{E}_x e^{-\lambda\tau_0} \left[u(0) - \mathbb{E}_0 \int_0^\infty e^{-\lambda t} f(B^{(\alpha)}_t) dt \right] \end{aligned}$$

where in the last step we used (2.5) with $\gamma = 0$.

Therefore, recalling $\mathbb{E}_x e^{-\lambda\tau_0} = e^{-\sqrt{2\lambda}|x|}$, and denoting the Laplace transform of $t \to p^{(\alpha)}(t,x,y)$ for fixed x, y by $\widehat{p}^{(\alpha)}(\lambda,x,y)$, one has

$$u(x) = \int_{-\infty}^\infty \widehat{p}^{(\alpha)}(\lambda,x,y) f(y) dy + e^{-\sqrt{2\lambda}|x|} \left\{ u(0) - \int_{-\infty}^\infty \widehat{p}^{(\alpha)}(\lambda,0,y) f(y) dy \right\}.$$

From (2.2) and (2.4), one has

$$\hat{p}^{(\alpha)}(\lambda,0,y) = \begin{cases} \frac{2\alpha}{\sqrt{2\lambda}} e^{-|y|\sqrt{2\lambda}} & \text{for } y > 0, \\ \frac{2(1-\alpha)}{\sqrt{2\lambda}} e^{-|y|\sqrt{2\lambda}} & \text{for } y < 0. \end{cases} \tag{2.6}$$

In particular, it follows that $\lambda u - \frac{1}{2} u'' = f$ and

$$-\left\{ \alpha u'(0^+) - (1-\alpha) u'(0^-) \right\} + \sqrt{2\lambda} \int_{-\infty}^\infty \widehat{p}^{(\alpha)}(\lambda,0,y) f(y) dy = \sqrt{2\lambda} u(0). \tag{2.7}$$

On the other hand, using the excursion definition of skew Brownian motion

$$
\begin{aligned}
u(0) &= \mathbb{E}_0 \int_0^\infty e^{-\lambda t} e^{-\gamma \ell_t^{(\alpha)}} f(B_t^{(\alpha)}) dt \\
&= \mathbb{E}_0 \int_0^\infty e^{-\lambda t} e^{-\gamma \ell_t^{(\alpha)}} \left[\sum_{n=1}^\infty \mathbf{1}_{J_n}(t) f(A_n |B_t|) \right] dt \\
&= \alpha \mathbb{E}_0 \int_0^\infty e^{-\lambda t} e^{-\gamma \ell_t^{(\alpha)}} \left[\sum_{n=1}^\infty \mathbf{1}_{J_n}(t) f(|B_t|) \right] dt \\
&\quad + (1-\alpha) \mathbb{E}_0 \int_0^\infty e^{-\lambda t} e^{-\gamma \ell_t^{(\alpha)}} \left[\sum_{n=1}^\infty \mathbf{1}_{J_n}(t) f(-|B_t|) \right] dt \\
&= \alpha \mathbb{E}_0 \int_0^\infty e^{-\lambda t} e^{-\gamma \ell_t^{(\alpha)}} f(|B_t^{(\alpha)}|) dt \\
&\quad + (1-\alpha) \mathbb{E}_0 \int_0^\infty e^{-\lambda t} e^{-\gamma \ell_t^{(\alpha)}} f(-|B_t^{(\alpha)}|) dt.
\end{aligned}
$$

Since the local time at 0 of reflected Brownian motion starting at zero coincides with the local time at 0 of skew Brownian motion with parameter α starting at zero, the last expression can be written in terms of the local time of Brownian motion and reflected Brownian motion to yield

$$
\begin{aligned}
u(0) &= \alpha \mathbb{E}_0 \int_0^\infty e^{-\lambda t} e^{-\gamma \ell_t^{(1/2)}} f(|B_t|) dt \\
&\quad + (1-\alpha) \mathbb{E}_0 \int_0^\infty e^{-\lambda t} e^{-\gamma \ell_t^{(1/2)}} f(-|B_t|) dt.
\end{aligned}
$$

Now, in view of Karatzas and Shreve (1984) ((1.5), p. 820), and after computing the indicated Laplace transform, one has

$$
\begin{aligned}
u(0) &= \frac{\alpha}{\gamma + \sqrt{2\lambda}} 2 \int_0^\infty e^{-y\sqrt{2\lambda}} f(y) dy + \frac{1-\alpha}{\gamma + \sqrt{2\lambda}} 2 \int_0^\infty e^{-y\sqrt{2\lambda}} f(-y) dy \\
&= \frac{\sqrt{2\lambda}}{\gamma + \sqrt{2\lambda}} \int_{-\infty}^\infty \widehat{p}^\alpha(\lambda, 0, y) f(y) dy,
\end{aligned}
$$

where in the last step we used (2.6). Thus, noting (2.7), we have $\alpha u'(0^+) - (1-\alpha) u'(0^-) = \gamma u(0)$. $\square$

3. Trivariate Density for Skew Brownian Motion. Here we first compute the Laplace transform of the density of the pair $(\ell_t^{(\alpha)}, \Gamma_t^{(\alpha)})$ on the event $\left[B_t^{(\alpha)} \geq y\right]$ for $y > 0$ for the skew Brownian motion starting at 0. Since for $\Gamma_t^{(\alpha)\,-} = t - \Gamma_t^{(\alpha)} = \text{meas}\left\{0 \leq s \leq t : B_s^{(\alpha)} < 0\right\}$, the triples

$(-B_t^{(\alpha)}, \ell_t^{(\alpha)}, \Gamma_t^{(\alpha)\,-})$ and $(B_t^{(1-\alpha)}, l_t^{(1-\alpha)}, \Gamma_t^{(1-\alpha)})$ are equivalent in law and $\Gamma_t^{(\alpha)} = t - \Gamma_t^{(1-\alpha)}$, we can easily find the Laplace transform of the density of the pair $(\ell_t^{(\alpha)}, \Gamma_t^{(\alpha)})$ on the event $\left[B_t^{(\alpha)} < y\right]$ for $y < 0$ using the previous case. The following is a direct extension of Karatzas and Shreve (1984) analysis of the case of standard Brownian motion ($\alpha = \frac{1}{2}$) to arbitrary $\alpha \in (0, 1)$.

LEMMA 3.1. *Let λ, β and γ be positive. Then*

$$\mathbb{E}_0 \int_0^\infty \mathbf{1}_{[y,\infty)}(B_t^{(\alpha)}) \exp\left\{-\lambda t - \beta\Gamma_t^{(\alpha)} - \gamma\ell_t^{(\alpha)}\right\} dt =$$

$$\begin{cases} \dfrac{2\alpha\ \exp\left\{-y\sqrt{2(\lambda+\beta)}\right\}}{\sqrt{2(\lambda+\beta)}\left[\gamma + (1-\alpha)\sqrt{2\lambda} + \alpha\sqrt{2(\lambda+\beta)}\right]} & \text{if } y > 0, \\ \dfrac{2(1-\alpha)\ \exp\left\{y\sqrt{2(\lambda+\beta)}\right\}}{\sqrt{2(\lambda+\beta)}\left[\gamma + \alpha\sqrt{2\lambda} + (1-\alpha)\sqrt{2(\lambda+\beta)}\right]} & \text{if } y < 0. \end{cases}$$

PROOF. For each $x \in \mathbb{R}$, define

$$u(x) = \mathbb{E}_x \int_0^\infty \mathbf{1}_{[y,\infty)}(B_t^{(\alpha)}) \exp\left(-\lambda t - \beta\Gamma_t^{(\alpha)} - \gamma\ell_t^{(\alpha)}\right) dt$$

According to Theorem 2.1, $u \in \mathcal{D}_{\alpha,\gamma}$ and satisfies

$$(\lambda + \beta\mathbf{1}_{[0,\infty)}(x))u(x) = \frac{1}{2}u''(x) + \mathbf{1}_{[y,\infty)}, \quad x \in \mathbb{R} \setminus \{0, y\} \tag{3.1}$$

$$\alpha u'(0^+) - (1-\alpha)u'(0^-) = \gamma u(0) \tag{3.2}$$

Considering the case $y > 0$, u has the form

$$u(x) = \begin{cases} c_1 \exp\left\{x\sqrt{2\lambda}\right\} & \text{if } x \le 0 \\ c_2 \exp\left\{x\sqrt{2(\lambda+\beta)}\right\} + c_3 \exp\left\{-x\sqrt{2(\lambda+\beta)}\right\} & \text{if } 0 \le x \le y \\ c_4 \exp\left\{-(x-y)\sqrt{2(\lambda+\beta)}\right\} + \frac{1}{\lambda+\beta} & \text{if } x \ge y \end{cases}$$

where the constants $c_i, 1 \le i \le 4$, are determined by the above conditions. The lemma follows in the case $y > 0$, from the computation of c_1 using the interface condition in (3.1), (3.2). For $y < 0$ simply use the observation above that the triples $(-B_t^{(\alpha)}, \ell_t^{(\alpha)}, \Gamma_t^{(\alpha)\,-})$ and $(B_t^{(1-\alpha)}, l_t^{(1-\alpha)}, \Gamma_t^{(1-\alpha)})$ have the same distribution. □

The expression in Lemma 3.1 is the Laplace transform of the density (if it exists) of the pair $(\ell_t^{(\alpha)}, \Gamma_t^{(\alpha)})$ on the event $\left[B_t^{(\alpha)} \ge y\right]$. As in Karatzas and

Shreve (1984), it is also possible to invert the Laplace transform to arrive at the trivariate density asserted in Theorem 1.2 in the introduction.

