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Octants Are Cover-Decomposable

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Abstract We prove that octants are cover-decomposable; i.e., any 12-fold covering of any subset of the space with a finite number of translates of a given octant can be decomposed into two coverings. As a corollary, we obtain that any 12-fold covering of any subset of the plane with a finite number of homothetic copies of a given triangle can be decomposed into two coverings. We also show that any 12-fold covering of the whole plane with the translates of a given open triangle can be decomposed into two coverings. However, we exhibit an indecomposable 3-fold covering with translates of a given triangle.

Keywords Cover-decomposability · Geometric hypergraph coloring

1 Introduction

Let $\mathcal{P} = \{P_i \mid i \in I\}$ be a collection of geometric sets in $\mathbb{R}^d$. We say that $\mathcal{P}$ is an *m-fold covering* of a set S if every point of S is contained in at least m members of $\mathcal{P}$. A 1-fold covering is simply called a *covering*.

Definition A geometric set $P \subset \mathbb{R}^d$ is said to be *cover-decomposable* if there exists a (minimal) constant $m = m(P)$ such that every m-fold covering of any subset of $\mathbb{R}^d$ with a finite number of translates of P can be decomposed into two coverings of the same subset. Define m as the cover-decomposability constant of P. We note that in the literature the definition is slightly different and the notion defined here is

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sometimes called *finite-cover-decomposable*; however, to avoid unnecessary complications, we simply call it *cover-decomposable*.

The simplest objects to examine are the orthants of $\mathbb{R}^d$. It is easy to see that a quadrant (two-dimensional orthant) is cover-decomposable. Cardinal [4] noticed that orthants in four and higher dimensions are not cover-decomposable as there is a plane on which their trace can be any family of axis-parallel rectangles, and it was shown by Pach, Tardos, and Tóth [12] that such families might not be decomposable into two coverings. Cardinal asked whether octants (three-dimensional orthants) are cover-decomposable. Our main result is an affirmative answer (the proof is given in Sect. 2).

Theorem 1 *Octants are cover-decomposable*; *i.e.*, *any* 12*-fold covering of any subset of* $\mathbb{R}^3$ *with a finite number of translates of a given octant can be decomposed into two coverings.*

The intersection of the translates of the octant containing $(-\infty, -\infty, -\infty)$ with the $x + y + z = 0$ plane gives the homothetic copies of an equilateral triangle. Since any triangle can be obtained by an affine transformation of the equilateral triangle we obtain the following.

Corollary 2 *Any* 12*-fold covering of any subset of the plane with a finite number of homothetic copies of any given triangle can be decomposed into two coverings.*

We say that a covering is *locally finite* if every compact set intersects only a finite number of covering sets, i.e., homothetic copies of the given triangle, in our case. Using standard compactness arguments, the previous corollary implies the following (the proofs are given in Sect. 4).

Theorem 3 *Any locally finite* 12*-fold covering of the whole plane with homothetic copies of a triangle is decomposable into two coverings.*

Theorem 4 *Any* 23*-fold covering of the whole plane with homothetic copies of an open triangle is decomposable into two coverings.*

The analogs of Corollary 2 and Theorem 3 for translates of a given triangle were proved with a bigger constant by Tardos and Tóth [17] using a more complicated argument (the original proof gave $m = 43$, which was later improved by Ács [1] to $m = 19$, which is still worse than our 12). Following their idea, using Theorem 3 for translates of a given open triangle, we obtain the following.

Corollary 5 *Any* 12*-fold covering of the whole plane with the translates of an open triangle is decomposable into two coverings.*

Our result brings the task to determine the exact cover-decomposability constant of triangles in range. Tardos and Tóth state that they cannot even rule out the possibility that the cover-decomposability constant of triangles is 3. To complement our upper bound, in Sect. 3 we show a construction proving that the constant is actually at least 4.

Our proof of Theorem 1 in fact proves the following equivalent, dual form of Theorem 1.

