Octonion Spectrum of 3D Short-time LCT Signals

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ABSTRACT. This work is devoted to the development of the octonion linear canonical transform (OLCT) theory proposed by Gao and Li in 2021 that has been designated as an emerging tool in the scenario of signal processing. The purpose of this work is to introduce octonion linear canonical transform of real-valued functions. Further more keeping in mind the varying frequencies, we used the proposed transform to generate a new transform called short-time octonion linear canonical transform (STOLCT). The results of this article focus on the properties like linearity, reconstruction formula and relation with 3D-short-time linear canonical transform (3D-STLCT). The crux of this paper lie in establishing well known uncertainty inequalities and convolution theorem for the proposed transform.

Keywords: Octonion; Octonion linear canonical transform(OLCT); Short-time octonion linear canonical transform(STOLCT); Uncertainty principle; Convolution.

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1. Introduction

The generalized integral transform called the linear canonical transform (LCT) has been designated as an emerging tool in the scenario of signal, image and video processing recently. The LCT provides a unified treatment of the generalized Fourier transforms in the sense that it is an embodiment of several well-known integral transforms including the Fourier transform, fractional Fourier transform, Fresnel transform. However, LCT has a drawback. Due to its global kernel it is not suitable for processing the signals with varying frequency content. The short-time linear canonical transform (STLCT) [18] with a local window function overcome this drawback. For nonstationary signals STLCT has been used widely and successfully in signal separation and linear time frequency representation.

The hyper-complex Fourier transform(FT) is of the great interest in the present era. It treats multi-channel signals as an algebraic whole without losing the spectral relations. Presently, many hyper-complex FTs exists in literature which are defined by different approaches, see [1, 2]. The developing interest in hyper-complex FTs including applications in watermarking, color image processing, image filtering, pattern recognition and edge detection [3]-[8]. Among the various hyper-complex FTs, the most basic ones are the quaternion Fourier transforms(QFTs). QFTs are most widely studied in recent years because of its wide applications in optics and signal processing. Various properties and applications of the QFT were established in [10]-[13]. The generalization of quaternion Fourier transform (QFT) is quaternion linear canonical transform (QLCT), which is more effective signal processing tool than QFT due to its extra parameters, see[14, 15, 16, 17, 19, 20, 22]. Later, the quaternion linear canonical transform (QLCT) with four parameters has been generalized to short-time quaternion linear canonical transform (STQLCT) [29]. It is useful in quaternion valued signals and is an alternative to 2D complex STLCT. Hence has found wide applications in image and signal processing, see [23, 24, 25, 27].

which deserve special attention in the hyper-complex signal processing. The octonion Fourier transform (OFT) was proposed by Hahn and Snopek in 2011[28]. From then OFT is becoming the hot area of research in modern signal processing. Some properties and uncertainty relations and applications associated with OFT have been studied, see[30, 31, 32, 33]. In 2021 Gao and Li [34] proposed octonion linear canonical transform (OLCT) as a generalization of OFT by substituting the Fourier kernel with the LCT kernel. They established some vital properties like inversion formula, isometry, Riemann-Lebesgue lemma and proved Heisenberg's and Donoho-Stark's uncertainty principles. Furthermore they [35] introduced octonion short-time Fourier transform, where they established classical properties besides establishing Pitt's, Lieb's and uncertainty inequalities. The generalization of OFT to other transforms is still in its infancy.

So motivated and inspired by this, we shall propose the novel octonion linear canonical transform of real-valued functions. Further more keeping in mind that the OLCT takes signals from time domain to the frequency domain but is unable to perform time-frequency localization simultaneously due to its global kernel. So to overcome this drawback, we used the proposed transform to generate a new transform called short-time octonion linear canonical transform (STOLCT). The results of this article focus on the properties like linearity, reconstruction formula and relation with 3D-short-time linear canonical transform (3D-STLCT). The crux of this paper lie in establishing well known uncertainty inequalities and convolution theorem for the proposed transform. The highlights of the paper are pointed out below:

- To introduce a novel integral transform coined as the octonion linear canonical transform (OLCT) for real-valued functions.
- To introduce short-time octonion linear canonical transform and decompose it in to components of different parity.
- To study the properties like linearity and reconstruction formula.
- To study and establish the relationship between short-time octonion linear canonical transform and 3D short-time linear canonical transform.
- To formulate several classes of uncertainty inequalities, such as the Hausdorff-Young inequality, Lieb's inequality and logarithmic uncertainty inequality.
- On the basis of classical convolution operation, we establish convolution theorem for the proposed transform.

The rest of the paper is organized as follows: In Section 2, some general definitions and basic properties of octonions are summarized. The definition and the properties of the OLCT are studied in Section 3. The concept of STOLCT and its associated properties are established in Section 4. In section 5, we develop a series of uncertainty inequalities such as the Hausdorff-Young inequality, Leibs inequality and logarithmic uncertainty inequality associated with the STOLCT. Also the convolution theorem for the STOLCT is obtained in this section. The potential applications of the STOLCT are presented in Section 5. In section 6, the conclusions of the proposed work are drawn.

2. Preliminaries

In this section, we collect some basic facts on the octonion algebra and the offset linear canonical transform(OLCT), which will be needed throughout the paper.

2.1. Octonion algebra.

The octonion algebra denoted by \mathbb{O} , [36] is generated by the eighth-order Cayley-Dickson construction. According to His construction, a hypercomplex number $o \in \mathbb{O}$ is an ordered pair of quaternions $q_0, q_1 \in \mathbb{H}$

$$o = (q_0, q_1)$$

= $((z_0, z_1), (z_2, z_3))$
= $q_0 + q_1 \cdot \mu_4$
= $(z_0 + z_1 \cdot \mu_2) + (z_2 + z_3 \cdot \mu_2) \cdot \mu_4$
(2.1)

which has equivalent form

$$o = s_o + \sum_{i=1}^{7} s_i \mu_i = s_0 + s_1 \mu_1 + s_2 \mu_2 + s_3 \mu_3 + s_4 \mu_4 + s_5 \mu_5 + s_6 \mu_6 + s_7 \mu_7$$
(2.2)

that is o is a hypercomplex number defined by eight real numbers s_i , i = 0, 1, ..., 7 and seven imaginary units μ_i where i = 1, 2, ..., 7. The octonion algebra is non-commutative and non-associative algebra. The multiplication of imaginary units in the Cayley-Dickson algebra of octonions are presented in Table I.[31]

Table	Ι
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•	1	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7
1	1	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7
μ_1	μ_1	-1	μ_3	$-\mu_2$	μ_5	$-\mu_4$	$-\mu_7$	μ_6
μ_2	μ_2	$-\mu_3$	-1	μ_1	μ_6	μ_7	$-\mu_4$	$-\mu_5$
μ_3	μ_3	μ_2	$-\mu_1$	-1	μ_7	$-\mu_6$	μ_5	$-\mu_4$
μ_4	μ_4	$-\mu_5$	$-\mu_6$	$-\mu_7$	-1	μ_1	μ_2	μ_3
μ_5	μ_5	μ_4	$-\mu_7$	μ_6	$-\mu_1$	-1	$-\mu_3$	μ_2
μ_6	μ_6	μ_7	μ_4	$-\mu_5$	$-\mu_2$	μ_3	-1	$-\mu_1$
μ_7	μ_7	$-\mu_6$	μ_5	μ_4	$-\mu_3$	$-\mu_2$	μ_1	-1

Multiplication Rules in Octonion Algebra.

The conjugate of an octonion is defined as

$$\overline{o} = s_0 - s_1 \mu_1 - s_2 \mu_2 - s_3 \mu_3 - s_4 \mu_4 - s_5 \mu_5 - s_6 \mu_6 - s_7 \mu_7 \tag{2.3}$$

Therefore norm is defined by $|o| = \sqrt{o\overline{o}}$ and $|o|^2 = \sum_{i=o}^7 s_i$. Also $|o_1o_2| = |o_1||o_2|, \forall o_1, o_2 \in \mathbb{O}$.

From (2.1) it is evident that every $o \in \mathbb{O}$ can be represented in quaternion form as

$$o = a + b\mu_4 \tag{2.4}$$

where $a = s_0 + s_1\mu_1 + s_2\mu_2 + s_3\mu_3$ and $b = s_4 + s_5\mu_1 + s_6\mu_2 + s_7\mu_3$ are both quaternions. By direct verification we have following lemma.

Lemma 2.1. [31] Let $a, b \in \mathbb{H}$, then (1) $\mu_4 a = \overline{a}\mu_4$; (2) $\mu_4(a\mu_4) = -\overline{a}$; (3) $(a\mu_4)\mu_4 = -a$; (4) $a(b\mu_4) = (ba)\mu_4$; (5) $(a\mu_4)b = (a\overline{b})\mu_4$; (6) $(a\mu_4)(b\mu_4) = -\overline{b}a$. It is clear from above Lemma that, for an octonion $a + b\mu_4, a, b \in \mathbb{H}$, we have

$$\overline{a+b\mu_4} = \overline{a} - b\mu_4 \tag{2.5}$$

and

$$|a + b\mu_4|^2 = |a|^2 + |b|^2.$$
(2.6)

Lemma 2.2. Let $\tilde{o}, \hat{o} \in \mathbb{O}$. Then $e^{\tilde{o}} \cdot e^{\hat{o}} = e^{\tilde{o}+\hat{o}}$ iff $\tilde{o} \cdot \hat{o} = \hat{o} \cdot \tilde{o}$.

An octonion-valued function $f: \mathbb{R}^3 \longrightarrow \mathbb{O}$ has following explicit form

$$f(x) = f_0 + f_1(x)\mu_1 + f_2(x)\mu_2 + f_3(x)\mu_3 + f_4(x)\mu_4 + f_5(x)\mu_5 + f_6(x)\mu_6 + f_7(x)\mu_7$$

= $f_0 + f_1\mu_1 + (f_2 + f_3\mu_1)\mu_2 + [f_4 + f_5\mu_1 + (f_6 + f_7\mu_1)\mu_2]\mu_4$
= $g(x) + h(x)\mu_4$ (2.7)

where each $f_i(x)$ is a real valued functions, $g, h \in \mathbb{H}$ are as in(2.1) and $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.