Proof of Theorem 1.2 & Corollary 1.1: We only consider the case $y > 0$. The case $y < 0$ follows by similar arguments. Using Laplace transforms, it is sufficient to establish

$$\int_0^\infty \int_0^t \int_0^\infty e^{-\lambda t-\beta\tau-\gamma l}\left[\frac{2\alpha(1-\alpha)l}{2\pi(t-\tau)^{3/2}\tau^{1/2}}\mathrm{e}^{\left(-\frac{((1-\alpha)l)^2}{2(t-\tau)}-\frac{(y+\alpha l)^2}{2\tau}\right)}\right] dld\tau dt$$
$$= \frac{2\alpha e^{\left(-y\sqrt{2(\lambda+\beta)}\right)}}{\sqrt{2(\lambda+\beta)}\left[\gamma+(1-\alpha)\sqrt{2\lambda}+\alpha\sqrt{2(\lambda+\beta)}\right]}, \quad y>0.$$

Reversing the order of integration, and using (2.4) and (2.3), we can write the left hand side as

$$\begin{aligned}
&2\alpha\int_0^\infty e^{-\gamma l}\int_0^\infty e^{-\beta\tau}\int_\tau^\infty e^{-\lambda t}h(t-\tau,(1-\alpha)l)\; dt\; g(\tau,y+\alpha l)\; d\tau\; dl\\
&= 2\alpha\int_0^\infty e^{-\gamma l}\exp\left(-(1-\alpha)l\sqrt{2\lambda}\right)\int_0^\infty e^{-(\lambda+\beta)\tau}\; g(\tau,y+\alpha l)\; d\tau\; dl\\
&= 2\alpha\int_0^\infty e^{-\gamma l}\frac{1}{\sqrt{2(\lambda+\beta)}}\exp\left(-(1-\alpha)l\sqrt{2\lambda}-(y+\alpha l)\sqrt{2(\lambda+\beta)}\right) dl\\
&= \frac{2\alpha e^{\left(-y\sqrt{2(\lambda+\beta)}\right)}}{\sqrt{2(\lambda+\beta)}\left[\gamma+(1-\alpha)\sqrt{2\lambda}+\alpha\sqrt{2(\lambda+\beta)}\right]}
\end{aligned}$$

□

The trivariate density of position, local time and occupation time of skew Brownian motion, can be obtained by differentiating with respect to y to yield:

$$P_0\left\{B_t^{(\alpha)}\in dy,\ell_t^{(\alpha)}\in dl,\Gamma_t^{(\alpha)}\in d\tau\right\} \tag{3.3}$$
$$= \begin{cases} \dfrac{\alpha\left[(1-\alpha)l\right]\left[\alpha l+y\right]}{\pi(t-\tau)^{3/2}\tau^{3/2}}\exp\left\{-\dfrac{((1-\alpha)l)^2}{2(t-\tau)}-\dfrac{(\alpha l+y)^2}{2\tau}\right\}dydld\tau \\ \qquad \text{if } y>0, l>0, 0<\tau<t, \\ \dfrac{(1-\alpha)\left[\alpha l\right]\left[(1-\alpha)l-y\right]}{\pi(t-\tau)^{3/2}\tau^{3/2}}\exp\left\{-\dfrac{(\alpha l)^2}{2\tau}-\dfrac{((1-\alpha)l-y)^2}{2(t-\tau)}\right\}dydld\tau \\ \qquad \text{if } y<0, l>0, 0<\tau<t. \end{cases}$$

Integrating out τ in (3.3) we obtain the joint distribution of skew Brownian motion with parameter α and its local time at 0 asserted in Corollary 1.1.

Integrating out y, l in (3.3) we recover the probability density function of the occupation time for skew Brownian motion with parameter α starting at 0; see Revuz and Yor (1991), and Ramirez et al. (2008) for alternative approaches to this particular case.

COROLLARY 3.1.

$$P_0\left(\Gamma_t^{(\alpha)} \in d\tau\right) = \frac{\alpha(1-\alpha)t}{\pi(t-\tau)^{1/2}\tau^{1/2}\left[(1-\alpha)^2\tau + \alpha^2(t-\tau)\right]} d\tau; \;\; 0 < \tau < t$$

Integrating out y, τ in (3.3) provides the distribution of local time of skew Brownian motion at zero. As expected, it coincides with that for reflected Brownian motion but is included here as a simple verification.

COROLLARY 3.2.

$$P_0\left(\ell_t^{(\alpha)} \in dl\right) = \frac{2}{\sqrt{2\pi t}} \exp\left\{-\frac{l^2}{2t}\right\} dl, \;\; t > 0.$$

Proof of Corollary 1.2:
The computation of $P_x\left(B_t^{(\alpha)} \in dy, \ell_t^{(\alpha)} \in dl, \Gamma_t^{(\alpha)} \in d\tau\right)$ from Theorem 1.2 for $x \neq 0$ will follow from standard convolution properties of first passage time densities at zero of Brownian motion since they coincide with those of skew Brownian motion. Recall that $h(\cdot; x)$ denotes the first passage time density to zero for (skew) Brownian motion starting at $x > 0$. Then, the strong Markov property for Brownian motion yields

$$h(\cdot; x_1 + x_2) = h(\cdot; x_1) * h(\cdot; x_2), \;\; x_1 x_2 > 0 \tag{3.4}$$

Thus, for $l > 0, 0 < \tau < t$, we can write the results of (3.3) as,

$$\begin{aligned} & P_0\left(B_t^{(\alpha)} \in dy, \ell_t^{(\alpha)} \in dl, \Gamma_t^{(\alpha)} \in d\tau\right) \\ = & \begin{cases} 2\alpha h\left(t-\tau; (1-\alpha)l\right) h\left(\tau; \alpha l + y\right) & \text{if } y > 0, \\ 2(1-\alpha) h\left(\tau; \alpha l\right) h\left(t-\tau; (1-\alpha)l - y\right) & \text{if } y < 0. \end{cases} \end{aligned}$$

To obtain the trivariate density when $B_0^{(\alpha)} = x$, we use the strong Markov property of skew Brownian motion and the already noted fact that $T_0^{(\alpha)} =$

$T_0^{(1/2)}$ in distribution under P_x, to obtain

$$\begin{aligned}
&P_x\left(B_t^{(\alpha)} \in dy, \ell_t^{(\alpha)} \in dl, \Gamma_t^{(\alpha)} \in d\tau\right)\\
&= P_x\left(B_t^{(\alpha)} \in dy, \ell_t^{(\alpha)} \in dl, \Gamma_t^{(\alpha)} \in d\tau, T_0^{(\alpha)} \leq \tau\right)\\
&= \int_{s=0}^{\tau} P_x\left(B_t^{(\alpha)} \in dy, \ell_t^{(\alpha)} \in dl, \Gamma_t^{(\alpha)} \in d\tau | T_0^{(\alpha)} = s\right) P_x\left(T_0^{(1/2)} \in ds\right)\\
&= \int_{s=0}^{\tau} P_0\left(B_{t-s}^{(\alpha)} \in dy, l_{t-s}^{(\alpha)} \in dl, \Gamma_{t-s}^{(\alpha)} \in d\tau - s\right) h(s;x)ds.
\end{aligned}$$

Considering the case $x \geq 0, y < 0$, and using Theorem (1.2) the last expression can be written as

$$\begin{aligned}
&= 2(1-\alpha)\int_0^{\tau} h\left(\tau - s; \alpha l\right) h\left(t-\tau; (1-\alpha)l - y\right) h\left(s; x\right) ds dy dl d\tau\\
&= 2(1-\alpha) h\left(\tau; \alpha l + x\right) h\left(t-\tau; (1-\alpha)l - y\right) dy dl d\tau,
\end{aligned}$$

using the convolution property (3.4).