Theorem 6 *Any finite set of points in* $\mathbb{R}^3$ *can be colored with two colors such that any translate of a given octant with at least* 12 *points contains both colors.*

To see that Theorem 1 implies Theorem 6, assume that the octant we fixed is the one containing $(-\infty, -\infty, -\infty)$ (the negative octant). Now for each vertex v, take the octant V having apex v and containing (∞, ∞, ∞) (a positive octant). Now a negative octant O contains v if and only if the positive octant V contains o, the apex of O. As containment is preserved, coloring the positive octants with apices of the original point set according to Theorem 1, the same coloring for the vertices gives a valid coloring for Theorem 6. The reverse implication is similar; it again uses that containment is preserved by this dualization (for more on dualization see the surveys [14] and [11]).

Finally, we mention the dual of Corollary 2, which is not equivalent to Corollary 2 but follows from Theorem 6 in the same way as Corollary 2 follows from Theorem 1.

Corollary 7 *Any finite planar point set can be colored with two colors such that any homothetic copy of a given triangle that contains at least* 12 *points contains both colors.*

We finally note that in this paper in many theorems it does not matter whether the respective underlying set (orthant, triangle, etc.) is open or not, and in the proofs, for simplicity, we consider the sets to be open (unless otherwise stated). Also, although the arrangement of the points is arbitrary, for simplicity, in the proofs we suppose that the objects are in general position, as a slight perturbation only increases the constraints that we have to satisfy.

For more about handling these issues and other results on cover-decomposability, see the recent surveys [14] and [11] and the papers [2, 3, 5, 9, 10, 13, 15–17].

2 Proof of Theorems 1 and 6

Denote by W the octant with apex at the origin containing $(-\infty, -\infty, -\infty)$. We will work in the dual setting; that is, we have a finite set of points, P, in the space, that we want to color with two colors such that any translate of W with at least 12 points contains both colors. We call such a two-coloring of a point set in the space a *good coloring*. If we can do such a coloring for any P, then it follows using a standard dualization argument (see [14] or [11]) that W (and thus any octant) is cover-decomposable. So from now on our goal will be to show the existence of such a coloring.

For simplicity, suppose that no number occurs multiple times among the coordinates of the points of P (otherwise, by a small perturbation of P we can get such a point set, and its coloring will also be good for P). Denote the point of P with the t^{th} smallest z coordinate by p_t and the union of $p_1, \ldots, p_t$ by P_t. First we will show how to reduce the coloring of P to a planar and thus more tractable problem.

Denote the projection of P on the $z = 0$ plane by P'. Similarly denote the projection of p_t by p'_t, the projection of P_t by P'_t, and the projection of W by W'. Therefore, W' is the quadrant with apex at the origin containing $(-\infty, -\infty)$.

For such an ordered planar point set P' we say that a coloring with two colors is a *good coloring*, if for any t and any translate of W' containing at least 12 points of P'_t, it is true that the intersection of this translate and P'_t contains both colors. To see why we use the same notation for two differently defined colorings, the next claim shows that good coloring of a spatial point set and good coloring of the corresponding planar point set are equivalent problems.

Claim 8 *The planar point set P' has a good coloring if and only if the spatial point set P has a good coloring.*

Proof Clearly, if we take a translate of W with apex w having z coordinate bigger than the z coordinate of p_t and smaller than the z coordinate of p_{t+1}, then the projection of the intersection of this translate with P is equal to the intersection of P'_t with the translate of W' having apex w', the projection of w. Thus, having a good coloring for one problem gives a good coloring for the other if we give p_t and p'_t the same color for every t. □

Now we will prove that any P' has a good coloring, thus establishing Theorem 6 and, since they are equivalent, also Theorem 1. To avoid going mad, we will omit the apostrophe in the following, so we will simply write W instead of W' and so on. Also, we will use the term *wedge* to denote a translate of W.