For each real-valued function f(x) over \mathbb{R}^k and $1 \leq p < \infty$, the L^p -norm of f is defined by

$$\|f\|_{L^{p}(\mathbb{R}^{k})} = \left(\int_{\mathbb{R}^{k}} |f(x)|^{p} dx\right)^{\frac{1}{p}},$$
(2.8)

where $x = (x_1, x_2, ..., x_k) \in \mathbb{R}^d$ And for $p = \infty$, then the L^{∞} -norm is defined by

$$||f||_{\infty} = esssup_{x \in \mathbb{R}^k} |f(x)|.$$
(2.9)

For any functions f(x), g(x) over \mathbb{R}^k , the innear product is given by

$$\langle f,g \rangle_{L^2(\mathbb{R}^k)} = \int_{\mathbb{R}^k} f(x)\overline{g(x)}dx.$$
 (2.10)

Let $f, g \in L^2(\mathbb{R}^k)$, the classic convolution operation is defined as

$$(f * g)(x) = \int_{\mathbb{R}^k} f(y)g(x - y)dy.$$
 (2.11)

2.2. Octonion Linear Canonical Transform of Octonion-valued Functions.

In 2021 Gao,W.B and Li,B.Z [34] introduced linear canonical transform in octonion setting they called it the octonion linear canonical transform (OLCT) and defined it as follows:

For $f \in L^1(\mathbb{R}^3, \mathbb{O})$, then the one dimensional OLCT with respect to the uni-modular matrix A = (a, b, c, d) is given by

$$\mathcal{L}^{A}_{\mu_{4}}\{f\}(w) = \int_{\mathbb{R}} f(x) K^{\mu_{4}}_{A}(x, w) dx, \qquad (2.12)$$

where

$$K_A^{\mu_4}(x,w) = \frac{1}{\sqrt{2\pi|b|}} e^{\frac{\mu_4}{2b} \left[ax^2 - 2xw - +dw^2 - \frac{\pi}{2}\right]}, \quad b \neq 0$$

with the inversion formula

$$f(x) = \int_{\mathbb{R}} \mathcal{L}_{\mu_4}^A \{f\}(w) K_A^{-\mu_4}(x, w) dx, \qquad (2.13)$$

where $K_A^{-\mu_4}(x, w) = K_{A^{-1}}^{\mu_4}(w, x)$ and $A^{-1} = (d, -b, -c, a)$.

And for octonion valued function $f \in L^1(\mathbb{R}^3, \mathbb{O}) \cap L^2(\mathbb{R}^3, \mathbb{O})$, the three dimensional OLCT with respect to the matrix parameter $A_k = (a_k, b_k, c_k, d_k)$, satisfying $det(A_k) = 1$, k = 1, 2, 3 is defined as

$$\mathcal{L}_{\mu_1,\mu_2,\mu_4}^{A_1,A_2,A_3}\{f\}(w) = \int_{\mathbb{R}^3} f(x) K_{A_1}^{\mu_1}(x_1,w_1) K_{A_1}^{\mu_2}(x_2,w_2) K_{A_3}^{\mu_4}(x_3,w_3) dx$$
(2.14)

where $x = (x_1, x_2, x_3)$, $w = (w_1, w_2, w_3)$, and multiplication in above integral is done from left to right and

$$K_{A_{1}}^{\mu_{1}}(x_{1},w_{1}) = \frac{1}{\sqrt{2\pi|b_{1}|}} e^{\frac{\mu_{1}}{2b_{1}} \left[a_{1}x_{1}^{2}-2x_{1}w_{1}+d_{1}w_{1}^{2}-\frac{\pi}{2}\right]}, \quad b_{1} \neq 0$$

$$K_{A_{2}}^{\mu_{2}}(x_{2},w_{2}) == \frac{1}{\sqrt{2\pi|b_{2}|}} e^{\frac{\mu_{2}}{2b_{2}} \left[a_{2}x_{2}^{2}-2x_{2}w_{2}+d_{2}w_{2}^{2}-\frac{\pi}{2}\right]}, \quad b_{2} \neq 0$$

and

$$K_{A_3}^{\mu_4}(x_3, w_3) = \frac{1}{\sqrt{2\pi |b_3|}} e^{\frac{\mu_4}{2b_3} \left[a_3 x_3^2 - 2x_3 w_3 + d_3 w_3^2 - \frac{\pi}{2}\right]}, \quad b_3 \neq 0.$$

with the inversion formula

$$f(x) = \int_{\mathbb{R}^3} \mathcal{L}^{A_1, A_2, A_3}_{\mu_1, \mu_2, \mu_4} \{f\}(w) K^{\mu_4}_{A_3^{-1}}(w_3, x_3) K^{\mu_2}_{A_2^{-1}}(w_2, x_2) K^{\mu_1}_{A_1^{-1}}(w_1, x_1) dx, \qquad (2.15)$$

where $A^{-1}_k = (d_k, -b_k, -c_k, a_k) \in \mathbb{R}^{2 \times 2}$, for $k = 1, 2, 3$.

3. Octonion Linear Canonical Transform of Real-valued Functions

According to the octonion Fourier transform(OFT)[30] of a real-valued functions of three variables and octonion linear canonical transform for octonion valued functions[34] we can obtain the definition of the octonion linear canonical transform(OLCT) of real valued function f(x) in three variables as follows:

Definition 3.1. The 3D-OLCT of real-valued function $f : \mathbb{R}^3 \to \mathbb{R}$ can be defined as

$$\mathcal{L}_{\mu_1,\mu_2,\mu_4}^{A_1,A_2,A_3}\{f\}(w) = \int_{\mathbb{R}^3} f(x) K_{A_1}^{\mu_1}(x_1,w_1) K_{A_1}^{\mu_2}(x_2,w_2) K_{A_3}^{\mu_4}(x_3,w_3) dx.$$
(3.1)

where $x = (x_1, x_2, x_3)$, $w = (w_1, w_2, w_3)$, and kernel signals

$$K_{A_1}^{\mu_1}(x_1, w_1) = \frac{1}{\sqrt{2\pi|b_1|}} e^{\frac{\mu_1}{2b_1} \left[a_1 x_1^2 - 2x_1 w_1 + d_1 w_1^2 - \frac{\pi}{2}\right]}, \quad b_1 \neq 0$$
(3.2)

$$K_{A_2}^{\mu_2}(x_2, w_2) == \frac{1}{\sqrt{2\pi|b_2|}} e^{\frac{\mu_2}{2b_2} \left[a_2 x_2^2 - 2x_2 w_2 + d_2 w_2^2 - \frac{\pi}{2}\right]}, \quad b_2 \neq 0$$
(3.3)

and

$$K_{A_3}^{\mu_4}(x_3, w_3) = \frac{1}{\sqrt{2\pi|b_3|}} e^{\frac{\mu_4}{2b_3} \left[a_3 x_3^2 - 2x_3 w_3 + d_3 w_3^2 - \frac{\pi}{2}\right]}, \quad b_3 \neq 0.$$
(3.4)

Since the octonion algebra is non-associative it should be noted that the multiplication in the above integrals is done from left to right. Also we assume that the above signal f is continuous and both signal and its OLCT are integrable(in Lebesgue sense) in this paper. **Theorem 3.1** (Inversion). Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a continuous and square-integrable function (in Lebesgue sense). Then the OLCT of f is an invertible, and its inverse if given by

$$f(x) = \{\mathcal{L}_{\mu_{1},\mu_{2},\mu_{4}}^{A_{1},A_{2},A_{3}}\}^{-1} \left(L_{\mu_{1},\mu_{2},\mu_{4}}^{A_{1},A_{2},A_{3}}\{f\}(x)\right)$$

$$= \int_{\mathbb{R}^{3}} \mathcal{L}_{\mu_{1},\mu_{2},\mu_{4}}^{A_{1},A_{2},A_{3}}\{f\}(w) K_{A_{3}^{-1}}^{\mu_{4}}(w_{3},x_{3}) K_{A_{2}^{-1}}^{\mu_{2}}(w_{2},x_{2}) K_{A_{1}^{-1}}^{\mu_{1}}(w_{1},x_{1}) dx, \quad (3.5)$$

where $A_k^{-1} = (d_k, -b_k, -c_k, a_k) \in \mathbb{R}^{2 \times 2}$, for k = 1, 2, 3.

Proof. Consider the octonion-valued function $f : \mathbb{R}^3 \to \mathbb{O}$, i.e.

 $f(x) = f_0 + f_1(x)\mu_1 + f_2(x)\mu_2 + f_3(x)\mu_3 + f_4(x)\mu_4 + f_5(x)\mu_5 + f_6(x)\mu_6 + f_7(x)\mu_7,$ where $f_i : \mathbb{R}^3 \to \mathbb{R}, i = 0, 1, 2, 3, 4, 5, 6, 7$. we have by (2.15)

$$f(x) = \int_{\mathbb{R}^3} \mathcal{L}^{A_1, A_2, A_3}_{\mu_1, \mu_2, \mu_4} \{f\}(w) K^{\mu_4}_{A_3^{-1}}(w_3, x_3) K^{\mu_2}_{A_2^{-1}}(w_2, x_2) K^{\mu_1}_{A_1^{-1}}(w_1, x_1) dx,$$

Thus for the special case $f : \mathbb{R}^3 \to \mathbb{R}$ result follows.

Note the result also follows by using the procedure of Theorem 3.1[30].