A similar computation yields that for $x \geq 0, \ y > 0, \ 0 < \tau < t$ and on the event $[T_0^{(\alpha)} < t]$,

$$\begin{aligned}
P_x&\left(B_t^{(\alpha)} \in dy, \ell_t^{(\alpha)} \in dl, \Gamma_t^{(\alpha)} \in d\tau, [T_0^{(\alpha)} < t]\right)\\
&= 2\alpha h\left(t-\tau; (1-\alpha)l\right) h\left(\tau; \alpha l + y + x\right) dy dl d\tau.
\end{aligned}$$

□

Remark 3.1. *We note that if $x \geq 0, y > 0$ one also needs to consider the case that the skew Brownian motion does not reach the origin. In this case one has*

$$\begin{aligned}
& P_x\left(B_t^{(\alpha)} \in dy, \ell_t^{(\alpha)} = 0, \Gamma_t^{(\alpha)} = t\right)\\
= \;& P_x\left(B_t^{(\alpha)} \in dy, T_0^{(\alpha)} \geq t\right)\\
= \;& \frac{1}{\sqrt{2\pi t}}\left[\exp\left\{-\frac{(y-x)^2}{2t}\right\} - \exp\left\{-\frac{(y+x)^2}{2t}\right\}\right] dy; \ x \geq 0, \ y \geq 0.
\end{aligned}$$

Corollary 3.3. *If $x \geq 0$ we have*

$$P_x(B_t^{(\alpha)} \in dy, \ell_t^{(\alpha)} \in d\ell) =$$

$$\begin{cases} \dfrac{(1-2\alpha)(l-y+x)}{\sqrt{2\pi t^3}} \mathrm{e}^{\left(-\frac{(l-y+x)^2}{2t}\right)} dydl + \dfrac{2(2l-y+x)}{\sqrt{2\pi t^3}} \mathrm{e}^{\left(-\frac{(2l-y+x)^2}{2t}\right)} dydl \\ \text{if} \;\; y \leq 0, \;\; l > 0 \\ \dfrac{(1-2\alpha)(l-y+x)}{\sqrt{2\pi t^3}} \mathrm{e}^{\left(-\frac{(l-y+x)^2}{2t}\right)} dydl + \dfrac{2(2l+y+x)}{\sqrt{2\pi t^3}} \mathrm{e}^{\left(-\frac{(2l+y+x)^2}{2t}\right)} dydl \\ + \dfrac{1}{\sqrt{2\pi t}} \left[\mathrm{e}^{\left(-\frac{(y-x)^2}{2t}\right)} - \mathrm{e}^{\left(-\frac{(y+x)^2}{2t}\right)} \right] \delta_0(dl) dy \\ \text{if} \;\; y \geq 0, \;\; l > 0, \end{cases}$$

whereas if $x \leq 0$, then

$$P_x(B_t^{(\alpha)} \in dy, \ell_t^{(\alpha)} \in d\ell) =$$

$$\begin{cases} \dfrac{(2\alpha-1)(l+y-x)}{\sqrt{2\pi t^3}} \mathrm{e}^{\left(-\frac{(l+y-x)^2}{2t}\right)} dydl + \dfrac{2(2l+y-x)}{\sqrt{2\pi t^3}} \mathrm{e}^{\left(-\frac{(2l+y-x)^2}{2t}\right)} dydl \\ \text{if} \;\; y \geq 0, \;\; l > 0 \\ \dfrac{(2\alpha-1)(l+y-x)}{\sqrt{2\pi t^3}} \mathrm{e}^{\left(-\frac{(l+y-x)^2}{2t}\right)} dydl + \dfrac{2(2l+y-x)}{\sqrt{2\pi t^3}} \mathrm{e}^{\left(-\frac{(2l+y-x)^2}{2t}\right)} dydl \\ + \dfrac{1}{\sqrt{2\pi t}} \left[\mathrm{e}^{\left(-\frac{(y-x)^2}{2t}\right)} - \mathrm{e}^{\left(-\frac{(y+x)^2}{2t}\right)} \right] \delta_0(dl) dy \\ \text{if} \;\; y \leq 0, \;\; l > 0 \end{cases}$$

While the following formula is relatively more complicated, it is easily computed and plays an essential role in the application given in the next section. For the application it is sufficient to consider $x < 0, y > 0$, moreover, the other cases may be obtained similarly obtained from the trivariate density.

Corollary 3.4. *For $x < 0$ and $y \geq 0$,*

$$
\begin{aligned}
&P_x\left(B_t^{(\alpha)} \in dy, \Gamma_t^{(\alpha)} \in d\tau\right) \\
&= \frac{(1-\alpha)}{\pi\sqrt{\tau(t-\tau)}} \frac{(1-\alpha)^3\tau y - \alpha^3(t-\tau)y}{\left[\alpha^2(t-\tau)+(1-\alpha)^2\tau\right]^2} \exp\left(\frac{-\xi(x^2,y^2,\tau,t)}{\xi(\tau,t-\tau,\tau,t)}\right) \\
&\quad + \sqrt{\frac{2}{\pi}} \frac{\alpha(1-\alpha)^2}{\left[\alpha^2(t-\tau)+(1-\alpha)^2\tau\right]^{3/2}} \times \Phi^c\Big(\frac{\sqrt{2}\xi(\alpha x, -(1-\alpha)y, \tau, t)}{\sqrt{\xi(\tau, t-\tau, \tau, t)}}\Big) \\
&\quad \times [1 - 2\frac{(\xi(x^2,y^2,\tau,t) - \xi^2(\alpha x, -(1-\alpha)y, \tau, t))}{\xi(\tau,t-\tau,\tau,t)}] \\
&\quad \times \exp(-\frac{(\xi(x^2,y^2,\tau,t) - \xi^2(\alpha x, -(1-\alpha)y, \tau, t))}{(\xi(\tau,t-\tau,\tau,t))}),
\end{aligned}
$$

where

$$
\xi(u,w,\tau,t) = \frac{u(t-\tau)+w\tau}{\alpha^2(t-\tau)+(1-\alpha)^2\tau}.
$$

Proof. Let

$$
A = \frac{\alpha x(t-\tau) - (1-\alpha)y\tau}{\alpha^2(t-\tau)+(1-\alpha)^2\tau}, B = \frac{\alpha^2(t-\tau)+(1-\alpha)^2\tau}{2\tau(t-\tau)},
$$

and

$$
C^2 = \frac{x^2(t-\tau)+y^2\tau}{\alpha^2(t-\tau)+(1-\alpha)^2\tau}.
$$

For $x \geq 0,\ y > 0,\ \ell > 0,\ 0 < \tau < t$, one has after rather lengthy differentiations

$$
\begin{aligned}
(3.5)\quad & P_x\left(B_t^{(\alpha)} \in dy, \Gamma_t^{(\alpha)} \in d\tau\right) = \\
& \frac{-2(1-\alpha)}{2\pi\sqrt{\tau(t-\tau)}} \frac{\partial}{\partial x}\frac{\partial}{\partial y} \int_0^\infty \exp\left\{-\frac{(\alpha\ell+x)^2}{2\tau}\right\} \exp\left\{-\frac{((1-\alpha)\ell-y)^2}{2(t-\tau)}\right\} d\ell \\
&= \frac{(1-\alpha)}{\pi\sqrt{\tau(t-\tau)}} \frac{(1-\alpha)^3\tau x - \alpha^3(t-\tau)y}{\left[\alpha^2(t-\tau)+(1-\alpha)^2\tau\right]^2} \exp(-BC^2) \\
&\quad + \frac{\alpha(1-\alpha)^2}{\pi\sqrt{\tau(t-\tau)}} \frac{1}{\left[\alpha^2(t-\tau)+(1-\alpha)^2\tau\right]} \frac{\sqrt{\pi}}{\sqrt{B}} \Phi^c(A\sqrt{2B}) \\
&\qquad \times \left[1 - 2B(C^2-A^2)\right] \exp(-B(C^2-A^2)).
\end{aligned}
$$

□

4. Elastic Change of Measure and Applications to Skew Brownian Motion with Drift. The objective of this section is to establish the elastic change of measure relation between skew Brownian motion with drift and the elastic (driftless) skew Brownian motion defined by Theorem 1.3, and apply it to obtain transition probabilities for skew Brownian motion with drift together with the stochastic ordering of passage times.