A possible way to imagine this planar problem is that in every step t we have a set of points, P_t, and our goal is to color the coming new point, p_{t+1}, such that we always have a good coloring. We note that this would be impossible in an online setting, i.e., without knowing in advance which points will come in which order. But using the fact that we know in advance every p_i makes the problem solvable.

We start by introducing some notation. If $p_x < q_x$ but $p_y > q_y$ then we say that p is NW from q and q is SE from p. In this case we call p and q incomparable. Similarly, p is SW from q (and q is NE from p) if and only if both coordinates of p are smaller than the respective coordinates of q.

Instead of coloring the points, we will rather define on them a bipartite graph G, whose proper two-coloring will give us a good coloring. Actually, as we will later see, this graph will be a forest.

We define G recursively, starting with the empty set and the empty graph. At any step j we define a graph G_j on the points of P_j and also maintain a set S_j of pairwise incomparable points, called the *staircase*. Thus, before the t^{th} step we have a graph G_{t-1} on the points of P_{t-1} and a set S_{t-1} of pairwise incomparable points. In the t^{th} step we add p_t to our point set obtaining P_t, and we will define the new staircase, S_t, and also the new graph, G_t, which will have G_{t-1} as a subgraph. Before the exact definition of S_t and G_t, we make some more definitions and fix some properties that will be maintained during the process.

In any step j, we say that a point p is *good* if any wedge containing p already contains two points of P_j connected by an edge of G_j; i.e., at any time after j a wedge containing p will contain points of both colors in the final coloring.

At any time j, consider the order of the points of S_j given by their x coordinates. If two points of S_j are consecutive in this order, then we say that these staircase points are *neighbors*. We note that this does not mean that they are connected in the graph. A point s of the staircase is *almost good* if for at least one of its neighbors s' it is true that any wedge containing s and s' contains two points of P_j connected by an edge of G_j. Notice that the good points and the neighbors of the good points are always almost good.

We say that a point p of P_j is *above* the staircase if there exists a staircase point $s \in S_j$ such that p is NE from s. If p is not above or on the staircase, then we say that p is *below* the staircase. Now we can state the properties we maintain.

At any time j:

1. All points above the staircase are good.
2. All points of the staircase except the first and last are almost good.
3. All points below the staircase are incomparable.
4. If a wedge only contains points that are below the staircase, then it contains at most 3 points.

For $t = 0$ all these properties are trivially true. Suppose that the properties hold at time $t - 1$. Now we proceed with point p_t according to the following algorithm maintaining all the properties. During the process, we denote the current graph by G and the current staircase by S.

Algorithm Step t

Set $G = G_{t-1}$ and $S = S_{t-1}$.

Step (a) If p_t is above the staircase S_{t-1}, then we do the following; otherwise skip to Step (b).

In this case $S_t = S_{t-1}$ and $G_t = G_{t-1} \cup \{e\}$, where e is an arbitrary edge between p_t and a point s of S_{t-1} which is SW from p_t. The properties will hold trivially by induction; the only thing we need to check is if p_t is a good point, but this is again guaranteed as any wedge containing p_t contains the edge e. The algorithm terminates.

Note that we proceed further if and only if p_t is below the staircase S_{t-1}.

Step (b) If there exist two points p and q that are below the staircase and p and q are comparable, then we do the following; otherwise skip to Step (c).

Without loss of generality suppose that q is SW from p. Notice that because of Property 3, either p or q is the last added point and there are no points below the staircase that are NE from p. Now, define the new staircase, S, as S minus the points of S that are NE from p, plus the point p. This way the points of the staircase remain pairwise incomparable, as we added p and deleted all the points that were comparable to p. Also, we add the edge pq to the graph, i.e. $G := G \cup \{pq\}$. For an illustration of repeated application of this step, see Fig. 1 (edges of G are drawn red). Thus, any wedge containing p contains the edge pq, i.e. p is a good point. Property 1