Further, we can expand the kernel of the OLCT in the form

$$\begin{aligned}
K_{A_{1}}^{\mu_{1}}(x_{1},w_{1})K_{A_{2}}^{\mu_{2}}(x_{2},w_{2})K_{A_{3}}^{\mu_{4}}(x_{3},w_{3}) &= \frac{1}{2\pi\sqrt{2\pi|b_{1}b_{2}b_{3}|}}e^{\mu_{1}\xi_{1}}e^{\mu_{2}\xi_{2}}e^{\mu_{4}\xi_{3}} \\
&= \frac{1}{2\pi\sqrt{2\pi|b_{1}b_{2}b_{3}|}}(c_{1}+\mu_{1}s_{1})(c_{2}+\mu_{2}s_{2})(c_{3}+\mu_{4}s_{3}) \\
&= \frac{1}{2\pi\sqrt{2\pi|b_{1}b_{2}b_{3}|}}(c_{1}c_{2}c_{3}+s_{1}c_{2}c_{3}\mu_{1}+c_{1}s_{2}c_{3}\mu_{2} \\
&+ s_{1}s_{2}c_{3}\mu_{3}+c_{1}c_{2}s_{3}\mu_{4}+s_{1}c_{2}s_{3}\mu_{5}+c_{1}s_{2}s_{3}\mu_{6}+s_{1}s_{2}s_{3}\mu_{7}), \\
\end{aligned}$$
(3.6)

where
$$\xi_k = \frac{1}{2b_k} \left[a_k x_k^2 - 2x_k w_k + d_k w_k^2 - \frac{\pi}{2} \right], \quad c_k = \cos \xi_k \text{ and } s_k = \sin \xi_k, \quad k = 1, 2, 3.$$

Now using (3.6) OLCT of a real-valued function $\mathcal{L}^{A_1,A_2,A_3}_{\mu_1,\mu_2,\mu_4}\{f\}(w)$ of three variables can be expressed as octonion sum of components of different parity ([3, 28]):

$$\mathcal{L}^{A_1,A_2,A_3}_{\mu_1,\mu_2,\mu_4}\{f\}(w) = L_{eee} + L_{oee}\mu_1 + L_{eoe}\mu_2 + L_{ooe}\mu_3 + L_{eeo}\mu_4 + L_{oeo}\mu_5 + L_{eoo}\mu_6 + L_{ooo}\mu_7$$
(3.7)

where

$$\begin{split} L_{eee}(w) &= \frac{1}{2\pi\sqrt{2\pi|b_1b_2b_3|}} \int_{\mathbb{R}^3} f_{eee}(x) \cos\xi_1 \cos\xi_2 \cos\xi_3 dx, \\ L_{oee}(w) &= \frac{1}{2\pi\sqrt{2\pi|b_1b_2b_3|}} \int_{\mathbb{R}^3} f_{oee}(x) \sin\xi_1 \cos\xi_2 \cos\xi_3 dx, \\ L_{eoe}(w) &= \frac{1}{2\pi\sqrt{2\pi|b_1b_2b_3|}} \int_{\mathbb{R}^3} f_{eoe}(x,u) \cos\xi_1 \sin\xi_2 \cos\xi_3 dx, \\ L_{ooe}(w) &= \frac{1}{2\pi\sqrt{2\pi|b_1b_2b_3|}} \int_{\mathbb{R}^3} f_{ooe}(x) \sin\xi_1 \sin\xi_2 \cos\xi_3 dx, \\ L_{eeo}(w) &= \frac{1}{2\pi\sqrt{2\pi|b_1b_2b_3|}} \int_{\mathbb{R}^3} f_{eeo}(x) \cos\xi_1 \cos\xi_2 \sin\xi_3 dx, \end{split}$$

$$L_{oeo}(w) = \frac{1}{2\pi\sqrt{2\pi|b_1b_2b_3|}} \int_{\mathbb{R}^3} f_{oeo}(x)\sin\xi_1\cos\xi_2\sin\xi_3dx,$$
$$L_{eoo}(w) = \frac{1}{2\pi\sqrt{2\pi|b_1b_2b_3|}} \int_{\mathbb{R}^3} f_{eoo}(x)\cos\xi_1\cos\xi_2\sin\xi_3dx,$$
$$L_{ooo}(w) = \frac{1}{2\pi\sqrt{2\pi|b_1b_2b_3|}} \int_{\mathbb{R}^3} f_{ooo}(x)\sin\xi_1\sin\xi_2\sin\xi_3dx.$$

Where $f_{lmn}(x, u), l, m, n \in \{e, o\}$ are eight terms of different parity with relation to x_1, x_2 and x_3 . In the above notation, we use subscripts e and o to indicate that a function is either even (e) or odd (o) with respect to an appropriate variable, i.e. $f_{eeo}(x)$ is even with respect to x_1 and x_2 and odd with respect to x_3 .

Before moving forward we introduce the 3D-LCT.

Definition 3.2. [34] The 3D-LCT is defined by

$$\mathcal{L}_{A_1,A_2,A_3}\{f\}(w) = \frac{1}{2\pi\sqrt{2\pi|b_1b_2b_3|}} \int_{\mathbb{R}^3} f(x)e^{\mu_1\xi_1}e^{\mu_1\xi_2}e^{\mu_1\xi_3}dx.$$
 (3.8)

Lemma 3.1. The relation between OLCT and 3-D LCT is that

$$\mathcal{L}_{\mu_{1},\mu_{2},\mu_{4}}^{A_{1},A_{2},A_{3}}\{f\}(w) = \frac{1}{4} \left\{ (\mathcal{L}_{A_{1},A_{2},A_{3}}\{f\}(w) + \mathcal{L}_{A_{1},A_{2},A_{3}}\{f\}(w))(1-\mu_{3}) + (\mathcal{L}_{A_{1},A_{2}',A_{3}}\{f\}(w) + \mathcal{L}_{A_{1},A_{2}',A_{3}'}\{f\}(w))(1+\mu_{3}) \right\} + \frac{1}{4} \left\{ (\mathcal{L}_{A_{1},A_{2},A_{3}}\{f\}(w) - \mathcal{L}_{A_{1},A_{2},A_{3}'}\{f\}(w))(1-\mu_{3}) + (\mathcal{L}_{A_{1},A_{2}',A_{3}}\{f\}(w) - \mathcal{L}_{A_{1},A_{2}',A_{3}'}\{f\}(w))(1+\mu_{3}) \right\} .\mu_{5} \qquad (3.9)$$

where $A'_k = (a_k, -b_k, -c_k, d_k), \quad k = 1, 2.$

Proof. The proof is similar to the Theorem 6[34].

3.1. Quaternion Short-Time Linear Canonical Transform.

The Quaternion Short-Time Linear Canonical Transform(QSTLCT) was introduced by Zhu and Zheng, which is a generalization of the Short-Time Linear Canonical Transform(STLCT) in the quaternion algebra setting[29]. Let μ_1, μ_2 and μ_3 (or equivalently i,j,k) denote the three imaginary units in quaternion algebra. For $A_i = (a_i, b_i, c_i, d_i) \in \mathbb{R}^{2 \times 2}$ be a matrix parameter satisfying det $(A_i) = 1$, for i = 1, 2.

For $A_i = (a_i, b_i, c_i, d_i) \in \mathbb{R}^{2 \times 2}$ be a matrix parameter satisfying det $(A_i) = 1$, for i = 1, 2. Let $\phi \in L^2(\mathbb{R}^2, \mathbb{H})$ be a non-zero quaternion window function. Then (QSTLCT) of a signal $f \in L^2(\mathbb{R}^2, \mathbb{H})$ can be defined as

$$\mathcal{G}_{\phi}^{A_1,A_2}f(w,u) = \int_{\mathbb{R}^2} f(x)\overline{\phi(x-u)}K_{A_1}^{\mu_1}(x_1,w_1)K_{A_2}^{\mu_2}(x_2,w_2)dx, \qquad (3.10)$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, $w = (w_1, w_2) \in \mathbb{R}^2$, $u = (u_1, u_2) \in \mathbb{R}^2$ and $K_{A_1}^{\mu_1}(x_1, w_1)$, $K_{A_2}^{\mu_2}(x_2, w_2)$ are given by equations (4.2) and (4.3) respectively.

4. Short-Time Octonion Linear Canonical Transform

In this section, we define the novel short-time octonion linear cananical transform (STOLCT) of real valued function of three variables and discuss several basic properties of the STOLCT. These properties play important roles in signal representation. **Definition 4.1.** Let $A_i = (a_i, b_i, c_i, d_i) \in \mathbb{R}^{2 \times 2}$ be a matrix parameter satisfying $det(A_i) = 1$, for i = 1, 2, 3. Let $\phi \in L^2(\mathbb{R}^3)$ be a non-zero real-valued window function. Then STOLCT of a real-valued signal $f \in L^2(\mathbb{R}^3)$ can be defined as

$$\mathcal{G}_{\phi}^{A_1,A_2,A_3}\{f\}(w,u) = \int_{\mathbb{R}^3} f(x)\overline{\phi(x-u)}K_{A_1}^{\mu_1}(x_1,w_1)K_{A_2}^{\mu_2}(x_2,w_2)K_{A_3}^{\mu_4}(x_3,w_3)dx, \quad (4.1)$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $w = (w_1, w_2, w_3) \in \mathbb{R}^3$, and $u = (u_1, u_2, u_3) \in \mathbb{R}^3$ and

$$K_{A_1}^{\mu_1}(x_1, w_1) = \frac{1}{\sqrt{2\pi|b_1|}} e^{\frac{\mu_1}{2b_1} \left[a_1 x_1^2 - 2x_1 w_1 + d_1 w_1^2 - \frac{\pi}{2}\right]}, \quad b_1 \neq 0$$
(4.2)

$$K_{A_2}^{\mu_2}(x_2, w_2) == \frac{1}{\sqrt{2\pi|b_2|}} e^{\frac{\mu_2}{2b_2} \left[a_2 x_2^2 - 2x_2 w_2 + d_2 w_2^2 - \frac{\pi}{2}\right]}, \quad b_2 \neq 0$$
(4.3)

and

$$K_{A_3}^{\mu_4}(x_3, w_3) = \frac{1}{\sqrt{2\pi|b_3|}} e^{\frac{\mu_4}{2b_3} \left[a_3 x_3^2 - 2x_3 w_3 + d_3 w_3^2 - \frac{\pi}{2}\right]}, \quad b_3 \neq 0.$$
(4.4)

are kernel signals. w and u represent frequency and time respectively. Since ϕ is realvalued so complex conjugate $\overline{\phi} = \phi$. On the basis of classical convolution STOLCT defined in (4.1) can be rewritten as

$$\mathcal{G}_{\phi}^{A_1,A_2,A_3}\{f\}(w,u) = \left(f(u)K_{A_1}^{\mu_1}(x_1,w_1)K_{A_2}^{\mu_2}(x_2,w_2)K_{A_3}^{\mu_4}(x_3,w_3)\right) * \left(\overline{\phi(-u)}\right).$$

Remark 4.1. It should be noted that the multiplication in the above integrals is performed from left to right, as the octonion algebra is non-associative.