Proof of Theorem 1.3: Recall that α-skew Brownian motion with drift v is denoted by $B^{(\alpha,v)}$ and its transition probability by $p^{(\alpha,v)}(t,x,y)$. Then for $c(t,x) = \mathbb{E}_x c_0(B^{(\alpha,v)}) = \int_{-\infty}^{\infty} c_0(y) p^{(\alpha,v)}(t,x,y)dy$ one has $c \in \mathcal{D}_{\alpha,0}$ and

$$\frac{\partial c}{\partial t} = \frac{1}{2}\frac{\partial^2 c}{\partial x^2} + v\frac{\partial c}{\partial x}$$
$$c(t,0^+) = c(t,0^-), \quad \alpha\frac{\partial c}{\partial x}(t,0^+) = (1-\alpha)\frac{\partial c}{\partial x}(t,0^-)$$
$$c(0,x) = c_0(x).$$

Defining $\tilde{c}(t,x) = \exp\{vx\}c(t,x)$ we obtain

$$\frac{\partial \tilde{c}}{\partial t} = \frac{1}{2}\frac{\partial^2 \tilde{c}}{\partial x^2} - \frac{v^2}{2}\tilde{c}$$
$$\tilde{c}(t,0^+) = \tilde{c}(t,0^-), \quad \alpha\frac{\partial \tilde{c}}{\partial x}(t,0^+) - (1-\alpha)\frac{\partial \tilde{c}}{\partial x}(t,0^-) = (1-2\alpha)v\tilde{c}(0)$$
$$\tilde{c}(0,x) = \exp\{vx\}c_0(x)$$

Let $\gamma = |(1-2\alpha)v|$. Then from the Feynman-Kac formula we have

$$\tilde{c}(t,x) = \mathbb{E}_x\tilde{c}_0({}^{(\alpha,\gamma)}B_t)\exp\Big\{-\frac{v^2}{2}t\Big\} = \int_{-\infty}^{\infty}\tilde{c}_0(y)\exp\Big\{-\frac{v^2}{2}t\Big\}q^{(\alpha,\gamma)}(t,x,y)dy$$

But since $c(t,x) = \exp\{-vx\}\tilde{c}(t,x)$, we have

$$\int_{-\infty}^{\infty} c_0(y)p^{(\alpha,v)}(t,x,y)dy = \int_{-\infty}^{\infty} c_0(y)\exp\Big\{-v(x-y) - \frac{v^2}{2}t\Big\}q^{(\alpha,\gamma)}(t,x,y)dy$$

Thus the elastic change of measure relation follows as

$$p^{(\alpha,v)}(t,x,y) = \exp\Big\{-v(x-y) - \frac{v^2}{2}t\Big\}q^{(\alpha,\gamma)}(t,x,y).$$

□

Proof of Theorem 1.4: Recall that ${}^{(\alpha,\gamma)}B_t$ agrees with $B_t^{(\alpha)}$ when $l_t^{(\alpha)} \leq R_\gamma$. Thus, letting $f^{(\alpha)}(t,x;\cdot,\cdot)$ denote the joint density of $(B_t^{(\alpha)}, \ell_t^{(\alpha)})$ with

$B_0^{(\alpha)} = x$,

$$\begin{aligned}
\mathbb{E}_x \tilde{c}_0({}^{(\alpha,\gamma)}B_t) &= \mathbb{E}_x\Big[\tilde{c}_0(B_t^{(\alpha)})\mathbf{1}_{[l_t^{(\alpha)}(B^{(\alpha)})\le R_\gamma]}\Big] \\
&= \int_0^\infty \mathbb{E}_x\Big[\tilde{c}_0(B_t^{(\alpha)})\mathbf{1}_{[l_t^{(\alpha)}\le r]}\Big]\gamma\exp\{-\gamma r\}dr \\
&= \int_0^\infty\int_{-\infty}^\infty\int_0^r \tilde{c}_0(y)f^{(\alpha)}(t,x;y,l)\gamma\exp\{-\gamma r\}dldydr \\
&= \int_{-\infty}^\infty\int_0^\infty\int_l^\infty \tilde{c}_0(y)f^{(\alpha)}(t,x;y,l)\gamma\exp\{-\gamma r\}drdldy \\
&= \int_{-\infty}^\infty\int_0^\infty \tilde{c}_0(y)f^{(\alpha)}(t,x;y,l)\exp\{-\gamma l\}dldy.
\end{aligned}$$

Also since

$$\mathbb{E}_x\tilde{c}_0({}^{(\alpha,\gamma)}B_t) = \int_{-\infty}^\infty \tilde{c}_0(y)q^{(\alpha,\gamma)}(t,x,y)dy,$$

one has

$$q^{(\alpha,\gamma)}(t,x,y) = \int_0^\infty f^{(\alpha)}(t,x;y,l)\exp\{-\gamma l\}dl. \tag{4.1}$$

From Corollary 1.1

(4.2)

$$f^{(\alpha)}(t,0;y,l) = \begin{cases} \dfrac{2\alpha(l+y)}{\sqrt{2\pi t^3}}\exp\left\{-\dfrac{(l+y)^2}{2t}\right\} & \text{if } y>0, l>0 \\[2ex] \dfrac{2(1-\alpha)(l-y)}{\sqrt{2\pi t^3}}\exp\left\{-\dfrac{(l-y)^2}{2t}\right\} & \text{if } y<0, l>0 \end{cases}$$

Recognizing from (4.1) that $q^{(\alpha,\gamma)}(t,0,y)$ is the Laplace transform with respect to local time l of $f^{(\alpha)}(t,0,y,l)$, the theorem follows by a direct calculation of this Laplace transform and Theorem 1.3. □

Our objective is to next prove the more fundamental stochastic ordering of Corollary 1.3 for skew Brownian motion. Although the relevance to the question raised in the introduction is given in the next section, Corollary 1.3 may also be of independent interest apart from the application.

Lemma 4.1. *Suppose that $X = X_1 + X_2$ and $Y = Y_1 + Y_2$ are respective sums of independent non-negative random variables. If X_i is stochastically smaller than Y_i, for $i = 1, 2$, then X is stochastically smaller than Y.*

PROOF. For $t > 0$,

$$
\begin{aligned}
P(X > t) &= \int_0^t P(X_1 > t - s)P(X_2 \in ds) \\
&\leq \int_0^t P(Y_1 > t - s)P(X_2 \in ds) \\
&= \int_0^t P(X_2 > t - s)P(Y_1 \in ds) \\
&\leq \int_0^t P(Y_2 > t - s)P(Y_1 \in ds) \\
&= P(Y > t). \qquad (4.3)
\end{aligned}
$$

□

Proof of Corollary 1.3: Let $T_0 \equiv T_0^{(1/2)}$ denote the first time for standard Brownian motion to reach zero. Also note that $T_0^{(\alpha)}$ is distributed as T_0 under P_y for $y \neq 0$, $0 < \alpha < 1$. So clearly for $t \geq 0$, one has

$$
P_y(T_0^{(\alpha)} > t) = P_y(T_0 > t) = P_{-y}(T_0 > t) = P_{-y}(T_0^{(\alpha)} > t). \qquad (4.4)
$$

Now observe, using the strong Markov property of skew Brownian motion,

$$
P_{-y}(T_y^{(\alpha)} > t) = \int_0^t P_0(T_y^{(\alpha)} > t - s)P_{-y}(T_0 \in ds), \qquad (4.5)
$$

and

$$
P_y(T_{-y}^{(\alpha)} > t) = \int_0^t P_0(T_y^{(1-\alpha)} > t - s)P_{-y}(T_0 \in ds). \qquad (4.6)
$$

Next consider the following coupled representations of the α-*skew Brownian motion* processes $B^{(\alpha)} = \{B_t^{(\alpha)} : t \geq 0\}$: Let $\{U_m : m = 1, 2, \dots\}$ be an i.i.d. sequence, independent of $B^{(\alpha)}$, of uniformly distributed random variables on $[0, 1]$ also defined on Ω by

$$
B_t^{(\alpha)} = \sum_{m=1}^{\infty} \mathbf{1}_{J_m}(t)\{2\mathbf{1}_{[0,\alpha)}(U_n) - 1\}|B_t|, \qquad (4.7)
$$

where $\mathbf{1}_S$ denotes the indicator function of the set S. Then, for any $t > 0$, one has for $1 > \alpha > 1/2$ that

$$
[T_y^{(\alpha)} > t] \subset [T_y^{(1-\alpha)} > t].
$$

The asserted stochastic ordering follows by application of the lemma to (4.5) and (4.6). □

