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Occurrence of Strange Axiom A Attractors Near Quasi Periodic Flows on T^m, $m \geqq 3$

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Abstract. It is shown that by a small C^2 (resp. C^∞) perturbation of a quasi-periodic flow on the 3-torus (resp. the m-torus, $m>3$), one can produce strange Axiom A attractors. Ancillary results and physical interpretation are also discussed.

1. Statement of Results

The main purpose of this note is to prove the following fact.

Theorem 1. *Let $a=(a_1, \dots, a_n)$ be a constant vector field on the torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$.*

If $n=3$, in every C^2 neighborhood of a there is a vector field satisfying Axiom A and having a non trivial attractor.

If $n \geqq 4$, in every C^∞ neighborhood of a there is a vector field satisfying Axiom A and having a non trivial attractor.

We say that an Axiom A attractor is non trivial (or "strange") if it does not consist of a single periodic orbit (for general definitions, see Smale [7]). The above theorem improves a result of Ruelle and Takens [6], and can be obtained by simple modifications of the proof given there. We nevertheless give here a complete proof, based on the following result which is of interest in itself.

Theorem 2. *Let M be a C^∞ compact manifold of dimension m.*

(a) *If $m=2$, in every C^1 neighborhood of the identity there is an Axiom A diffeomorphism with a non trivial attractor.*

(b) *If $M=T^2$, in every C^2 neighborhood of the identity there is an Axiom A diffeomorphism with a non trivial attractor.*

(c) *If $m \geqq 3$, in every C^∞ neighborhood of the identity there is an Axiom A diffeomorphism with a non trivial attractor.*

The proof of these theorems is given in Sect. 2. In Sect. 3 we discuss non trivial attractors for Axiom A diffeomorphisms of two-dimensional manifolds. Section 4 is devoted to physical interpretation of Theorem 1.

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2. Proofs

To prove Theorem 1 notice that a can be approximated in the C^∞ topology by a constant vector field $b=(k_1c, \dots, k_nc) \neq 0$ with $c>0$ and the $k_i \in \mathbb{Z}$. One may assume that $|k_1|, \dots, |k_n|$ have no common divisor. There is then an automorphisms of T^n which transforms b to $(0, \dots, 0, c)$. It suffices thus to prove Theorem 1 for a of that form.

Choose a diffeomorphism φ of T^{n-1} according to Theorem 2 (b) or (c) and let $\tilde{\varphi}$ be the diffeomorphism of $\mathbb{R}^{n-1}$ close to the identity which induces φ on $\mathbb{R}^{n-1}/\mathbb{Z}^{n-1}$. Let α be a C^∞ function with compact support in $(0,1)$ such that $\alpha \geqq 0$ and $\int \alpha(t)dt = 1$; define $\beta(u) = \int_0^u \alpha(t)dt$.

Observe that $\beta(u)=\beta'(u)=0$ for $u \leqq 0$; $\beta(u)=1$ and $\beta'(u)=0$ for $u \geqq 1$. Let (x,u) be coordinates on $\mathbb{R}^{n-1} \times \mathbb{R}$ with $x \in \mathbb{R}^{n-1}$, $u \in \mathbb{R}$. The mapping

$$F\begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} x + (\tilde{\varphi}(x) - x)\beta(u) \\ u \end{pmatrix}$$

is a diffeomorphism from $\mathbb{R}^{n-1} \times [0,1]$ to $\mathbb{R}^{n-1} \times [0,1]$ which is as smooth as $\tilde{\varphi}$. For each x, the curve $u \mapsto F\begin{pmatrix} x \\ u \end{pmatrix}$, $0 \leqq u \leqq 1$, joins $\begin{pmatrix} x \\ 0 \end{pmatrix}$ to $\begin{pmatrix} \tilde{\varphi}(x) \\ 1 \end{pmatrix}$. The vector field $F_*\left(\frac{\partial}{\partial u}\right)$ on $\mathbb{R}^{n-1} \times [0,1]$ is given by

$$F_*\left(\frac{\partial}{\partial u}\right)\begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} (\tilde{\varphi}(x) - x)\beta'(u) \\ 1 \end{pmatrix}$$

for $x \in \mathbb{R}^{n-1}$ and $0 \leqq u \leqq 1$. Since $\beta'(u)=0$ for $u \leqq 0$ and $u \geqq 1$, $F_*\left(\frac{\partial}{\partial u}\right)$ may be periodically extended to a vector field on $\mathbb{R}^{n-1} \times \mathbb{R}$ which projects down to one on $T^{n-1} \times T^1 = T^n$. Call this last vector field X. The Poincaré map of X – or equivalently cX – from $T^{n-1} \times \{0\}$ to itself is $\tilde{\varphi}$, and cX is near $(0, \dots, 0, c)$, so cX satisfies the conditions of Theorem 1.