Remark 4.2. With the help of quaternion and octonion algebra the formula (4.1) can be rewritten as

$$\mathcal{G}_{\phi}^{A_1,A_2,A_3}\{f\}(w,u) = \langle f, \Phi_{x,w,u} \rangle,$$

where $\Phi_{x,w,u} = \phi(x-u) K_{A_3}^{\mu_4}(x_3, w_3) \left(K_{A_2}^{\mu_2}(x_2, w_2) K_{A_1}^{\mu_1}(x_1, w_1) \right)$ is the kernel of the STOLCT.

Moreover Definition 4.1 can be expressed as

$$\mathcal{G}_{\phi}^{A_{1},A_{2},A_{3}}\{f\}(w,u) = \mathcal{L}_{\mu_{1},\mu_{2},\mu_{4}}^{A_{1},A_{2},A_{3}}\{f(x)\overline{\phi(x-u)}\}(w)$$
$$= \mathcal{L}_{\mu_{1},\mu_{2},\mu_{4}}^{A_{1},A_{2},A_{3}}\{h\}(w),$$
(4.5)

where $h(x, u) = f(x)\overline{\phi(x - u)}$.

It is clear from (4.5) that STOLCT of a signal is a two step process. In first step signal is multiplied by a window function and then in second step we obtain OLCT of multiplied signal. Thus all the results for OLCT can be extended to the novel STOLCT, and vice versa.

4.1. Decomposition of STOLCT in to components of different parity. The STOLCT of a real-valued function $\mathcal{G}_{\phi}^{A_1,A_2,A_3}\{f\}(w,u)$ of three variables can be expressed (using (3.6) and (3.7)) as octonion sum of components of different parity :

$$\mathcal{G}_{\phi}^{A_{1},A_{2},A_{3}}\{f\} = G_{eee}^{\phi} + G_{oee}^{\phi}\mu_{1} + G_{eoe}^{\phi}\mu_{2} + G_{oeo}^{\phi}\mu_{3} + G_{eeo}^{\phi}\mu_{4} + G_{oeo}^{\phi}\mu_{5} + G_{eoo}^{\phi}\mu_{6} + G_{ooo}^{\phi}\mu_{7}$$

$$(4.6)$$

where

$$G_{eee}^{\phi}(w,u) = \frac{1}{2\pi\sqrt{2\pi|b_1b_2b_3|}} \int_{\mathbb{R}^3} h_{eee}(x,u)\cos\xi_1\cos\xi_2\cos\xi_3dx,$$
(4.7)

$$G_{oee}^{\phi}(w,u) = \frac{1}{2\pi\sqrt{2\pi|b_1b_2b_3|}} \int_{\mathbb{R}^3} h_{oee}(x,u)\sin\xi_1\cos\xi_2\cos\xi_3dx, \tag{4.8}$$

$$G_{eoe}^{\phi}(w,u) = \frac{1}{2\pi\sqrt{2\pi|b_1b_2b_3|}} \int_{\mathbb{R}^3} h_{eoe}(x,u)\cos\xi_1\sin\xi_2\cos\xi_3dx,$$
(4.9)

$$G_{ooe}^{\phi}(w,u) = \frac{1}{2\pi\sqrt{2\pi|b_1b_2b_3|}} \int_{\mathbb{R}^3} h_{ooe}(x,u)\sin\xi_1\sin\xi_2\cos\xi_3dx,$$
(4.10)

$$G_{eeo}^{\phi}(w,u) = \frac{1}{2\pi\sqrt{2\pi|b_1b_2b_3|}} \int_{\mathbb{R}^3} h_{eeo}(x,u)\cos\xi_1\cos\xi_2\sin\xi_3dx,$$
(4.11)

$$G_{oeo}^{\phi}(w,u) = \frac{1}{2\pi\sqrt{2\pi|b_1b_2b_3|}} \int_{\mathbb{R}^3} h_{oeo}(x,u)\sin\xi_1\cos\xi_2\sin\xi_3dx,$$
(4.12)

$$G_{eoo}^{\phi}(w,u) = \frac{1}{2\pi\sqrt{2\pi|b_1b_2b_3|}} \int_{\mathbb{R}^3} h_{eoo}(x,u)\cos\xi_1\cos\xi_2\sin\xi_3dx,$$
(4.13)

$$G_{ooo}^{\phi}(w,u) = \frac{1}{2\pi\sqrt{2\pi|b_1b_2b_3|}} \int_{\mathbb{R}^3} h_{ooo}(x,u)\sin\xi_1\sin\xi_2\sin\xi_3dx.$$
(4.14)

Where $h(x, u) = f(x)\phi(x - u)$ is a real valued function and it can be expressed as sum eight terms:

$$h(x, u) = h_{eee}(x, u) + h_{eeo}(x, u) + h_{eoe}(x, u) + h_{eoo}(x, u) + h_{oee}(x, u) + h_{oeo}(x, u) + h_{ooe}(x, u) + h_{ooo}(x, u),$$
(4.15)

where $h_{lmn}(x, u), l, m, n \in \{e, o\}$ are eight terms of different parity with relation to x_1, x_2 and x_3 . Again using subscripts e and o to indicate that a function is either even (e) or odd (o) with respect to an appropriate variable, i.e. $h_{eeo}(x)$ is even with respect to x_1 and x_2 and odd with respect to x_3 .

Now, we discuss several basic properties of the ST-QOLCT given by (3.1). These properties play important roles in signal representation.

Theorem 4.3 (Linearity). Let $f, g \in L^2(\mathbb{R}^3)$ be two real-valued signals and ϕ be a real-valued non-zero window function in $L^2(\mathbb{R}^3)$. Then for $\alpha, \beta \in \mathbb{R}$, we have

$$\mathcal{G}_{\phi}^{A_1,A_2,A_3}\{\alpha f + \beta g\} = \alpha \mathcal{G}_{\phi}^{A_1,A_2,A_3}\{f\} + \beta \mathcal{G}_{\phi}^{A_1,A_2,A_3}\{g\}.$$
(4.16)

Proof. This follows directly from the linearity of the product and the integration involved in Definition 4.1. \Box

Lemma 4.1. Let $\phi \in L^p(\mathbb{R}^3)$, $f \in L^1(\mathbb{R}^3)$. Then we have

$$\|\mathcal{G}_{\phi}^{A_{1},A_{2},A_{3}}\{f\}(w,u)\|_{L^{p}(\mathbb{R}^{3})} \leq \frac{1}{2\pi\sqrt{2\pi|b_{1}b_{2}b_{3}|}}\|f\|_{L^{1}(\mathbb{R}^{3})} \|\phi\|_{L^{q}(\mathbb{R}^{3})}.$$
(4.17)

Proof. By the virtue of Minkowski's inequality, we have

$$\begin{aligned} \|\mathcal{G}_{\phi}^{A_{1},A_{2},A_{3}}\{f\}(w,u)\|_{L^{p}(\mathbb{R}^{3})} &\leq \int_{\mathbb{R}^{3}} \left(\int_{\mathbb{R}^{3}} \left| f(x)\overline{\phi(x-u)}K_{A_{1}}^{\mu_{1}}(x_{1},w_{1})K_{A_{2}}^{\mu_{2}}(x_{2},w_{2}) \right. \\ &\times K_{A_{3}}^{\mu_{4}}(x_{3},w_{3}) \Big|^{p} \, du \Big)^{1/p} \, dx \\ &= \frac{1}{2\pi\sqrt{2\pi|b_{1}b_{2}b_{3}|}} \int_{\mathbb{R}^{3}} \left(\int_{\mathbb{R}^{3}} \left| f(x)\overline{\phi(x-u)} \right|^{p} \, du \right)^{1/p} \, dx. \end{aligned}$$

On setting x - u = z in the above inequality, we obtain

$$\begin{aligned} \|\mathcal{G}_{\phi}^{A_{1},A_{2},A_{3}}\{f\}(w,u)\|_{L^{p}(\mathbb{R}^{3})} &\leq \frac{1}{2\pi\sqrt{2\pi|b_{1}b_{2}b_{3}|}} \int_{\mathbb{R}^{3}} \left(\left|f(x)\overline{\phi(z)}\right|^{p}dz\right)^{1/p}dx \\ &= \frac{1}{2\pi\sqrt{2\pi|b_{1}b_{2}b_{3}|}} \left(\int_{\mathbb{R}^{3}}|\overline{\phi(z)}|^{p}dz\right)^{1/p}\int_{\mathbb{R}^{3}}|f(x)|dx \\ &= \frac{1}{2\pi\sqrt{2\pi|b_{1}b_{2}b_{3}|}} \|f\|_{L^{1}(\mathbb{R}^{3})} \|\phi\|_{L^{q}(\mathbb{R}^{3})}.\end{aligned}$$

Which completes the proof.

The next theorem guarantees the reconstruction of the input signal from the corresponding STOLCT.

Theorem 4.4 (Reconstruction formula). Let $\phi \in L^2(\mathbb{R}^3)$ be a non-zero window function, then every real-valued signal $f \in L^2(\mathbb{R}^3)$ can be fully reconstructed by the formula

$$f(x) = \frac{1}{\|\phi\|_{L^{2}(\mathbb{R}^{3})}^{2}} \int_{\mathbb{R}^{3}} \int_{\mathbb{R}^{3}} \mathcal{G}_{\phi}^{A_{1},A_{2},A_{3}}\{f\}(w,u) K_{A_{3}^{-1}}^{\mu_{4}}(w_{3},x_{3}) K_{A_{2}^{-1}}^{\mu_{2}}(w_{2},x_{2}) \times K_{A_{1}^{-1}}^{\mu_{1}}(w_{1},x_{1})\phi(x-u)dwdu.$$

$$(4.18)$$

Proof. From (4.5), we have

$$\mathcal{G}_{\phi}^{A_1,A_2,A_3}\{f\}(w,u) = \mathcal{L}_{\mu_1,\mu_2,\mu_4}^{A_1,A_2,A_3}\{f(x)\overline{\phi(x-u)}\}(w).$$
(4.19)

Applying Theorem 3.1 to (4.19), we get

$$f(x)\overline{\phi(x-u)} = \{\mathcal{L}^{A_1,A_2,A_3}_{\mu_1,\mu_2,\mu_4}\}^{-1} \left(\mathcal{G}^{A_1,A_2,A_3}_{\phi}\{f\}\right)(x)$$

$$= \int_{\mathbb{R}^3} \mathcal{G}^{A_1,A_2,A_3}_{\phi}\{f\}(w,u)K^{\mu_4}_{A_3^{-1}}(w_3,x_3)K^{\mu_2}_{A_2^{-1}}(w_2,x_2)K^{\mu_1}_{A_1^{-1}}(w_1,x_1)dw.20)$$