5. Application to Solute Dispersion Across an Interface. In analogy with the Fourier flux law for heat conduction, the standard model of advection-dispersion is based on Ficks' linear flux law together with continuity of concentration and flux across the interface. For the application treated here the flux is aligned with the $y-$axis, and the concentration field is uniform in the other two orthogonal directions. Thus the Fickian (macroscale) conservation laws can be reduced to an advection-dispersion equation of the form

$$\frac{\partial c}{\partial t} = \frac{1}{2}\frac{\partial}{\partial y}(D(y)\frac{\partial c}{\partial y}) - \frac{\partial vc}{\partial y} \tag{5.1}$$

$$c(t,0^+) = c(t,0^-), \qquad D^+\frac{\partial c}{\partial y}(t,0^+) = D^-\frac{\partial c}{\partial y}(t,0^-), \tag{5.2}$$

where

$$D(y) = \begin{cases} D^- & \text{if} \quad y < 0. \\ D^+ & \text{if} \quad y \geq 0. \end{cases} \tag{5.3}$$

Note 5.1. *In treating the application in this paper we have adhered to the standard probability notation of $\frac{1}{2}D$ in place of D in the concentration equation (5.1).*

In particular, from a probabilistic point of view the particle motion is given by the unique strong solution to

$$dY_t = \frac{D^+ - D^-}{D^+ + D^-} dL_t^0(Y) + v\,dt + \sqrt{D(Y_t)}\,dB_t \tag{5.4}$$

where $B = \{B_t : t \geq 0\}$ is standard Brownian motion. In the case $v = 0$, $Y_t \equiv S_t^*$ is given by

$$S_t^* = s(B_t^{(\alpha^*)}) = \begin{cases} \sqrt{D^+}B_t^{(\alpha^*)} & \text{if} \quad B_t^{(\alpha^*)} \geq 0 \\ \sqrt{D^-}B_t^{(\alpha^*)} & \text{if} \quad B_t^{(\alpha^*)} < 0. \end{cases} \tag{5.5}$$

where $\alpha^* = \frac{\sqrt{D^+}}{\sqrt{D^+}+\sqrt{D^-}}$ and $s(x) = \sqrt{D^+}x^+ - \sqrt{D^-}x^-$ is the natural scale function; see Ramirez et al. (2007). In general we refer to the process Y given by (5.4) as the *physical skew diffusion with drift v* for the problem (5.1).

Remark 5.1. *The advective-dispersive movement of solutes through a porous medium in the presence of a discontinuous interface separating fine* *and coarse regions is a topic of both active experimental and theoretical interest (Hoteit, Mose, Younes, Lehmann and Ackerer (2002); Kuo, Irwin, Greenkorn and Cushman (1999); LaBolle, Quastel and Fogg (1998); LaBolle, Quastel, Fogg and Gravner (2000), Berkowitz et al. (2009); Ramirez et al. (2008, 2006)). The results obtained in Ramirez et al. (2006) for a rescaling of α-skew Brownian motion for a uniquely determined value of α as the underlying stochastic particle motion governing the (deterministic) Fickian advection-dispersion concentration in the presence of the sharp interface parallel to the flow made it possible to obtain the time-asymptotic central limit theorem and effective dispersion rate, extending the classic Taylor-Aris formula to this setting. This also provided the theoretical foundation for the correct Monte-Carlo approach among those considered in connection with experiments parallel to the flow by Hoteit et al. (2002); see Ramirez et al. (2008). However, the absence of drift in the particle coordinate associated with the α-skew Brownian motion was essential for those developments.*

The results of the previous sections now provide an approach to explicitly compute the concentration curves for flow orthogonal to an interface; see Berkowitz et al. (2009). The following bivariate distribution of the position and occupation time is algebraically more complicated than for that of position and local time, e.g., Corollary 1.1, but it is most relevant to our application in this section.

The transition probabilities of the physical skew dispersion with drift v are obtained in the following theorem. Note that from the definition of S^*_t in (5.5), the occupation time of the positive semiaxis of S^* equals $\Gamma^{(\alpha^*)}_t$, the occupation time by $B^{(\alpha^*)}_t$ of the positive semiaxis. Likewise, the joint density of position and occupation time of S^* and $B^{(\alpha^*)}$ satisfy

$$f_{(S^*_t,\Gamma^{(\alpha^*)}_t)}(y;z,\tau) = \frac{1}{\sqrt{D(z)}} f_{(B^{(\alpha^*)}_t,\Gamma^{(\alpha^*)}_t)}\Big(\frac{y}{\sqrt{D(y)}};\frac{z}{\sqrt{D(z)}},\tau\Big), \tag{5.6}$$

where $f_{(B^{(\alpha^*)}_t,\Gamma^{(\alpha^*)}_t)}$ is given in Corollary 3.4.

THEOREM 5.1. *The transition probability densities for the physical skew diffusion process Y with drift v defined by (5.4) are given by*

$$p(t,y,z) = \mathrm{e}^{\left(\frac{v}{D(z)}z - \frac{v^2}{2D^-}t\right)} \mathrm{e}^{\left(-\frac{v}{D(y)}y\right)} \hat{f}_{(S^*_t,\Gamma^{(\alpha^*)}_t)}\Big(z;y,\frac{v^2}{2}\big(\frac{1}{D^+}-\frac{1}{D^-}\big)\Big),$$

where $\hat{f}_{(S^*_t,\Gamma^{(\alpha^*)}_t)}(y;z,\lambda), \lambda \geq 0$, is given by

$$\hat{f}_{(S^*_t,\Gamma^{(\alpha^*)}_t)}(y;z,\lambda) = \int_0^t e^{-\lambda\tau} f_{(S^*_t,\Gamma^{(\alpha^*)}_t)}(y;z,\tau)d\tau.$$

PROOF. For $c(t,y)$ defined by (5.1), consider the change of concentration given by

$$\tilde{c}(t,y) = e^{-\frac{v}{D(y)}y} c(t,y). \tag{5.7}$$

Then, it is straightforward to show that $\tilde{c}(t,y)$ evolves according to the following skew reaction-dispersion equation

$$\frac{\partial \tilde{c}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial y}(D(y)\frac{\partial \tilde{c}}{\partial y}) - \frac{v^2}{2D(y)}\tilde{c} \tag{5.8}$$

$$\tilde{c}(t,0^+) = \tilde{c}(t,0^-), \qquad D^+\frac{\partial \tilde{c}}{\partial y}(t,0^+) = D^-\frac{\partial \tilde{c}}{\partial y}(t,0^-). \tag{5.9}$$

Moreover, from (5.7),

$$\tilde{c}(0,y) = c_0(y)e^{-\frac{v}{D(y)}y}. \tag{5.10}$$

It now follows from the Feynman-Kac formula that

$$\begin{aligned} c(t,y) &= \mathrm{e}^{\frac{v}{D(y)}y}\tilde{c}(t,y) \\ &= \mathrm{e}^{\frac{v}{D(y)}y}\mathbb{E}_y\left(c_0(S_t^*)\mathrm{e}^{-\frac{v}{D(S_t^*)}S_t^*}\mathrm{e}^{\left(-\int_0^t \frac{v^2}{2D(S_r^*)}dr\right)}\right), \end{aligned} \tag{5.11}$$

where $S_t^*, t \geq 0$, denotes the driftless rescaled (physical) skew Brownian motion defined by (5.5).

In view of the special form (5.3) of the dispersion coefficient, this formula may be re-expressed in terms of the occupation time,

$$\Gamma_t^{(\alpha)} = \int_0^t \mathbf{1}_{[0,\infty)}(B_s^{(\alpha)})ds$$

of skew Brownian motion on the positive axis. Namely,

$$c(t,y) = \mathrm{e}^{\frac{v}{D(y)}y}e^{-\frac{v^2}{2D^-}t}\mathbb{E}_y\left(c_0(S_t^*)\mathrm{e}^{-\frac{v}{D(S_t^*)}S_t^*}\mathrm{e}^{\left(-\frac{v^2}{2}(\frac{1}{D^+}-\frac{1}{D^-})\Gamma_t^{(\alpha^*)}\right)}\right). \tag{5.12}$$

From the relation (5.6), it follows that (5.12) may be expressed as

$$\begin{aligned} c(t,y) \;&=\; \mathrm{e}^{\left(\frac{v}{D(y)}y - \frac{v^2}{2D^-}t\right)} \times \\ &\int_0^t\int_{-\infty}^{\infty} c_0(z)\mathrm{e}^{\left(-\frac{v}{D(z)}z\right)}\mathrm{e}^{\left(-\frac{v^2}{2}(\frac{1}{D^+}-\frac{1}{D^-})\tau\right)} f_{(S_t^*,\Gamma_t^{(\alpha^*)})}(y;z,\tau)\;dy\;d\tau. \end{aligned}$$