We shall see in Sect. 3 that there are Axiom A diffeomorphisms on 2-dimensional manifolds, which are isotopic to the identity, and where the non wandering set consists of a non trivial attractor and a finite number of periodic points. For the 2-torus we can in particular construct a diffeomorphism viewed as a map $\varphi: \mathbb{R}^2 \to \mathbb{R}^2$, such that the square $[0,1] \times [0,1]$ is preserved, its boundary mod$\mathbb{Z}^2$ containing one source (at $(0,0)$), two saddle points (at $(0, \frac{1}{2})$ and $(\frac{1}{2}, 0)$), and no other point of the nonwandering set. This situation is depicted in Fig. 1.

The fact that φ is isotopic to the identity reflects itself in the existence of a map $t \to \varphi_t$ of $[0,1]$ into C^∞ diffeomorphisms of $\mathbb{R}^2$ preserving the square $[0,1] \times [0,1]$ such that, for all k, $t \to \varphi_t$ is C^∞ from $[0,1]$ to the C^k diffeomorphisms. We can also arrange that (φ_t), restricted to a neighborhood of the boundary of the square $[0,1] \times [0,1]$ satisfies the flow condition

$$\varphi_{a+b} = \varphi_b \circ \varphi_a .$$

We shall now use the existence of the family (φ_t) to prove Theorem 2.

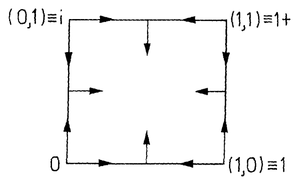

Fig. 1

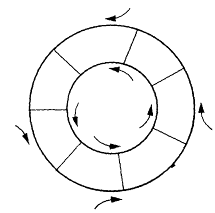

Fig. 2

(a) It will be convenient here to identity $\mathbb{R}^2$ to $\mathbb{C}$ so that φ becomes a complex function. For any integer $N \geqq 2$, and real $\varepsilon > 0$, let a map of the annulus

$$\{z \in \mathbb{C} : \varepsilon \leqq |z| \leqq \varepsilon e^{2\pi/N}\}$$

to itself be defined by

$$f_N(z) = \varepsilon \exp \frac{2\pi}{N} \left\{ k_i + \varphi_{\frac{k}{N}} \circ \left(\varphi_{\frac{k-1}{N}}\right)^{-1} \left[\frac{N}{2\pi} \left(\log \frac{z}{\varepsilon} \right) - (k-1)i \right] \right\}$$

if $(k-1)\dfrac{2\pi}{N} \leqq \arg z \leqq k\dfrac{2\pi}{N}$ for $k = 1, \ldots, N$.

This map turns the annulus by $\dfrac{2\pi}{N}$, transforming the "square"

$$\left\{ z : \varepsilon \leqq |z| \leqq \varepsilon e^{\frac{2\pi}{N}}, (k-1)\frac{2\pi}{N} \leqq \arg z \leqq k\frac{2\pi}{N} \right\}$$

by a map related to $\varphi_{\frac{k}{N}} \circ \left(\varphi_{\frac{k-1}{N}}\right)^{-1}$. The effect of the N-th iterate of f_N in each one of the above "squares" is thus conjugate to $\varphi_1 = \varphi$. It is now easy to extend f_N (defined in a local chart) to an Axiom A diffeomorphism of any 2-dimensional manifold M (see Fig. 2). If $N \to \infty$, then the differentiability of $t \to \varphi_t$ shows that f_N tends to the identity in C^1; the same can be obtained of its extension to M, proving the first part of Theorem 2.