On multiplying both sides of (4.20) from right by $\phi(x-u)$, we get

$$f(x)\overline{\phi(x-u)}\phi(x-u) = \int_{\mathbb{R}^3} \mathcal{G}_{\phi}^{A_1,A_2,A_3}\{f\}(w,u)K_{A_3^{-1}}^{\mu_4}(w_3,x_3)K_{A_2^{-1}}^{\mu_2}(w_2,x_2)K_{A_1^{-1}}^{\mu_1}(w_1,x_1)\phi(x-u)dw$$
(4.21)

Now integrating both sides of (4.22) with respect du and using Fubini's theorem, we obtain

$$f(x) \int_{\mathbb{R}^3} \|\phi(x-u)\|^2 du = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathcal{G}_{\phi}^{A_1,A_2,A_3} \{f\}(w,u) K_{A_3^{-1}}^{\mu_4}(w_3,x_3) K_{A_2^{-1}}^{\mu_2}(w_2,x_2) \times K_{A_1^{-1}}^{\mu_1}(w_1,x_1) \phi(x-u) dw du.$$
(4.22)

Hence

$$f(x) \|\phi\|_{L^{2}(\mathbb{R}^{3})}^{2} = \int_{\mathbb{R}^{3}} \int_{\mathbb{R}^{3}} \mathcal{G}_{\phi}^{A_{1},A_{2},A_{3}} \{f\}(w,u) K_{A_{3}^{-1}}^{\mu_{4}}(w_{3},x_{3}) K_{A_{2}^{-1}}^{\mu_{2}}(w_{2},x_{2}) \times K_{A_{1}^{-1}}^{\mu_{1}}(w_{1},x_{1})\phi(x-u)dwdu.$$

$$(4.23)$$

Which completes the proof.

Prior to the next theorem we define the 3D-STLCT corresponding to the 3D-STFT[?] as

Definition 4.2. Let $f, \phi \in L^2(\mathbb{R}^3)$ be a real-valued function, where ϕ is non-zero window function. Then 3D-STLCT is defined by

$$\mathcal{V}_{\phi}^{A_{1},A_{2},A_{3}}\{f\}(w,u) = \frac{1}{2\pi\sqrt{2\pi|b_{1}b_{2}b_{3}|}} \int_{\mathbb{R}^{3}} f(x)\overline{\phi(x-u)}e^{\mu_{1}\xi_{1}}e^{\mu_{1}\xi_{2}}e^{\mu_{1}\xi_{3}}dx.$$
(4.24)

Now, we shall show that the STOLCT is related to 3D-STLCT.

Theorem 4.5. Let $f, \phi \in L^2(\mathbb{R}^3)$ be a real-valued function, where ϕ is non-zero window function. Let $\mathcal{V}_{\phi}^{A_1,A_2,A_3}\{f\}(w,u)$ be the 3D-STLCT of function f with respect to ϕ . Then the following equation is satisfied

$$\mathcal{G}_{\phi}^{A_{1},A_{2},A_{3}}\{f\}(w,u) = \frac{1}{4} \left\{ (\mathcal{V}_{\phi}^{A_{1},A_{2},A_{3}}\{f\}(w,u) + \mathcal{L}_{A_{1},A_{2},A_{3}'}\{f\}(w,u))(1-\mu_{3}) + (\mathcal{V}_{\phi}^{A_{1},A_{2}',A_{3}}\{f\}(w,u) + \mathcal{V}_{\phi}^{A_{1},A_{2}',A_{3}'}\{f\}(w,u))(1+\mu_{3}) \right\} \\
+ \frac{1}{4} \left\{ (\mathcal{V}_{\phi}^{A_{1},A_{2},A_{3}}\{f\}(w,u) - \mathcal{V}_{\phi}^{A_{1},A_{2},A_{3}'}\{f\}(w,u))(1-\mu_{3}) + (\mathcal{V}_{\phi}^{A_{1},A_{2}',A_{3}}\{f\}(w,u) - \mathcal{V}_{\phi}^{A_{1},A_{2}',A_{3}'}\{f\}(w,u))(1+\mu_{3}) \right\} .\mu_{5} \\$$
(4.25)

where $A'_{k} = (a_{k}, -b_{k}, -c_{k}, d_{k}), \quad k = 2, 3.$

Proof. From Definition 4.2, we have

$$\mathcal{V}_{\phi}^{A_1,A_2,A_3}\{f\}(w,u) = \frac{1}{2\pi\sqrt{2\pi|b_1b_2b_3|}} \int_{\mathbb{R}^3} f(x)\overline{\phi(x-u)}e^{\mu_1\xi_1}e^{\mu_1\xi_2}e^{\mu_1\xi_3}dx.$$
(4.26)

Now for $A'_2 = (a_2, -b_2, -c_2, d_2)$, then

$$\mathcal{V}_{\phi}^{A_1,A_2',A_3}\{f\}(w,u) = \frac{1}{2\pi\sqrt{2\pi|b_1b_2b_3|}} \int_{\mathbb{R}^3} f(x)\overline{\phi(x-u)}e^{\mu_1\xi_1}e^{-\mu_1\xi_2}e^{\mu_1\xi_3}dx.$$
(4.27)

By equivalent definition of sine and cosine functions, we obtain

$$\frac{1}{2} \left(\mathcal{V}_{\phi}^{A_{1},A_{2},A_{3}} \{f\}(w,u) + \mathcal{V}_{\phi}^{A_{1},A_{2}',A_{3}} \{f\}(w,u) \right) \\
= \frac{1}{2\pi \sqrt{2\pi |b_{1}b_{2}b_{3}|}} \int_{\mathbb{R}^{3}} f(x) \overline{\phi(x-u)} e^{\mu_{1}\xi_{1}} \cos \xi_{2} e^{\mu_{1}\xi_{3}} dx.$$
(4.28)

And

$$\frac{1}{2} \left(\mathcal{V}_{\phi}^{A_{1},A_{2}',A_{3}} \{f\}(w,u) - \mathcal{V}_{\phi}^{A_{1},A_{2},A_{3}} \{f\}(w,u) \right) \\
= \frac{1}{2\pi \sqrt{2\pi |b_{1}b_{2}b_{3}|}} \int_{\mathbb{R}^{3}} f(x) \overline{\phi(x-u)} e^{\mu_{1}\xi_{1}} (-\mu_{1}\sin\xi_{2}) e^{\mu_{1}\xi_{3}} dx.$$
(4.29)

On multiplying (4.29) from right by μ_3 and using multiplication rules from Table 2.1, we have

$$\frac{1}{2} \left(\mathcal{V}_{\phi}^{A_1, A_2', A_3} \{f\}(w, u) - \mathcal{V}_{\phi}^{A_1, A_2, A_3} \{f\}(w, u) \right) \mu_3 \\
= \frac{1}{2\pi \sqrt{2\pi |b_1 b_2 b_3|}} \int_{\mathbb{R}^3} f(x) \overline{\phi(x - u)} e^{\mu_1 \xi_1} (\mu_2 \sin \xi_2) e^{-\mu_1 \xi_3} dx.$$
(4.30)

Adding (4.28) and (4.30), we get

$$\frac{1}{2} \left(\mathcal{V}_{\phi}^{A_{1},A_{2},A_{3}} \{f\}(w,u) + \mathcal{V}_{\phi}^{A_{1},A_{2}',A_{3}} \{f\}(w,u) \right) \\
+ \frac{1}{2} \left(\mathcal{V}_{\phi}^{A_{1},A_{2}',A_{3}} \{f\}(w,u) - \mathcal{V}_{\phi}^{A_{1},A_{2},A_{3}} \{f\}(w,u) \right) \mu_{3} \\
= \frac{1}{2\pi \sqrt{2\pi |b_{1}b_{2}b_{3}|}} \int_{\mathbb{R}^{3}} f(x) \overline{\phi(x-u)} e^{\mu_{1}\xi_{1}} e^{\mu_{2}\xi_{2}} e^{-\mu_{1}\xi_{3}} dx.$$
(4.31)

To simplify we introduce the following notation:

$$\mathbb{V}_{A_{1},A_{2}',A_{3}}^{A_{1},A_{2},A_{3}}(w,u) = \frac{1}{2} \left(\mathcal{V}_{\phi}^{A_{1},A_{2},A_{3}}\{f\}(w,u) + \mathcal{V}_{\phi}^{A_{1},A_{2}',A_{3}}\{f\}(w,u) \right) \\
+ \frac{1}{2} \left(\mathcal{V}_{\phi}^{A_{1},A_{2}',A_{3}}\{f\}(w,u) - \mathcal{V}_{\phi}^{A_{1},A_{2},A_{3}}\{f\}(w,u) \right) \mu_{3}.$$
(4.32)

Now for $A'_3 = (a_3, -b_3, -c_3, d_3)$, then

$$\mathbb{V}_{A_{1},A_{2}',A_{3}'}^{A_{1},A_{2},A_{3}'}(w,u) = \frac{1}{2} \left(\mathcal{V}_{\phi}^{A_{1},A_{2},A_{3}'}\{f\}(w,u) + \mathcal{V}_{\phi}^{A_{1},A_{2}',A_{3}'}\{f\}(w,u) \right) \\
+ \frac{1}{2} \left(\mathcal{V}_{\phi}^{A_{1},A_{2}',A_{3}'}\{f\}(w,u) - \mathcal{V}_{\phi}^{A_{1},A_{2},A_{3}'}\{f\}(w,u) \right) \mu_{3}. \quad (4.33)$$

$$= \frac{1}{2\pi\sqrt{2\pi|b_1b_2b_3|}} \int_{\mathbb{R}^3} f(x)\overline{\phi(x-u)}e^{\mu_1\xi_1}e^{\mu_2\xi_2}e^{\mu_3\xi_3}dx.$$
(4.34)

By following similar steps as before we get

$$\frac{1}{2} \left(\mathbb{V}_{A_1,A_2,A_3}^{A_1,A_2,A_3} \{f\}(w,u) + \mathbb{V}_{A_1,A_2,A_3}^{A_1,A_2,A_3'} \{f\}(w,u) \right) \\
= \frac{1}{2\pi \sqrt{2\pi |b_1 b_2 b_3|}} \int_{\mathbb{R}^3} f(x) \overline{\phi(x-u)} e^{\mu_1 \xi_1} e^{\mu_2 \xi_2} \cos \xi_3 dx.$$
(4.35)