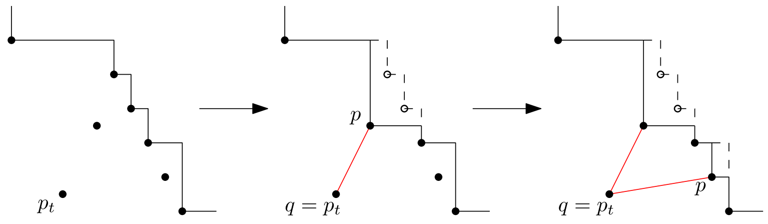

Fig. 1 Repeated application of Step (b) of the algorithm

is true for the points that were above the old S by induction. All other points above S are exactly the points that were deleted from the staircase in this step. All such points are NE from p and thus any wedge containing them contains the edge pq. Property 2 holds for p as it is a good point, and it holds for the 2 neighbors of p as any point neighboring a good point is an almost good point. For any other s from the staircase its neighbors remain the same, so it remains almost good.

Go back to Step (b) until Property 3 is satisfied, then proceed to Step (c).

Step (c) If there exist 4 points below the staircase such that these 4 points are pairwise incomparable and there exists a wedge V such that V contains these 4 points but no points of the staircase, then do the following; otherwise skip to Step (d).

Denote these 4 points by q_1, q_2, q_3, q_4 in increasing order of their x coordinates. Notice that there are no points below the staircase that are comparable because of Step (b). Now define the new S as the old S minus the points of S that are NE from q_2 or q_3, plus the points q_2 and q_3. This way the points of the staircase remain pairwise incomparable as we added q_2 and q_3 and deleted all the points that were comparable to them. Also, we add the edges q_1q_2 and q_3q_4 to the graph, i.e. $G_t = G_{t-1} \cup \{q_1q_2, q_3q_4\}$. For an illustration see Fig. 2(a). Property 1 is true for the points that were above the old S by induction. All other points above the new S are exactly the points that were deleted from the staircase in this step. It is easy to check if such a point is either NE from both of q_1 and q_2 or it is NE from both of q_3 and q_4 (we use that V was completely below the staircase, see Fig. 2(b)). Thus, a wedge containing such a point contains the edge q_1q_2 or the edge q_3q_4, and Property 1 will be true. If q_2 has a preceding neighbor, then Property 2 is true for q_2 and also for its neighbor which is not q_3 as a wedge covering them must cover q_1 as well and thus the edge q_1q_2, i.e. they are almost good. If q_2 becomes the first point of the staircase, then we don't need Property 2 to hold for q_2. By symmetry q_3 is either the last point of the staircase or q_3 and its neighbor which is not q_2 are also almost good. For any other s from the rest of the staircase (except the first and last point), s remains almost good by induction as its neighbors do not change.

Go back to Step (c) until Property 4 is satisfied, then proceed to Step (d).

Step (d) Set $S_t = S$ and $G_t = G$ and the algorithm terminates.

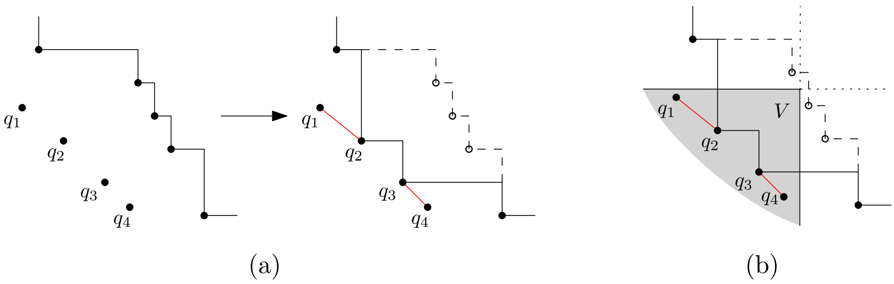

Fig. 2 Application of Step (c) of the algorithm

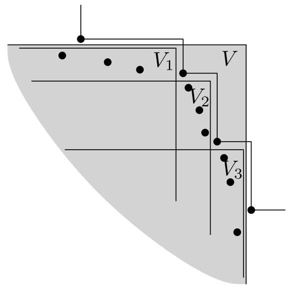

Fig. 3 At most 11 points can be in a monochromatic wedge

Adding p_t below the staircase and proceeding as in Step (b) or Step (c) always maintains Properties 1 and 2. As neither Step (b) nor Step (c) can be applied anymore, Properties 3 and 4 must hold as well. Now let us examine the graph G.