(b) Let $h_N : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$h_N \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \dfrac{x_1}{N} \\[2ex] \dfrac{x_2}{N} + \dfrac{x_1}{N^2} \end{pmatrix}, \qquad h_N^{-1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} N x_1 \\ N x_2 - x_1 \end{pmatrix}.$$

For any integer $N \geqq 2$ a map f_N of $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ is defined by

$$f_N \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \dfrac{k_1}{N} \\[2ex] \dfrac{k_2 - 1}{N} + \dfrac{k_1}{N^2} \end{pmatrix}$$

$$+ h_N \circ \varphi_{\frac{k_1}{N^2} + \frac{k_2 - 1}{N}} \circ \left(\varphi_{\frac{k_1 - 1}{N} + \frac{k_2 - 1}{N}}\right)^{-1} \circ h_N^{-1} \begin{pmatrix} x_1 - \dfrac{k_1 - 1}{N} \\[2ex] x_2 - \dfrac{k_2 - 1}{N} - \dfrac{k_1 - 1}{N^2} \end{pmatrix}$$

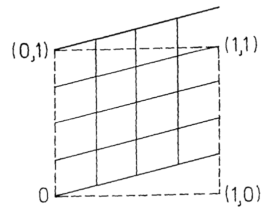

Fig. 3

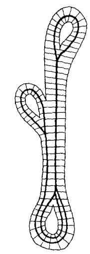

Fig. 4

in the "square"

$$\frac{k_1-1}{N} \leqq x_1 \leqq \frac{k_1}{N}, \quad \frac{k_2-1}{N} \leqq x_2 - \frac{x_1}{N} \leqq \frac{k_2}{N}$$

for $k_1, k_2 = 1, \ldots, N$. The map f_N permutes cyclically the above "squares", and the effect of the N^2-th iterate on each "square" is conjugate to $\varphi_1 = \varphi$ (see Fig. 3). Thus, f_N is an Axiom A diffeomorphisms, and the differentiability of $t \to \varphi_t$ shows that f_N tends to the identity in C^2 when $N \to \infty$, proving the second part of Theorem 2.

(c) For every integer $N \geqq 2$ there is an Axiom A diffeomorphism h_N of the circle $S^1 = \mathbb{R}/\mathbb{Z}$ with nonwandering set consisting of the attracting periodic orbit $\left\{0, \frac{1}{N}, \ldots, \frac{N-1}{N}\right\}$ and the repulsive periodic orbit $\left\{\frac{1}{2N}, \frac{3}{2N}, \ldots, \frac{2N-1}{2N}\right\}$. We assume that $h_N^k\left(\frac{i}{N}\right) = \frac{i+k}{N}$ and we also assume that h_N tends to the identity in the C^∞ topology when $N \to \infty$. A map $f_N : [0,1] \times [0,1] \times S^1 \to [0,1] \times [0,1] \times S^1$ is now defined by

$$\begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix} = \begin{pmatrix} \varphi_{h_N(y)} \circ \varphi_y^{-1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ h_N(y) \end{pmatrix}$$

and the differentiability of $t \to \varphi_t$ shows that f_N tends to the identity in C^∞ when $N \to \infty$. It is easy to extend f_N to an Axiom A diffeomorphism of any manifold M of dimension $m \geqq 3$, such that this diffeomorphism is close to the identity in the C^∞ topology for large N. This proves the last part of Theorem 2.

Remark. It would be possible to modify the above constructions so that the Axiom A diffeomorphisms constructed have only non trivial attractors.

3. Axiom A Attractors in Two Dimensions

An Axiom A diffeomorphism with a non trivial attractor has been constructed on T^2 by Smale [7]: The DA diffeomorphism. It is not isotopic to the identity, and its existence is due to the particular topology of T^2. It was not clear that a non trivial Axiom A attractor could exist on the sphere S^2 until Plykin [5] produced an example. Plykin's example is orientation reversing, but its square is orientation preserving and therefore isotopic to the identity. In general if an Axiom A diffeomorphism of S^2 has a nontrivial attractor, we can find a disc neighborhood