And

$$\frac{1}{2} \left(\mathbb{V}_{A_1,A_2,A_3}^{A_1,A_2,A_3} \{f\}(w,u) - \mathbb{V}_{A_1,A_2',A_3'}^{A_1,A_2,A_3'} \{f\}(w,u) \right) \\
= \frac{1}{2\pi \sqrt{2\pi |b_1 b_2 b_3|}} \int_{\mathbb{R}^3} f(x) \overline{\phi(x-u)} e^{\mu_1 \xi_1} e^{\mu_2 \xi_2} (-\mu_1 \sin \xi_3) dx.$$
(4.36)

On multiplying (4.36) from right by μ_5 and using multiplication rules from Table 2.1, we have

$$\frac{1}{2} \left(\mathbb{V}_{A_1,A_2,A_3}^{A_1,A_2,A_3} \{f\}(w,u) - \mathbb{V}_{A_1,A_2,A_3'}^{A_1,A_2,A_3'} \{f\}(w,u) \right) \mu_5 \\
= \frac{1}{2\pi \sqrt{2\pi |b_1 b_2 b_3|}} \int_{\mathbb{R}^3} f(x) \overline{\phi(x-u)} e^{\mu_1 \xi_1} e^{\mu_2 \xi_2} (\mu_4 \sin \xi_3) dx.$$
(4.37)

Adding (4.35) and (4.37), we get

$$\frac{1}{2} \left(\mathbb{V}_{A_{1},A_{2},A_{3}}^{A_{1},A_{2},A_{3}} \{f\}(w,u) + \mathbb{V}_{A_{1},A_{2}',A_{3}'}^{A_{1},A_{2},A_{3}'} \{f\}(w,u) \right) \\
+ \frac{1}{2} \left(\mathbb{V}_{A_{1},A_{2}',A_{3}}^{A_{1},A_{2},A_{3}} \{f\}(w,u) - \mathbb{V}_{A_{1},A_{2}',A_{3}'}^{A_{1},A_{2},A_{3}'} \{f\}(w,u) \right) \mu_{5} \\
= \frac{1}{2\pi \sqrt{2\pi |b_{1}b_{2}b_{3}|}} \int_{\mathbb{R}^{3}} f(x) \overline{\phi(x-u)} e^{\mu_{1}\xi_{1}} e^{\mu_{2}\xi_{2}} e^{\mu_{4}\xi_{3}} dx.$$
(4.38)

On substituting (4.32) and (4.33) in (4.38), we get the desired result.

5. Uncertainty Inequalities for the STOLCT

We know that in signal processing there are different types of uncertainty principles in the QFT, QLCT and QOLCT domains. In [34, 26] authors investigate Heisenberg's uncertainty principle and Donoho-Stark's uncertainty principle, Pitt's inequality, logarithmic uncertainty inequality, Hausdorff-Young inequality and local uncertainty inequality for the octonion linear canonical transform and octonion offset linear canonical transform. Recently in [35] authors establish Pitt's inequality, Lieb's inequality and logarithmic uncertainty principle for the STOFT. Considering that the STOLCT is a generalized version of the OLCT, it is natural and interesting to study uncertainty principles of a real-valued function and its STOLCT. So in this section we shall investigate some uncertainty inequalities for the STOLCT.

Lemma 5.1. Let $f \in L^p(\mathbb{R}^3)$, $\phi \in L^q(\mathbb{R}^3)$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$|\mathcal{G}_{\phi}^{A_1,A_2,A_3}\{f\}(w,u)| \le \frac{1}{2\pi\sqrt{2\pi|b_1b_2b_3|}} \|f\|_{L^p(\mathbb{R}^3)} \cdot \|\phi\|_{L^q(\mathbb{R}^3)}$$
(5.1)

Proof. By the virtue of Hölders inequality, we have

$$\begin{aligned} |\mathcal{G}_{\phi}^{A_{1},A_{2},A_{3}}\{f\}(w,u)| &= \left| \int_{\mathbb{R}^{3}} f(t)\overline{\phi(x-u)}K_{A_{1}}^{\mu_{1}}(x_{1},w_{1})K_{A_{2}}^{\mu_{2}}(x_{2},w_{2})K_{A_{3}}^{\mu_{4}}(x_{3},w_{3})dx \right| \\ &\leq \left(\int_{\mathbb{R}^{3}} \left| f(t)\overline{\phi(x-u)}K_{A_{1}}^{\mu_{1}}(x_{1},w_{1})K_{A_{2}}^{\mu_{2}}(x_{2},w_{2})K_{A_{3}}^{\mu_{4}}(x_{3},w_{3}) \right| dx \right) \\ &= \frac{1}{2\pi\sqrt{2\pi|b_{1}b_{2}b_{3}|}} \left(\int_{\mathbb{R}^{3}} \left| f(t)\overline{\phi(x-u)} \right| \right) dx \\ &\leq \frac{1}{2\pi\sqrt{2\pi|b_{1}b_{2}b_{3}|}} \left(\int_{\mathbb{R}^{3}} \left| f(t) \right|^{p} dt \right)^{1/p} \left(\int_{\mathbb{R}^{3}} \left| \overline{\phi(t-u)} \right|^{q} dt \right)^{1/q} \\ &= \frac{1}{2\pi\sqrt{2\pi|b_{1}b_{2}b_{3}|}} \|f\|_{L^{p}(\mathbb{R}^{3})} \cdot \|\phi\|_{L^{q}(\mathbb{R}^{3})}. \end{aligned}$$

Which completes the proof.

Theorem 5.1. Let Ω be a measurable set $\subset \mathbb{R}^3 \times \mathbb{R}^3$ and suppose that $\phi, f \in L^2(\mathbb{R}^3)$ be two signals with $||f||_{L^p(\mathbb{R}^3)} = 1 = ||\phi||_{L^q(\mathbb{R}^3)}$, with $\epsilon \geq 0$ and

$$\int \int_{\Omega} |\mathcal{G}_{\phi}^{A_{1},A_{2},A_{3}}\{f\}(w,u)|^{2} dw du \ge 1 - \epsilon.$$
(5.2)

We have $2\pi\sqrt{2\pi|b_1b_1b_3|}(1-\epsilon) \leq m(\Omega)$, where $m(\Omega)$ is Lebesgue measure of Ω . *Proof.* From Lemma 5.1, we have

$$\|\mathcal{G}_{\phi}^{A_{1},A_{2},A_{3}}\{f\}(w,u)\|_{L^{\infty}(\mathbb{R}^{3})} \leq \frac{1}{2\pi\sqrt{2\pi|b_{1}b_{2}b_{3}|}}\|f\|_{L^{p}(\mathbb{R}^{3})} \|\phi\|_{L^{q}(\mathbb{R}^{3})}.$$
(5.3)

On inserting (5.3) in (5.2), we obtain

$$1 - \epsilon \leq \int \int_{\Omega} |\mathcal{G}_{\phi}^{A_{1},A_{2},A_{3}}\{f\}(w,u)|^{2} dw du$$

$$\leq m(\Omega) \|\mathcal{G}_{\phi}^{A_{1},A_{2},A_{3}}\{f\}(w,u)\|_{L^{\infty}(\mathbb{R}^{3})}$$

$$\leq m(\Omega) \frac{1}{2\pi\sqrt{2\pi|b_{1}b_{2}b_{3}|}} \|f\|_{L^{p}(\mathbb{R}^{3})} \cdot \|\phi\|_{L^{q}(\mathbb{R}^{3})}$$

$$= \frac{m(\Omega)}{2\pi\sqrt{2\pi|b_{1}b_{2}b_{3}|}}$$

implies $2\pi\sqrt{2\pi |b_1 b_2 b_3|}(1-\epsilon) \leq m(\Omega)$. Which completes the proof.

Theorem 5.2 (Lieb's inequality to the STOLCT). Let $2 \le p \le \infty$ and $\phi \in L^2(\mathbb{R}^3)$ be a non-zero real-valued window function. For every real-valued signal $f \in L^2(\mathbb{R}^3)$, we have

$$\|\mathcal{G}_{\phi}^{A_{1},A_{2},A_{3}}\{f\}(w,u)\|_{L^{q}(\mathbb{R}^{3})} \leq \frac{|b_{1}b_{2}|^{\frac{-q}{2}+1}}{(2\pi)^{q+\frac{1}{2}}|b_{3}|^{\frac{1}{2}}}E_{p,q}\|f\|_{L^{2}(\mathbb{R}^{3})}\|\phi\|_{L^{2}(\mathbb{R}^{3})}$$
(5.4)

where $E_{p,q} = \left(\frac{4}{q}\right)^{\frac{1}{q}} \left(\frac{4}{p}\right)^{\frac{1}{p}}$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Before proving this theorem we shall recall the following lemma.

Lemma 5.2 (Hausdorff-Young inequality for QLCT). [21] For $1 \le p \le 2$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\|\mathcal{L}_{\mu_{1},\mu_{2}}^{A_{1},A_{2}}\{f\}(w)\|_{q} \leq \frac{|b_{1}b_{2}|^{-\frac{1}{2}+\frac{1}{q}}}{2\pi}\|f(x)\|_{p}.$$
(5.5)

Lemma 5.3 (Hausdorff-Young inequality for OLCT). For $1 \le p \le 2$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\|\mathcal{L}_{\mu_{1},\mu_{2},\mu_{4}}^{A_{1},A_{2},A_{3}}\{f\}(w)\|_{q} \leq \frac{|b_{1}b_{2}|^{-\frac{1}{2}+\frac{1}{q}}}{(2\pi)^{\frac{1}{2q}+1}|b_{3}|^{\frac{1}{2q}}}\|f(x)\|_{p}.$$
(5.6)

Proof. We omit proof as it is similar to the proof of the Theorem 4.3[26].