Claim 9 *The final graph G is a forest.*

Proof We prove by induction a stronger statement, that G will be such a forest that the components of the points below the staircase are disjoint trees.

When we add an edge in Step (a), then the newly added point that goes above the staircase will be one of the endpoints; thus this property is maintained.

When we add an edge in Step (b) or (c), then it connects two points below the staircase, one of which we immediately move to the staircase, so we are done by induction. □

Claim 10 *Any two-coloring of G is a good coloring of P.*

Proof Take an arbitrary two-coloring of G. Take an arbitrary wedge V at time t that contains at least 12 points of P_t. If V contains a point from above the staircase S_t, then by Property 1 V contains points of both colors. If V contains at least 3 points from the staircase, then V also contains 3 consecutive points; thus by applying Property 2 to the middle one V contains both colors (as it contains both neighbors of this middle point). Finally, if a wedge V does not contain a point from above the staircase and contains at most 2 points from the staircase, then all the points below the staircase that are covered by V can be covered by 3 wedges containing points only from below the staircase (see wedges V_1, V_2, and V_3 in Fig. 3). By Property 4 each of these wedges cover at most 3 points; thus V can contain altogether at most 11 points (2 from the staircase and $3 \cdot 3$ from below the staircase), a contradiction. □

The preceding claim finishes the proof of Theorem 6 and thus also of Theorem 1.

As noted by our anonymous reviewer, when applying this algorithm in the special case of homothetic triangles in the plane, as the spatial point set is on the $x+y+z=0$ plane, Step (a) can never occur during the algorithm. Also, as in this special case no new point comes NE to a previous point, it is true for all edges of G that its end-vertices can be contained in an octant that contains no other points. Thus, when in the $x+y+z=0$ plane, for all edges of G its end-vertices can be contained in a homothet of the fixed triangle that contains no other points. In other words, the final forest is a subgraph of the Delaunay-graph w.r.t. that fixed triangle (for direct applications of different Delaunay-graphs for such problems, see, e.g., [7]).

3 Miscellany and a Lower Bound

We have seen in the Introduction that if the point set of Theorem 6 is from the $x+y+z=0$ plane, then the problem is equivalent to the cover-decomposability of homothetic copies of an equilateral triangle. Another special case is if the point set is on the $x+y=0$ plane. The intersection family of the octants with this plane is the family of *bottomless axis-parallel rectangles* (a set is a *bottomless axis-parallel rectangle* if it is the homothetic copy of the set $\{(x,y): 0<x<1, y<0\}$). Bottomless rectangles were examined by the first author in [6], where it was proved that any 3-fold covering with bottomless rectangles is decomposable into two coverings and also that any finite point set can be colored with two colors such that every bottomless rectangle containing at least 4 points contains both colors. It was also shown that these results are sharp. We will use the ideas from [6] to prove the following claims, the first of which is a strengthening of Theorem 6 in a special case and the second giving a sharp lower bound for this special case, which also holds for the general case.

Claim 11 *If the projection of the original point set from R^3 onto the $z=0$ plane yields a point set P having only pairwise incomparable points, then it admits a two-coloring such that any translate of a given octant that contains at least* 4 *points contains points with both colors.*

Proof We use the same notation as in Sect. 2. Now at any time the points of P_t are pairwise incomparable. Order them according to their x coordinate. We will maintain a partial coloring such that, at any time t:

1. There are no two consecutive points in this order that are not colored.
2. The colored points are colored alternatingly.