$\square$

In addition to concentration curves, as noted in the introduction, breakthrough fluxes have been the subject of recent experiments in which a particularly distinguished asymmetry in breakthrough curves have been reported under mirror symmetric flows from fine to coarse and coarse to fine geometries. The experiments of Berkowitz et al. (2009) were intended to explore the so-called flux-averaged breakthrough concentration curves $c_f(t) = c_f(t, y)$, for fixed y as a function of t, defined by

$$c_f(t, y) = c(t, y) - \frac{D(y)}{2v}\frac{\partial c}{\partial y}, \tag{5.13}$$

in the presence of the two different configurations (fine-to-coarse and coarse-to-fine) under mirror symmetric reversed flow conditions. Using Theorem 5.1, one may explicitly analyze the resulting fluxes defined by (5.13) for observed asymmetries; e.g., see plots in Appuhamillage et al. (2009). While such curves are consistent with experiment, from a probabilistic point of view first passage times provide a more natural formulation of this phenomena. Since the mirror image of velocity is used in the fine to coarse and coarse to fine arrangements, we take $v = 0$ to focus on the pure effect of the interface on dispersion. Also one recall from Remark 1.1 that the parameter $\gamma \equiv \gamma(\alpha, v)$ specifying the elastic change of measure is invariant under the transformations $\alpha \to 1 - \alpha$, $v \to -v$.

The first symmetry result for concentration profiles provides a point of contrast to first passage times. In addition, it highlights a symmetrization of Walsh's formula (2.2) by the physical diffusion; i.e., rescaling space by the respective diffusivities symmetrizes the transition probabilities when $\alpha = \alpha^*$.

PROPOSITION 5.1. *Let $p^{(\alpha)}(t, x, y)$ denote the transition probabilities for $\alpha-$skew Brownian motion given in (2.2) and let $p^*(t, x, y)$ denote the transition probabilities for Y in the case $v = 0$. Then $p^{(\alpha)}(t, x, y)$ is asymmetric and discontinuous across the interface, while $p^*(t, x, y)$ is symmetric and continuous across the interface.*

PROOF. The first assertion follows from inspection of Walsh's formula (2.2) and the second by the indicated change of variables to obtain the transition probabilities of $Y_t = s(B_t^{(\alpha^*)}), t \geq 0$, for $\alpha^* = \frac{\sqrt{D^+}}{\sqrt{D^+}+\sqrt{D^-}}$. Namely, if $y \geq 0$ then

$$p^*(t,x,y) = \begin{cases} \dfrac{2}{\sqrt{D^+}+\sqrt{D^-}} \dfrac{1}{\sqrt{2\pi t}} \mathrm{e}^{\left(-\frac{(y\sqrt{D^-}-x\sqrt{D^+})^2}{2D^-D^+t}\right)} & \text{if } x \leq 0, \\ \dfrac{1}{\sqrt{2\pi D^+ t}} \left[\mathrm{e}^{\left(-\frac{(y-x)^2}{2D^+t}\right)} + \dfrac{\sqrt{D^+}-\sqrt{D^-}}{\sqrt{D^-}+\sqrt{D^+}} \mathrm{e}^{\left(-\frac{(y+x)^2}{2D^+t}\right)} \right] & \text{if } x > 0, \end{cases}$$

whereas if $y \leq 0$, then

$$p^*(t,x,y) = \begin{cases} \dfrac{1}{\sqrt{2\pi D^- t}} \left[\mathrm{e}^{\left(-\frac{(y-x)^2}{2D^-t}\right)} - \dfrac{\sqrt{D^+}-\sqrt{D^-}}{\sqrt{D^-}+\sqrt{D^+}} \mathrm{e}^{\left(-\frac{(y+x)^2}{2D^-t}\right)} \right] & \text{if } x < 0. \\ \dfrac{2}{\sqrt{D^+}+\sqrt{D^-}} \dfrac{1}{\sqrt{2\pi t}} \mathrm{e}^{\left(-\frac{(y\sqrt{D^+}-x\sqrt{D^-})^2}{2D^-D^+t}\right)} & \text{if } x \geq 0, \end{cases}$$

□

The resulting corollaries to Theorem 5.1 establish a simple probabilistic basis for the symmetries and asymmetries predicted by experimental results of Berkowitz et al (2009) and Kuo et al. (2000).

COROLLARY 5.1. *Let $Y^{(\pm v)}$ denote the respective physical skew diffusions with drift v. Then for any $y, t \geq 0$,*

$$P_{-y}(Y_t^{(v)} \in dy) = P_y(Y_t^{(-v)} \in -dy).$$

PROOF. In the case $v = 0$ this is the previously noted symmetrization of Walsh's formula given by Proposition 5.1, i.e., $p^*(t,-y,y) = p^*(t,y,-y)$. The extension to $v \neq 0$ may be checked from Theorem 5.1. □

On the other hand, as suggested by Corollary 1.3, mirror symmetry of the geometric configuration results in an asymmetric stochastic ordering of the breakthrough times. In particular, *fine to coarse breakthrough is faster than coarse to fine breakthrough*! To isolate the role of the interface in the first passage time between symmetrically configured fine to coarse and coarse to fine media, we take $v = 0$ and consider the fine to coarse configuration.

LEMMA 5.1. *For $c > 0$, $0 < \alpha < 1$, let $B^{(\alpha)}$ be skew Brownian motion starting at $B_0^{(\alpha)} = 0$. Then the process $\{B_{ct}^{(\alpha)} : t \geq 0\}$ is distributed as $c^{\frac{1}{2}} B^{(\alpha)}$.*

PROOF. This follows immediately from the formula (2.2) for the transition probabilities through the finite dimensional distributions of the process started at zero. ☐

COROLLARY 5.2. *Suppose that* $\sqrt{D^-} < \sqrt{D^+}, v = 0$. *Let* $Y = s(B^{(\alpha^*)})$ *denote the corresponding physical diffusion. Also let*

$$T_y^* = \inf\{t \geq 0 : Y_t = y\}.$$

Then for each $t > 0$,

$$P_{-y}(T_y^* > t) < P_y(T_{-y}^* > t).$$

PROOF. Without loss of generality take $y = 1$. Using the scaling property from Lemma 5.1 and symmetry of Brownian motion, one has

$$T_0^* =^{P_1-\text{dist}} \frac{1}{D^+} T_0, \tag{5.14}$$

and

$$T_0^* =^{P_{-1}-\text{dist}} \frac{1}{D^-} T_0. \tag{5.15}$$

Next, one similarly has

$$T_1^* =^{P_0-\text{dist}} \frac{1}{D^+} T_1^{(\alpha^*)}, \tag{5.16}$$

and

$$T_{-1}^* =^{P_0-\text{dist}} \frac{1}{D^-} T_1^{(1-\alpha^*)}. \tag{5.17}$$

In particular, using the strong Markov property to obtain convolutions, one has

$$P_{-1}(T_1^* > t) = \int_0^t P_0(\frac{1}{D^+} T_1^{(\alpha^*)} > t - s) P_0(\frac{1}{D^-} T_1 \in ds), \tag{5.18}$$

and

$$P_1(T_{-1}^* > t) = \int_0^t P_0(\frac{1}{D^-} T_1^{(1-\alpha^*)} > t - s) P_0(\frac{1}{D^+} T_1 \in ds). \tag{5.19}$$

Now, for $D^- < D^+$, $1 > \alpha^* > 1/2$.. Thus, in view of Corollary 1.3, the term $\frac{1}{D^-} T_1 \equiv \frac{1}{D^-} T_1^{(1/2)}$ is stochastically smaller than $\frac{1}{D^-} T_1^{(1-\alpha^*)}$ under P_0, and similarly the term $\frac{1}{D^+} T_1^{(\alpha^*)}$ is stochastically smaller than $\frac{1}{D^+} T_1$ under P_0. The assertion follows by an application of Lemma 4.1. ☐

At the (macro) scale of particle concentrations, the determination of the appropriate parameter $\alpha = \alpha^*$ can be deduced from conservation (continuity) principles; Uffink (1985), Ramirez et al. (2006). However from a probabilistic point of view we will see that one may also arrive at α^* by two different but related "stochastic balancing principles."One may be viewed in terms of a martingale property, and the other is equivalent to a continuity correction to a local time by the physical skew diffusion. While simple, such principles at the scale of individual particle motions provide a probabilistic basis for possible extensions of the theory to more complex geometries not available at the scale of (5.1). We close by establishing these two "principles".