Proof of main theorem

Proof. We have $\mathcal{G}_{\phi}^{A_1,A_2,A_3}\{f\}(w,u) = \mathcal{L}_{\mu_1,\mu_2,\mu_4}^{A_1,A_2,A_3}\{f(x)\overline{\phi(x-u)}\}(w)$, by Lemma 5.3, we have $\left(\int_{\mathbb{R}^3} \left|\mathcal{G}_{\phi}^{A_1,A_2,A_3}\{f\}(w,u)\right|^q dw\right)^{\frac{1}{q}} = \left(\int_{\mathbb{R}^3} \left|\mathcal{L}_{\mu_1,\mu_2,\mu_4}^{A_1,A_2,A_3}\{f(x)\overline{\phi(x-u)}\}(w)\right|^q dw\right)^{\frac{1}{q}}$ $\leq \frac{|b_1b_2|^{-\frac{1}{2}+\frac{1}{q}}}{(2\pi)^{\frac{1}{2q}+1}|b_3|^{\frac{1}{2q}}} \|f(x)\overline{\phi(x-u)}\|_p$ $= \frac{|b_1b_2|^{-\frac{1}{2}+\frac{1}{q}}}{(2\pi)^{\frac{1}{2q}+1}|b_3|^{\frac{1}{2q}}} \left(\int_{\mathbb{R}^3} |f(x)\overline{\phi(x-u)}|^p dx\right)^{\frac{1}{p}}$

then,

$$\begin{split} \int_{\mathbb{R}^3} \left| \mathcal{G}_{\phi}^{A_1,A_2,A_3}\{f\}(w,u) \right|^q dw \\ &\leq \frac{|b_1b_2|^{\frac{-q}{2}+1}}{(2\pi)^{q+\frac{1}{2}}|b_3|^{\frac{1}{2}}} \left(|f|^p * |\tilde{\phi}|^p(u) \right)^{\frac{q}{p}}, \end{split}$$

where $\tilde{\phi}(x) = \phi(-x)$ Thus

$$\begin{split} \|\mathcal{G}_{\phi}^{A_{1},A_{2},A_{3}}\{f\}(w,u)\|_{q} &= \left(\int_{\mathbb{R}^{3}} \left(\int_{\mathbb{R}^{3}} \left|\mathcal{G}_{\phi}^{A_{1},A_{2},A_{3}}\{f\}(w,u)\right|^{q} dw\right) du\right)^{\frac{1}{q}} \\ &\leq \frac{|b_{1}b_{2}|^{\frac{-q}{2}+1}}{(2\pi)^{q+\frac{1}{2}}|b_{3}|^{\frac{1}{2}}} \left(\int_{\mathbb{R}^{3}} \left(|f|^{p} * |\tilde{\phi}|^{p}(u)\right)^{\frac{q}{p}} du\right)^{\frac{1}{q}} \\ &= \frac{|b_{1}b_{2}|^{\frac{-q}{2}+1}}{(2\pi)^{q+\frac{1}{2}}|b_{3}|^{\frac{1}{2}}} \left\||f|^{p} * |\tilde{\phi}|^{p}\right\|_{L^{\frac{q}{p}}(\mathbb{R}^{3})}^{\frac{1}{p}} \end{split}$$

If $k = \frac{2}{p}$, $l = \frac{q}{p}$ and $\frac{1}{k} + \frac{1}{k'} = 1$, $\frac{1}{l} + \frac{1}{l'} = 1$ then $\frac{1}{k} + \frac{1}{k} = 1 + \frac{1}{l}$ and $|f|^p$, $|\tilde{\phi}|^p \in L^2(\mathbb{R}^3)$ and by Young inequality, we have

$$\left\| |f|^{p} * |\tilde{\phi}|^{p} \right\|_{L^{l}(\mathbb{R}^{3})} \leq B_{k}^{4} B_{l}^{2} \||f|^{p} \|_{L^{k}(\mathbb{R}^{3})} \||\tilde{\phi}|^{p} \|_{L^{l}(\mathbb{R}^{3})},$$

where $B_s = \left(\frac{s^{\frac{1}{s}}}{s'^{\frac{1}{s'}}}\right)^{\frac{1}{2}}$, $\frac{1}{s} + \frac{1}{s'} = 1$. However

$$|||f|^{p}||_{L^{k}(\mathbb{R}^{3})} = \left(\int_{\mathbb{R}^{3}} |f(x)|^{p \cdot \frac{2}{p}} dx\right)^{\frac{p}{2}} = ||f||_{L^{2}(\mathbb{R}^{3})}^{p}$$

and

$$\||\tilde{\phi}|^p\|_{L^k(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} |\phi(x-u)|^{p,\frac{2}{p}} dx\right)^{\frac{p}{2}} = \|\phi\|_{L^2(\mathbb{R}^3)}^p.$$

Hence

$$\begin{aligned} \|\mathcal{G}_{\phi}^{A_{1},A_{2},A_{3}}\{f\}(w,u)\|_{q} &\leq \frac{|b_{1}b_{2}|^{\frac{-q}{2}+1}}{(2\pi)^{q+\frac{1}{2}}|b_{3}|^{\frac{1}{2}}} \left\||f|^{p} * |\tilde{\phi}|^{p}\right\|_{L^{\frac{q}{p}}(\mathbb{R}^{3})}^{\frac{1}{p}} \\ &\leq \frac{|b_{1}b_{2}|^{\frac{-q}{2}+1}}{(2\pi)^{q+\frac{1}{2}}|b_{3}|^{\frac{1}{2}}} \left(B_{k}^{4}B_{l}^{\prime 2}\|f\|_{L^{2}(\mathbb{R}^{3})}^{p}\|\phi\|_{L^{2}(\mathbb{R}^{3})}\right)^{\frac{1}{p}} \\ &= \frac{|b_{1}b_{2}|^{\frac{-q}{2}+1}}{(2\pi)^{q+\frac{1}{2}}|b_{3}|^{\frac{1}{2}}}B_{k}^{\frac{4}{p}}B_{l}^{\prime\frac{2}{p}}\|f\|_{L^{2}(\mathbb{R}^{3})}\|\phi\|_{L^{2}(\mathbb{R}^{3})} \\ &= \frac{|b_{1}b_{2}|^{\frac{-q}{2}+1}}{(2\pi)^{q+\frac{1}{2}}|b_{3}|^{\frac{1}{2}}}E_{p,q}\|f\|_{L^{2}(\mathbb{R}^{3})}\|\phi\|_{L^{2}(\mathbb{R}^{3})}.\end{aligned}$$

Which completes the proof.

Lemma 5.4 (Logarithmic uncertainty principle for the OLCT). Let $f \in \mathcal{S}(\mathbb{R}^3)$, then the following inequality is satisfied:

$$2\pi |b_3| \int_{\mathbb{R}^3} \ln |w| \left| \mathcal{L}^{A_1, A_2, A_3}_{\mu_1, \mu_2, \mu_3} \{f\}(w) \right|^2 dw + \int_{\mathbb{R}^3} \ln |x| |f(x)|^2 dx \ge E \int_{\mathbb{R}^3} |f(x)|^2 dx \qquad (5.7)$$

with $D = \ln(2) + \Gamma'(\frac{1}{2}) / \Gamma(\frac{1}{2})$.

Proof. We avoid proof as it follows by using the procedure of the Theorem 4.2[26]. Also see Theorem 5[31]. \Box

Theorem 5.3 (Logarithmic uncertainty principle for the STOLCT). Let $f, \phi \in \mathcal{S}(\mathbb{R}^3)$, then the following inequality is satisfied:

$$2\pi |b_{3}| \int_{\mathbb{R}^{3}} \int_{\mathbb{R}^{3}} \ln |w| \left| \mathcal{G}_{\phi}^{A_{1},A_{2},A_{3}} \{f\}(w,u) \right|^{2} dw du + \|\phi\|_{L^{2}(\mathbb{R}^{3})}^{2} \int_{\mathbb{R}^{3}} \ln |x| |f(x)|^{2} dx$$
$$\geq E \|f\|_{L^{2}(\mathbb{R}^{3})}^{2} \|\phi\|_{L^{2}(\mathbb{R}^{3})}^{2}$$
(5.8)

with $D = \ln(2) + \Gamma'(\frac{1}{2}) / \Gamma(\frac{1}{2})$.

Proof. As $f, \phi \in \mathcal{S}(\mathbb{R}^3)$ implies $f(x)\overline{\phi(x-u)} = h(x,u) \in \mathcal{S}(\mathbb{R}^3)$. Thus replacing f(x) by h(x,u) in Lemma 5.4, we obtain

$$2\pi |b_3| \int_{\mathbb{R}^3} \ln |w| \left| \mathcal{L}^{A_1, A_2, A_3}_{\mu_1, \mu_2, \mu_3} \{h\}(w) \right|^2 dw + \int_{\mathbb{R}^3} \ln |x| |h(x, u)|^2 dx \ge E \int_{\mathbb{R}^3} |h(x, u)|^2 dx.$$
(5.9)

On integrating both sides of (5.9) with respect to du, we have

$$2\pi |b_{3}| \int_{\mathbb{R}^{3}} \int_{\mathbb{R}^{3}} \ln |w| \left| \mathcal{L}_{\mu_{1},\mu_{2},\mu_{3}}^{A_{1},A_{2},A_{3}} \{h\}(w) \right|^{2} dw du + \int_{\mathbb{R}^{3}} \int_{\mathbb{R}^{3}} \ln |x| |h(x,u)|^{2} dx du$$
$$\geq E \int_{\mathbb{R}^{3}} \int_{\mathbb{R}^{3}} |h(x,u)|^{2} dx du.$$
(5.10)

Now using (4.5) in (5.10), we get

$$2\pi |b_3| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \ln|w| \left| \mathcal{G}_{\phi}^{A_1, A_2, A_3} \{f\}(w, u) \right|^2 dw du + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \ln|x| |f(x)|^2 |\phi(x - u)|^2 dx du$$
$$\geq E \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |f(x)|^2 |\phi(x - u)|^2 dx du.$$
(5.11)

Which implies

$$2\pi |b_3| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \ln |w| \left| \mathcal{G}_{\phi}^{A_1, A_2, A_3} \{f\}(w, u) \right|^2 dw du + \int_{\mathbb{R}^3} |\phi(x - u)|^2 du \int_{\mathbb{R}^3} \ln |x| |f(x)|^2 dx \\ \ge E \int_{\mathbb{R}^3} |f(x)|^2 dx \int_{\mathbb{R}^3} |\phi(x - u)|^2 du.$$
(5.12)

Hence

$$2\pi |b_{3}| \int_{\mathbb{R}^{3}} \int_{\mathbb{R}^{3}} \ln |w| \left| \mathcal{G}_{\phi}^{A_{1},A_{2},A_{3}} \{f\}(w,u) \right|^{2} dw du + \|\phi\|_{L^{2}(\mathbb{R}^{3})}^{2} \int_{\mathbb{R}^{3}} \ln |x| |f(x)|^{2} dx$$

$$\geq E \|f\|_{L^{2}(\mathbb{R}^{3})}^{2} \|\phi\|_{L^{2}(\mathbb{R}^{3})}^{2}.$$
(5.13)
ch completes the proof.