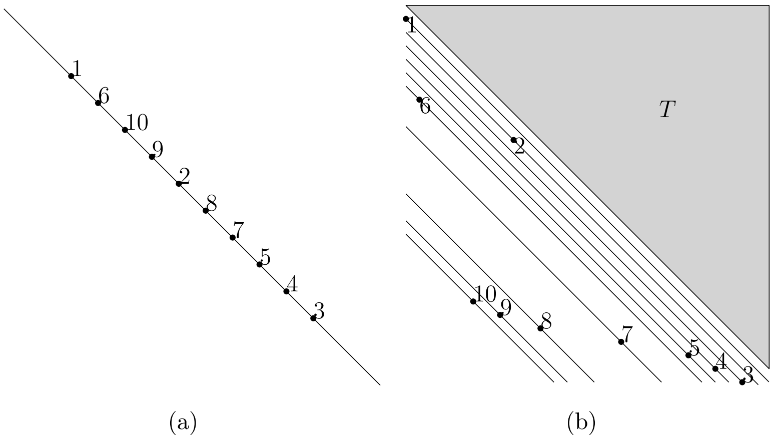

Fig. 4 Lower bound constructions

We start with the empty set and then in a general step we add the point p_t. If in the order it goes between two colored points, then we leave it uncolored. If it comes next to an uncolored point, then we color these two points maintaining the alternating coloring. At the end we color the remaining uncolored points arbitrarily. We claim that at any time t any wedge covering at least 4 points covers points from both colors already at step t of the coloring. Indeed, any wedge covers consecutive points; it covers at least 2 (consecutive) colored points by Property 1, and any two consecutive colored points are colored differently by Property 2. □

Claim 12 *For any octant there exists a* 10 *point set* $P \subset R^3$ *such that its projection onto the* $z = 0$ *plane yields a point set having only pairwise incomparable points, yet in any two-coloring of* P *there exists a translate of a given octant that contains* 3 *points with the same color and no other points.*

Proof The point set on Fig. 4(a) has the needed properties (for simplicity, the projection of the point having the t^{th} biggest z coordinate is denoted by t instead of p_t). Indeed, suppose on the contrary that there exists a two-coloring with no monochromatic wedge covering exactly 3 points. It is easy to check that the triples $(1, 2, 3)$, $(1, 2, 4)$, $(1, 2, 5)$, $(3, 4, 5)$, $(6, 2, 5)$, $(6, 2, 7)$, $(6, 2, 8)$, $(5, 7, 8)$, $(6, 1, 2)$, $(6, 1, 9)$, $(6, 1, 10)$, $(2, 9, 10)$ can all be covered by some wedge at some time t. By the pigeonhole principle there are two points from $(1, 2, 6)$ that have the same colors. If, e.g., 1 and 2 are colored red, then by the first three triples 3, 4, and 5 all must be colored blue, but then the fourth triple is monochromatic, a contradiction. The analysis is similar if 2 and 6 have the same color. Finally, if 1 and 6 are, e.g., red and 2 is blue, then we obtain a contradiction from the last three triples, as 9 and 10 should be both blue because of the penultimate and antepenultimate triples, but then the ultimate triple is monochromatic. □

This construction can be modified a bit to imply the same result for translates of a given triangle.

Claim 13 *There exists a* 10 *point set* $P \subset R^2$ *and a given triangle* T *such that in any two-coloring of* P *there exists a translate of* T *that contains* 3 *points of the same color and no other points.*

Proof The point set and the triangle on Fig. 4(b) have the needed properties; the proof of this is exactly the same as that of the previous claim. □

Finally we note that this construction is a bit smaller than the one in [6], which had size 12, so we obtain a smaller construction for that problem too by taking the same points as in Claim 11 projected onto the $y = 0$ plane.