In establishing these principles, the Itô - Tanaka and the occupation time formulae are repeatedly used. We find it convenient to use the versions of these formulae utilizing the *right local time* of the processes involved. Namely, given a semimartingale Y with quadratic variation denoted by $\langle Y, Y\rangle$ its right local time at a is defined by

$$A_+^Y(t,a) = \lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\int_0^t \mathbf{1}_{[a\le Y_s<a+\epsilon]}d\left\langle Y,Y\right\rangle_s.$$

Recall also that the *left local time*, A_-^Y and the *symmetric local time* L^Y are respectively defined as

$$A_-^Y(t,a) = \lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\int_0^t \mathbf{1}_{[a-\epsilon< Y_s<a]}d\left\langle Y,Y\right\rangle_s,$$

and

$$L_t^Y(a) = \frac{1}{2}\left[A_+^Y(t,a)+A_-^Y(t,a)\right].$$

In the particular case of skew Brownian motion, the following relations among one sided and the symmetric local times at 0 are known; e.g. see Ouknine (1990).

$$2\alpha L_t^{B^{(\alpha)}}(0) = A_+^{B^{(\alpha)}}(t,0), \quad 2(1-\alpha)L_t^{B^{(\alpha)}}(0) = A_-^{B^{(\alpha)}}(t,0) \tag{5.20}$$

In particular,

$$dB_t^{(\alpha)} = dB_t + \frac{2\alpha-1}{2\alpha}dA_+^{B^{(\alpha)}}(t,0) \tag{5.21}$$

Proof of Theorem 1.5 (Martingale Determination of α^*): Recall that $S_t^* = \sqrt{D^+}(B_t^{(\alpha^*)})^+ - \sqrt{D^-}(B_t^{(\alpha^*)})^-$. First, note that with $A_t^*(a) = A_+^{S^*}(t,a)$ one has that at $a=0$,

$$A_t^*(0) = \sqrt{D^+}\ A_+^{B^{(\alpha^*)}}(t,0). \tag{5.22}$$

Indeed, from the definitions,

$$\begin{aligned} A_t^*(0) &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^t \mathbf{1}_{[0\le S_s^*<\epsilon]}\ d\langle S^*, S^*\rangle_s \\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^t \mathbf{1}_{[0\le B_s^{(\alpha^*)}<\frac{\epsilon}{\sqrt{D^+}}]} D^+\ ds \\ &= \sqrt{D^+}\ A_+^{B^{(\alpha^*)}}(t,0). \end{aligned}$$

We make the claim that in terms of $A_t^*(0)$,

$$dS_t^* = \sqrt{D(B_t^{(\alpha^*)})}\ dB_t + \Big(\frac{D^+ - D^-}{2D^+}\Big)\ dA_t^*(0). \tag{5.23}$$

To see this, apply the Itô-Tanaka formula to $(S_t^*)^+$ to get

$$d(S_t^*)^+ = \sqrt{D^+}\Big[\mathbf{1}_{[B_t^{(\alpha^*)}>0]}dB_t + \frac{1}{2}dA_+^{B^{(\alpha^*)}}(t,0)\Big]. \tag{5.24}$$

Similarly, an application of the Itô-Tanaka formula in connection with $(S_t^*)^-$, (5.21), and the fact that the local time of skew Brownian motion is supported at $x=0$, yields

$$d(S_t^*)^- = \sqrt{D^-}\Big[-\mathbf{1}_{[B_t^{(\alpha^*)}\le 0]}dB_t - \frac{(2\alpha^*-1)}{2\alpha^*}dA_+^{B^{(\alpha^*)}}(t,0) + \frac{1}{2}dA_+^{B^{(\alpha^*)}}(t,0)\Big]. \tag{5.25}$$

Thus, recalling that $\alpha^* = \sqrt{D^+}/[\sqrt{D^-}+\sqrt{D^+}]$, it follows from (5.24), (5.25) that

$$dS_t^* = d(S_t^*)^+ - d(S_t^*)^- = \sqrt{D(B_t^{(\alpha^*)})}\ dB_t + \Big(\frac{D^+ - D^-}{2\sqrt{D^+}}\Big)\ dA_+^{B^{(\alpha^*)}}(t,0). \tag{5.26}$$

The claim now follows as a consequence of (5.26) and (5.22).

Now suppose $f\in\mathcal{D}_{D^\pm}$. Hence, f is the difference of two convex functions and its second generalized derivative is

$$f''(da) = f''(a)da + [f'(0^+) - f'(0^-)]\delta_0 \tag{5.27}$$

From the Itô-Tanaka formula, and using (5.23) (5.27), it follows that

$$\begin{aligned} f(S_t^*) &= f(S_0^*) + \int_0^t f_-'(S_s^*)dS_s^* + \frac{1}{2}\int_{\mathbb{R}} A_t^*(a)f''(da) \\ &= f(x) + \int_0^t f'(S_s^*)\sqrt{D(S_s^*)}\ dB_s + \frac{1}{2}\int_{\mathbb{R}} A_t^*(a)\ f''(a)da \\ &\quad + \frac{D^+ - D^-}{2D^+}\int_0^t f_-'(S_s^*))dA_t^*(0) + \frac{1}{2}\int_{\mathbb{R}} A_t^*(a)\,[f'(0^+) - f'(0^-)]\,\delta_0. \end{aligned} \tag{5.28}$$

By the occupation time formula, and noting that the quadratic variation of S^* is given by $D(S_t^*)dt$,

$$\frac{1}{2}\int_{\mathbb{R}} A_t^*(a) f''(da) = \frac{1}{2}\int_0^t D(S_s^*) f''(S_s^*) ds.$$

The theorem is established once we show that the expression in (5.28) vanishes. To see this, note that this expression is the local time at the origin multiplied by

$$\frac{D^+ - D^-}{2D^+} f'(0^-) + \frac{1}{2}\left[f'(0^+) - f'(0^-)\right] = \frac{1}{2}\left(f'(0^+) - \frac{D^-}{D^+} f'(0^-)\right).$$

This vanishes in light of the interface condition imposed on $f \in \mathcal{D}_{D^\pm}$. $\square$

Proof of Theorem 1.6(Continuity Correction to Local Time): Note that as a consequence of (5.20)

$$\frac{A_+^{B^{(\alpha)}}(t,0)}{A_-^{B^{(\alpha)}}(t,0)} = \frac{\alpha}{1-\alpha}.$$

Also, recalling the definition of $s(B^{(\alpha)})$, one obtains

$$\begin{aligned}\tilde{A}_+^{s(B^{(\alpha)})}(t,0) &= \lim_{\epsilon\downarrow 0} \frac{1}{\sqrt{D^+}} \frac{\sqrt{D^+}}{\epsilon} \int_0^t \mathbf{1}_{[0\le B_s^{(\alpha)} < \epsilon/\sqrt{D^+}]} ds \\ &= \frac{1}{\sqrt{D^+}} A_+^{B^{(\alpha)}}.\end{aligned}$$

Similarly,

$$\begin{aligned}\tilde{A}_-^{s(B^{(\alpha)})}(t,0) &= \lim_{\epsilon\downarrow 0} \frac{1}{\sqrt{D^-}} \frac{\sqrt{D^-}}{\epsilon} \int_0^t \mathbf{1}_{[\epsilon/\sqrt{D^-} < B_s^{(\alpha)} < 0]} ds \\ &= \frac{1}{\sqrt{D^-}} A_-^{B^{(\alpha)}}.\end{aligned}$$

Thus,

$$\frac{\tilde{A}_+^{s(B^{(\alpha)})}(t,0)}{\tilde{A}_-^{s(B^{(\alpha)})}(t,0)} = \frac{\sqrt{D^-}}{\sqrt{D^+}} \frac{\alpha}{1-\alpha}. \tag{5.29}$$

The continuity follows if and only if $\alpha = \alpha^* := \frac{\sqrt{D^+}}{\sqrt{D^+}+\sqrt{D^-}}$. $\square$

6. Summary and Open Problems. The foundational component of this paper provides an extension of basic probability laws governing the trivariate density of Brownian motion, local time and occupation time and their coordinate projections to those for skew Brownian motion. Along the way the basic Feynman-Kac formula for elastic skew Brownian motion was also obtained. The elastic change of concentration measure to a reaction-dispersion concentration was shown to lead to a closed form determination of the concentration curves and transition probabilities for the physical skew diffusion with drift.