Which completes the proof.

5.1. Convolution Theorem for the STOLCT. The convolution has wide has wide range of applications in various areas of Mathematics like linear algebra, numerical analysis and signal processing. So in this subsection we establish the convolution theorem for the STOLCT on the bases of the classical convolution operator (2.11).

Theorem 5.4 (Convolution). Let $\phi, \psi \in L^2(\mathbb{R}^3)$ be two non-zero real-valued window functions then for any real-valued functions $f, g \in L^2(\mathbb{R}^3)$, we have

$$\begin{aligned} \mathcal{G}_{\phi*\psi}^{A_{1},A_{2},A_{3}}\{f*g\}(w,u) \\ &= \int_{\mathbb{R}^{3}} \mathcal{G}_{\psi}^{A_{1},A_{2},A_{3}}\{g\}(w,m)G_{eee}^{\phi}(w,u-m) + G_{\psi}^{A_{1},A_{2},A_{3}}\{g\}(t,m)G_{oee}^{\phi}(w,u-m)\mu_{1} \\ &+ G_{\psi}^{A_{1},A_{2},A_{3}}\{g\}(s,m)G_{eoe}^{\phi}(w,u-m)\mu_{2} + G_{\psi}^{A_{1},A_{2},A_{3}}\{g\}(t,m)G_{ooe}^{\phi}(w,u-m)\mu_{3} \\ &+ G_{\psi}^{A_{1},A_{2},A_{3}}\{g\}(w,m)G_{eeo}^{\phi}(w,u-m)\mu_{4} + G_{\psi}^{A_{1},A_{2},A_{3}}\{g\}(s,m)G_{oeo}^{\phi}(w,u-m)\mu_{5} \\ &+ G_{\psi}^{A_{1},A_{2},A_{3}}\{g\}(t',m)G_{eoo}^{\phi}(w,u-m)\mu_{6} + G_{\psi}^{A_{1},A_{2},A_{3}}\{g\}(t',m)G_{ooo}^{\phi}(w,u-m)\mu_{7}dm, \end{aligned}$$

$$(5.14)$$

where $t = (w_1, -w_2, -w_3) \in \mathbb{R}^3$, $s = (w_1, w_2, -w_3) \in \mathbb{R}^3$, $t' = (-w_1, w_2, -w_3) \in \mathbb{R}^3$ and $G^{\phi}_{lmn}, l, m, n \in \{e, o\}$ are given by equations (4.7) to (4.14).

Proof. By Definition 4.1 we have by classic convolution operator

$$\mathcal{G}_{\phi*\psi}^{A_1,A_2,A_3}\{f*g\}(w,u) = \int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} f(y)g(x-y)dy \right) \cdot \int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} \overline{\phi(z)\psi(x-u-z)}dz \right) \\ \times K_{A_1}^{\mu_1}(x_1,w_1)K_{A_2}^{\mu_2}(x_2,w_2)K_{A_3}^{\mu_4}(x_3,w_3) \tag{5.15}$$

Setting
$$(q_1, q_2, q_3) = q = x - y$$
, $(m_1, m_2, m_3) = m = u + z - y$, in (5.15), we obtain

$$\begin{aligned}
\mathcal{G}_{\phi*\psi}^{A_1, A_2, A_3} \{f * g\}(w, u) \\
&= \int_{\mathbb{R}^9} f(y)g(q)\overline{\phi(y - (u - m))}.\overline{\psi(q - m)}K_{A_1}^{\mu_1}(q_1 + y_1, w_1) \\
&\times K_{A_2}^{\mu_2}(q_2 + y_2, w_2)K_{A_3}^{\mu_4}(q_3 + y_3, w_3)dqdydm \\
&= \int_{\mathbb{R}^6} f(y)\overline{\phi(y - (u - m))} \left(\int_{\mathbb{R}^3} g(q)\overline{\psi(q - m)}K_{A_1}^{\mu_1}(q_1 + y_1, w_1) \\
&\times K_{A_2}^{\mu_2}(q_2 + y_2, w_2)K_{A_3}^{\mu_4}(q_3 + y_3, w_3)dq\right)dydm
\end{aligned}$$
(5.16)

Now,

let

$$\xi_k = \frac{1}{2b_k} \left[a_k q_k^2 - 2q_k w_k + d_k w_k^2 - \frac{\pi}{2} \right]$$

and

$$\gamma_k = \frac{1}{2b_k} \left[a_k y_k^2 + 2a_k q_k y_k - 2y_1 w_1 \right], \quad k = 1, 2, 3$$

then

$$K_{A_1}^{\mu_1}(q_1+y_1,w_1)K_{A_2}^{\mu_2}(q_2+y_2,w_2)K_{A_3}^{\mu_4}(q_3+y_3,w_3) = \frac{1}{2\pi\sqrt{2\pi|b_1b_2b_3|}}e^{\mu_1(\xi_1+\gamma_1)}e^{\mu_2(\xi_2+\gamma_2)}e^{\mu_4(\xi_3+\gamma_3)}$$

(5.17)

Now by applying multiplication rules of Table 2.1, we obtain

$$e^{\mu_{1}(\xi_{1}+\gamma_{1})}e^{\mu_{2}(\xi_{2}+\gamma_{2})}e^{\mu_{4}(\xi_{3}+\gamma_{3})} = ((e^{\mu_{1}\xi_{1}}.\cos(\gamma_{1})).(e^{\mu_{2}\xi_{2}}.\cos(\gamma_{2}))).(e^{\mu_{4}\xi_{3}}.\cos(\gamma_{3})) \\ + ((e^{\mu_{1}\xi_{1}}.\mu_{1}\sin(\gamma_{1})).(e^{\mu_{2}\xi_{2}}.\cos(\gamma_{2}))).(e^{\mu_{4}\xi_{3}}.\cos(\gamma_{3})) \\ + ((e^{\mu_{1}\xi_{1}}.\cos(\gamma_{1})).(e^{\mu_{2}\xi_{2}}.\mu_{2}\sin(\gamma_{2}))).(e^{\mu_{4}\xi_{3}}.\cos(\gamma_{3})) \\ + ((e^{\mu_{1}\xi_{1}}.\cos(\gamma_{1})).(e^{\mu_{2}\xi_{2}}.\cos(\gamma_{2}))).(e^{\mu_{4}\xi_{3}}.\mu_{4}\sin(\gamma_{3})) \\ + ((e^{\mu_{1}\xi_{1}}.\cos(\gamma_{1})).(e^{\mu_{2}\xi_{2}}.\cos(\gamma_{2}))).(e^{\mu_{4}\xi_{3}}.\mu_{4}\sin(\gamma_{3})) \\ + ((e^{\mu_{1}\xi_{1}}.\cos(\gamma_{1})).(e^{\mu_{2}\xi_{2}}.\mu_{2}\sin(\gamma_{2}))).(e^{\mu_{4}\xi_{3}}.\mu_{4}\sin(\gamma_{3})) \\ + ((e^{\mu_{1}\xi_{1}}.\cos(\gamma_{1})).(e^{\mu_{2}\xi_{2}}.\mu_{2}\sin(\gamma_{2}))).(e^{\mu_{4}\xi_{3}}.\mu_{4}\sin(\gamma_{3})) \\ + ((e^{\mu_{1}\xi_{1}}.\mu_{1}\sin(\gamma_{1})).(e^{\mu_{2}\xi_{2}}.\mu_{2}\sin(\gamma_{2}))).(e^{\mu_{4}\xi_{3}}.\mu_{4}\sin(\gamma_{3})) \\ + ((e^{\mu_{1}\xi_{1}}.\mu_{1}\sin(\gamma_{1})).(e^{\mu_{2}\xi_{2}}.\mu_{2}\sin(\gamma_{2}$$

On substituting (5.18),(5.17) in (5.16) and noting $t = (w_1, -w_2, -w_3) \in \mathbb{R}^3$, $s = (w_1, w_2, -w_3) \in \mathbb{R}^3$, $t' = (-w_1, w_2, -w_3) \in \mathbb{R}^3$, we get the desired result.

6. Potential Applications

As for as generalization of transformation in to octonion algebra, the OLCT transforms a octonion 3D signal into a octonion-valued frequency domain signal, which is an effective processing tool for signal, image and color analysis. The hypercomplex LCT that treats the mutichannel signls as a algebraic whole without losing the spectral relations for color image processing. But there is drawback that hypercomplex LCT canot reveal the local information of a signal due to its global kernel. The STOLCT is a new tool for time frequncy analysis which overcomes this drawback by using a sliding window. Another potential application of STOLCT is that it can also reconstruct each monocomponent mode from a multicomponent signal.

Moreover, the uncertainty principle makes a tradeoff between temporal and spectral resolutions unavoidable, i.e. the new uncertainty principles for the STOLCT describe the relation of one octonion-valued signal in spatial and another octonion-valued signal in frequency domain. They could further contribute to solving problems of signal processing, optics, color image processing, quantum mechanics, electrodynamics, electromagnetism, etc.

In addition, Lieb's uncertainty principle for the STOLCT could analyze the non-stationary signal and time-varying system, which has a significant application in the study of signal local frequency spectrum.

7. Conclusions

In this paper, we introduce octonion linear canonical transform of real-valued functions. Further more keeping in mind the varying frequencies, we used the proposed transform to generate a new transform called short-time octonion linear canonical transform (STOLCT). Then, the various properties of the proposed STOLCT are explored, such as linearity, inversion formulas, decomposition into components of different parity and relation with the 3D-STLCT. Furthermore, Lieb's inequality, logarithmic uncertainty inequality associated with the STOLCT are investigated. Also based on classical convolution operation, the convolution theorem for the STOLCT is derived. Finally, some potential applications of the STLCT are presented. In our future works, we will discuss the physical significance and engineering background of this paper.

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