4 Coverings of the Whole Plane

In this section we prove Theorem 3, Theorem 4, and Corollary 5.

To prove Theorem 3, we need the following well-known lemma.

Lemma 14 (König's Infinity Lemma, [8]) *Let* $V_0, V_1, \ldots$ *be an infinite sequence of disjoint nonempty finite sets, and let* G *be a graph on their union. Assume that every vertex* v_n *in a set* V_n *with* $n \geq 1$ *has a neighbor* $f(v_n)$ *in* V_{n-1}. *Then* G *contains an infinite path,* $v_0 v_1 \ldots$ *with* $v_n \in V_n$ *for all* n.

Proof of Theorem 3 We need to prove that any locally finite 12-fold covering of the whole plane with homothetic copies of a triangle is decomposable into two coverings. Take $K_1 \subset K_2 \subset \cdots$ compact sets such that their union is the whole plane. Let each $v_n \in V_n$ be a possible coloring of those finitely many triangles that intersect K_n such that every point of K_n is covered by both colors. In this case V_n is nonempty because of Corollary 2. The function f is the natural restriction to the triangles that intersect K_{n-1}. The infinite path gives a partition to two coverings. □

Proof of Theorem 4 We need to prove that any 23-fold covering of the whole plane with homothetic copies of a triangle is decomposable into two coverings. First take a compact set K_1. Select $\mathcal{T}_1$, a family of finitely many homothetic copies of the triangle that already give a 12-fold covering of K_1. Denote by K_1' the (open) set that is 12-fold covered by $\mathcal{T}_1$ and by K_1^* the union of the triangles from $\mathcal{T}_1$. So we have $K_1 \subset K_1' \subset K_1^*$. Take any coloring of $\mathcal{T}_1$ such that every point of K_1' is covered by both colors (such a coloring exists because of Corollary 2).

Now select a K_2 compact set such that $K_1^* \subset K_2$. Select $\mathcal{T}_2$, a family of finitely many homothetic copies of the triangle that already give a 12-fold covering of the compact set $K_2 \setminus K_1'$. Note that such a family exists because the points outside K_1' are covered by at most 11 members of $\mathcal{T}_1$. Define K_2' as the set that is 12-fold covered by $\mathcal{T}_1 \cup \mathcal{T}_2$ and by K_2^* the union of the triangles from $\mathcal{T}_1 \cup \mathcal{T}_2$. Take any coloring of $\mathcal{T}_2$ such that every point of K_2' is covered by both colors (such a coloring exists because of Corollary 2).

Similarly define $K_3, \ldots$ such that their union is the whole plane. Since the $\mathcal{T}_i$ families are disjoint, we get a good coloring. □

Finally, for completeness we show how Theorem 3 implies Corollary 5, using the ideas from [17].

Proof of Corollary 5 We need to prove that any 12-fold covering of the whole plane with translates of a triangle is decomposable into two coverings. Take any such covering $\mathcal{T}$; the integer grid defines a decomposition of the plane to integer squares. From the compactness of each such closed square it easily follows that there is a finite subset of $\mathcal{T}$ that is a 12-fold cover of this square. Take the union of such finite coverings for all squares of the grid. It is easy to see that this subset $\mathcal{T}'$ of $\mathcal{T}$ is a 12-fold covering of the whole plane. We claim that $\mathcal{T}'$ is also locally finite. Indeed, for an arbitrary compact set K, the translates of a given triangle that intersect K can intersect only finitely many squares of the grid, and thus finitely many sets from $\mathcal{T}'$. □

5 Remarks

Several important questions remain open. Our method could only provide a decomposition into two coverings. Is it possible to decompose any covering of the space with octants/plane with homothets of a triangle into more than two coverings if the original covering is thick enough? Only some weaker bounds are known for related problems [18].

We still do not know anything about (infinite) coverings of the plane/space with translates of closed polygons/octants. Is it possible to decompose such coverings?

For more related questions, see the recent surveys [14] and [11].

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