The presence of local time and drift presents new mathematical challenges for both Monte–Carlo numerical simulations and other schemes for numerical computation of the fundamental solution to the advection-dispersion equation or, equivalently, the transition probabilities of the process Y. The Zvonkin transformation, e.g., see Rogers and Williams (1987) for background, was explored in Lejay and Martinez (2006) to remove drift for Monte-Carlo purposes. For the present problem (5.1) this transformation results in a coefficient $\rho(x)$ of the second order operator that their theory requires to be bounded. However, in the present application the coefficient $\rho(x)$ is unbounded, in fact, it grows exponentially. In particular, while some interesting examples are included to illustrate their approach, the particle methods developed in Lejay and Martinez (2006) do not apply to (5.1). Another interesting alternative to the more rigorously developed Itô-Tanaka stochastic calculus, that avoids the use of local time and deals directly with generalized stochastic processes, was somewhat formally explored in LaBolle, Quastel, Fogg and Gravner (2000). A companion analytic approach has also been developed by Portenko (1990) in the context of pde's. The results of the present paper together with those of Portenko (1990) may prove useful in putting some of the ideas in LaBolle et al. (2000) on a more rigorous foundation. It certainly illustrates a rich general problem area.

The main point of the application considered in this paper was to (1) precisely determine the structure of concentration as predicted by Fickian advection-dispersion conservation laws in the presence of a sharp interface orthogonal to the flow direction, and (2) to analyze the role of the interface on breakthrough in terms of first passage times. Part of the goal was to dispel speculation among scientists of a need for refinements to the Fickian conservation laws in this context; see Berkowitz et al. (2008). In particular, it has been rigorously established that the Fickian laws provide general qualitative agreement with symmetries and asymmetries observed in experiments.

A number of interesting directions are possible in connection with applications of this type. Having resolved the principal coordinate directions of flow, it is natural to pursue applications to more complicated geometries; example advective-dispersive flow in media with spherical intrusions of contrasting dispersion rates.

The solution provided here also provides a benchmark to test various possible numerical and/or Monte–Carlo particle tracking schemes designed to address interfacial discontinuities. The role of local time presents one of the biggest challenges to Monte–Carlo simulation of particle tracking schemes. The transformation to a skew reaction-dispersion equation together with the Feynman–Kac formula and importance sampling methods may make this theory amenable to Monte–Carlo techniques.

We close by mentioning an important unsolved probability problem related to other types of breakthrough measurements, namely, the explicit determination of first passage time density of a particle injected at -1 to reach 1. An explicit formula for this density is unknown to our best knowledge.

References.

Appuhamillage, T., Bokil, V. A., Thomann, E., Waymire, E. and Wood, B. (2009). Solute transport across an interface: A Fickian theory for skewness in breakthrough curves . *Water Resour. Res. (to appear)*.

Barlow, M., Burdzy, K., Kaspi, H. and Mandelbaum, A. (2000). Variably skewed Brownian motion. *Electron. Comm. Probab.*, **5** 57–66 (electronic).

Barlow, M., Burdzy, K., Kaspi, H. and Mandelbaum, A. (2001). Coalescence of skew Brownian motions. *Lecture Notes in Math.* 202–205.

Barlow, M., Pitman, J. and Yor, M. (1989). On Walsh's Brownian motions. In *Séminaire de Probabilités, XXIII*, vol. 1372 of *Lecture Notes in Math.* Springer, Berlin, 275–293.

Berkowitz, B., Cortis, A., Dror, I. and Scher, H. (2008). Laboratory experiments on dispersive transport across interfaces: The role of flow direction. *EOS, Transactions of the American Geophysical Union* Fall Meeting Supplement, Abstract H23H–05.

Berkowitz, B., Cortis, A., Dror, I. and Scher, H. (2009). Laboratory experiments on dispersive transport across interfaces: The role of flow direction. *Water Resour. Res.*, **45** http://doi:10.1029/2008WR007342.

Bhattacharya, R. and Waymire, E. (2009). *Stochastic processes with applications.* Classics in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM).

Burdzy, K. and Chen, Z. (2001). Local time flow related to skew Brownian motion. *Annals of Probability* 1693–1715.

Decamps, M., Goovaerts, M. and Schoutens, W. (2006). Asymmetric skew Bessel processes and their applications to finance. *Journal of Computational and Applied Mathematics*, **186** 130–147.

Freidlin, M. and Sheu, S.-J. (2000). Diffusion processes on graphs: stochastic differential equations, large deviation principle. *Probab. Theory Related Fields*, **116** 181–220. URL http://dx.doi.org/10.1007/PL00008726.

Harrison, J. M. and Shepp, L. A. (1981). On skew Brownian motion. *The Annals of probability*, **9** 309–313.

Hoteit, H., Mose, R., Younes, A., Lehmann, F. and Ackerer, P. (2002). Threedimensional modeling of mass transfer in porous media using the mixed hybrid finite elements and the random-walk methods. *Mathematical Geology*, **34** 435–456.

Itô, K. and McKean, H. (1963). Brownian motions on a half line. *Ill. J. Math.*, **7** 181–231.

Itô, K. and McKean, H. J. (1996). *Diffusion Processes and their Sample Paths.* Classics in Mathematics, Springer, NY.

Karatzas, I. and Shreve, S. (1984). Trivariate density of Brownian motion, its local and occupation times, with application to stochastic control. *The Annals of Probability*, **12** 819–828.

Kuo, R., Irwin, N., Greenkorn, R. and Cushman, J. (1999). Experimental Investigation of Mixing in Aperiodic Heterogeneous Porous Media: Comparison with Stochastic Transport Theory. *Transport in Porous Media*, **37** 169–182.

LaBolle, E., Quastel, J. and Fogg, G. (1998). Diffusion theory for transport in porous media: Transition-probability densities of diffusion processes corresponding to advection-dispersion equations. *Water Resour. Res.*, **34** 1685–1693.

LaBolle, E., Quastel, J., Fogg, G. and Gravner, J. (2000). Diffusion processes in composite porous media and their numerical integration by random walks: Generalized stochastic differential equations with discontinuous coefficients. *Water Resour. Res.*, **36** 651–662.

Le Gall, J.-F. (1982). *Temps locaux et equations differentielles stochastiques.* Ph.D. thesis, Th'ese de troisi'eme cycle, Universite Pierre et Marie Curie (Paris VI), Paris.

Le Gall, J.-F. (1984). One-dimensional stochastic differential equations involving the local times of the unknown process. In *Stochastic analysis and applications (Swansea, 1983)*, vol. 1095 of *Lecture Notes in Math.* Springer, Berlin, 51–82.

Lejay, A. and Martinez, M. (2006). A scheme for simulating one-dimensional diffusion processes with discontinuous coefficients. *Annals of Applied Probability*, **16** 107–139.

Ouknine, Y. (1990). Le "Skew-Brownian motion" et les processus qui en dérivent. *Teor. Veroyatnost. i Primenen.*, **35** 173–179.

Portenko, N. (1990). *Generalized Diffusion Processes.* 1st ed. Translations of Mathematical monographs, American Mathematical Society.

Ramirez, J. (2009). Multi-skewed brownian motion and diffusion. *(under review).*

Ramirez, J., Thomann, E., Waymire, E., Chastanet, J. and Wood, B. (2008). A note on the theoretical foundations of particle tracking methods in heterogeneous porous media. *Water Resour. Res.*, **44** W01501,doi:10.1029/2007WR005914.

Ramirez, J., Thomann, E., Waymire, E., Haggerty, R. and Wood, B. (2006). A generalized Taylor-Aris formula and skew diffusion. *SIAM J. Multiscale Modeling and Simulation*, **5** 786–801.

Revuz, D. and Yor, M. (1991). *Continuous martingales and Brownian motion*, vol. 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin.

Rogers, L. and Williams, D. (1987). *Diffusions, Markov processes, and martingales, Volume 2: Itô Calculus.* Wiley, Chichester.

Trotter, H. (1958). A property of brownian motion paths. *Ill. J. Math.*, **2** 425–433.

Uffink, G. J. M. (1985). A random walk method for the simulation of macrodispersion in a stratified aquifer,. *Relation of Groundwater Quantity and Quality, IAHS Publ.* . 103–114.

Walsh, J. B. (1978). A diffusion with a discontinuous local time. *Asterisque*, **52–53** 37–45.

Department of Mathematics
Oregon State University
Corvallis, OR 97331 USA.
E-mail: ireshara@math.oregonstate.edu
bokilv@math.oregonstate.edu
thomann@math.oregonstate.edu
waymire@math.oregonstate.edu

School of Chemical, Biological
& Environmental Engineering
Oregon State University
Corvallis, OR 97331 USA.
E-mail: brian.wood@oregonstate.edu