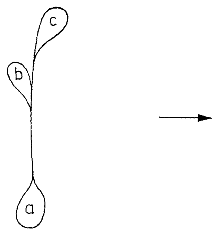

Fig. 5

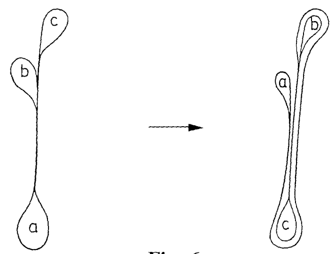

Fig. 6

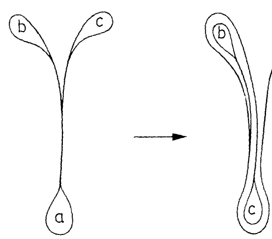

Fig. 7

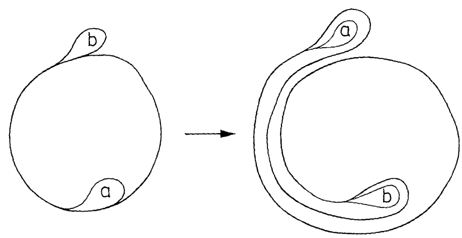

Fig. 8

D^2 of this attractor which is mapped into itself and study the corresponding map $D^2 \to D^2$. If it is orientation preserving this map can be used to introduce a non trivial Axiom A attractor into any compact 2-manifold. Figure 4 gives an example of a non trivial Axiom A attractor different from Plykin's example. The shaded area is mapped into the dark area in the manner indicated in Fig. 5. This map is not orientation preserving, but its square, pictured in Fig. 6, is. The collapsing of the shaded area of Fig. 4 to a line in Fig. 5 and Fig. 6 (more precisely a "branched manifold") is an idea due to Williams [8]. Other Axiom A attractors are pictured in Fig. 7 and Fig. 8, where Fig. 8 is in fact Plykin's example.

4. Physical Interpretation

Theorem 1 is relevant to the discussion of the bifurcation theory of turbulence by Ruelle and Takens [6]. Let the time evolution of a viscous flow be described by a differential equation

$$\frac{dx}{dt} = X_\mu(x)$$

where μ is a parameter ("Reynolds number"). Suppose that x_μ is a steady state which is stable for $\mu < \mu_c$, and looses its stability as μ increases because pairs of complex conjugate eigenvalues of $\frac{\partial X_\mu}{\partial x}(x_\mu, \mu)$ cross the imaginary axis. Then Theorem 1 implies that when three pairs of complex conjugate eigenvalues have crossed, a motion asymptotic to a non trivial Axiom A attractor *may* appear. The time dependence of the flow then becomes chaotic, with sensitive dependence on initial condition, a situation which one may call turbulent (see Lorenz [2], Ruelle and Takens [6]).

Theorem 1 is also relevant to another physical situation, that of a weak nonlinear coupling between oscillators. A system of n independent oscillating systems may be described by the equations

$$\frac{dx_1}{dt}=a_1,\ldots,\frac{dx_n}{dt}=a_n$$

where $x_1 \in \mathbb{R}/\mathbb{Z}, \ldots, x_n \in \mathbb{R}/\mathbb{Z}$. Theorem 1 shows that if $n \geqq 3$, a weak nonlinear coupling may produce a "turbulent" behavior. Theorem 1 fails for $n \leqq 2$. In fact flows on two-dimensional manifolds have rather special properties; for instance it is known that their topological entropy always vanishes (see Young [9]).

The examples of non trivial Axiom A attractors discussed in Sect. 3 are relatively complicated and would not occur in very simple polynomial maps of $\mathbb{R}^2$. It is not known if "turbulent" behavior could be produced by non Axiom A diffeomorphisms; for instance the mathematical status of the "Hénon attractor" (see [1]) is in doubt.

A phenomenon which does appear quite naturally in diffeomorphisms of 2-manifolds is the persistent occurrence of infinitely many sinks (see Newhouse [3], [4]). One has an open set in the C^r topology ($r \geqq 2$) and in this open set a residual subset of diffeomorphisms which have an infinite number of attracting periodic orbits. Such open sets can be found arbitrarily C^r near the time-one map of a flow with a hyperbolic rest point whose stable and unstable manifolds coincide. Thus they can be found arbitrarily near the identity. The method of proof of Theorem 1 yields then the following result.

Theorem 3. *Let $a=(a_1,a_2,a_3)$ be a constant vector field on the torus T^3. In every C^∞ neighborhood of a there is a vector field X with infinitely many attracting periodic orbits. This situation is persistent in the sense that in some open C^r neighborhood of X there is a residual set of vector fields with infinitely many attracting periodic orbits (any $r \geqq 2$).*

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