



Odd Hyperbolic Cosine Exponential-Exponential (OHC-EE) Distribution

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Abstract

In the present paper, we introduce a new lifetime distribution based on the general odd hyperbolic cosine-FG model. Some important properties of proposed model including survival function, quantile function, hazard function, order statistic are obtained. In addition estimating unknown parameters of this model will be examined from the perspective of classic and Bayesian statistics. Moreover, an example of real data set is studied; point and interval estimations of all parameters are obtained by maximum likelihood, bootstrap (parametric and non-parametric) and Bayesian procedures. Finally, the superiority of proposed model in terms of parent exponential distribution over other fundamental statistical distributions is shown via the example of real observations.

Keywords Lifetime distribution · Hazard rate · Bayesian method

1 Introduction

Alzaatreh et al. [2] have introduced a new model of lifetime distributions, which the researchers refer to its especial case as odd- G distribution. It is based on the combination of an arbitrary cumulative distribution function CDF with odd ratio of the baseline CDF as G . The integration form of the new CDF is

$$H(x) = \int_{-\infty}^{\frac{G(x)}{1-G(x)}} f(t)dt, \quad (1)$$

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where f is the probability density function PDF of arbitrary CDF . This interesting method attracted the attention of some researchers. We refer the reader to [1,4]. Generating new model based on this method resulted in creating very flexible statistical model. Recently, Kharazmi et al. [7] have introduced a new general family of lifetime distribution based on the odd ratio of a parent distribution G for general hyperbolic cosine-F (HCF) family of lifetime distributions that recently proposed by Kharazmi and Saadatinik [6]. This new model will be denoted by $odd-HCF-G$ (or $OHCFG$) distribution.

In the present paper, we introduce a special case of the $odd-HCF-G$ model by applying two exponential distributions instead of F and G models. It is referred to as $OHCEE$ distribution. One of our main motivation to introduce this new distribution is to provide more flexibility for fitting real datasets compare to other well-known classic statistical distributions.

We provide a comprehensive discussion about statistical and reliability properties of new $OHCEE$ model. Furthermore, we consider maximum likelihood and bootstrap estimation procedures in order to estimate the unknown parameters of the new model for a complete data set. In addition, parametric and non-parametric bootstrap confidence intervals are calculated.

The rest of the paper organized as follows. In the Sect. 2, we review the main statistical and reliability properties of recently proposed model by Kharazmi et al. [7]. In Sect. 3, by considering the two exponential distributions as the base distributions, a new model is presented according to the general model discussed in Sect. 1 and its prominent characteristics are studied. This new model refer to as $OHCEE$ distribution. In Sect. 4, we examine the inferential procedures for estimation unknown parameters of the $OHCEE$ model. In this Section, we provide discussions about the maximum likelihood, Bayesian and bootstrap procedures. Applications and numerical analysis of a real data set are presented in Sect. 5. Finally, in Sect. 6 the paper is concluded.

2 Preliminaries of HCF and $OHCFG$ Lifetime Models

In this section, we review the structure of two general HCF and $OHCFG$ models. Kharazmi and Saadatinik [6] introduced a family of distributions using hyperbolic c function. The hyperbolic cosine has similar name to the trigonometric functions, but it is defined in terms of the exponential function as follows:

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \quad (2)$$

According to Kharazmi and Saadatinik [6] a random variable X has a hyperbolic cosine-F (HCF) distribution if its cumulative distribution function (cdf) is given by

$$G(x) = \frac{2e^a}{e^{2a} - 1} \sinh(aF(x)), \quad (3)$$

where $x > 0$, $a > 0$.

Motivated by idea of Alzaatreh et al. [2], a new class of statistical distributions have proposed by Kharazmi et al [7]. The new model is constructed by applying Alzaatreh idea to the hyperbolic cosine-F (HCF) family of lifetime distributions.

The CDF of new model is defined as

$$\begin{aligned} H(x) &= \int_0^{\frac{G(x)}{1-G(x)}} \frac{2a e^a}{e^{2a} - 1} f(t) \cosh(aF(t)) dt \\ &= \frac{2e^a}{e^{2a} - 1} \sinh \left(aF \left(\frac{G(x)}{1 - G(x)} \right) \right) \end{aligned}$$

The density of OHC-FG model can be obtained as

$$h(x) = \frac{2a e^a}{e^{2a} - 1} \frac{g(x)}{(1 - G(x))^2} f \left(\frac{G(x)}{1 - G(x)} \right) \cosh \left(aF \left(\frac{G(x)}{1 - G(x)} \right) \right)$$

The survival reliability $\bar{H}(x)$ and the hazard rate function $r(x)$ for OHC-FG distribution are in the following form

$$\bar{H}(x) = 1 - \frac{2e^a}{e^{2a} - 1} \sinh \left(aF \left(\frac{G(x)}{1 - G(x)} \right) \right)$$

and

$$r(x) = \frac{\frac{2a e^a}{e^{2a} - 1} \frac{g(x)}{(1 - G(x))^2} f \left(\frac{G(x)}{1 - G(x)} \right) \cosh \left(aF \left(\frac{G(x)}{1 - G(x)} \right) \right)}{1 - \frac{2e^a}{e^{2a} - 1} \sinh \left(aF \left(\frac{G(x)}{1 - G(x)} \right) \right)},$$

respectively. The p th quantile x_p of the OHC-FG distribution can be obtained as

$$x_p = G^{-1} \left(\frac{F^{-1} \left(\frac{1}{a} \operatorname{arcsinh} \left(\frac{e^{2a} - 1}{2e^a} p \right) \right)}{1 + F^{-1} \left(\frac{1}{a} \operatorname{arcsinh} \left(\frac{e^{2a} - 1}{2e^a} p \right) \right)} \right); \quad 0 \leq p \leq 1.$$

$\operatorname{arcsinh}(x) = \ln(x + \sqrt{x^2 + 1})$, then we get Since

$$x_p = G^{-1} \left(\frac{F^{-1} \left(\frac{1}{a} \log \left(\frac{e^{2a} - 1}{2e^a} p + \sqrt{\left(\frac{e^{2a} - 1}{2e^a} p \right)^2 + 1} \right) \right)}{1 + F^{-1} \left(\frac{1}{a} \log \left(\frac{e^{2a} - 1}{2e^a} p + \sqrt{\left(\frac{e^{2a} - 1}{2e^a} p \right)^2 + 1} \right) \right)} \right).$$

Hence, If the baseline F and G distributions are invertable then we can easily generate random samples from the OHC-FG distribution.

3 An Especial Case of OHCFG Model

we apply the OHC-FG method to a specific class of distribution functions, namely to an exponential distribution and call this new distribution, three-parameter OHC-EE distribution.

Definition A random variable X has OHC-EE(a, λ_1, λ_2), if its probability density function (PDF) is given by

$$h(x) = \frac{2a e^a}{e^{2a} - 1} \lambda_1 e^{\lambda_1 x} \lambda_2 e^{-\lambda_2(e^{\lambda_1 x} - 1)} \cosh \left(a \left(1 - e^{-\lambda_2(e^{\lambda_1 x} - 1)} \right) \right);$$

$$x > 0, a, \lambda_1, \lambda_2 > 0. \quad (4)$$

The CDF of (4) can be written as

$$H(x) = \frac{2 e^a}{e^{2a} - 1} \sinh \left(a \left(1 - e^{-\lambda_2(e^{\lambda_1 x} - 1)} \right) \right),$$

also, survival and hazard rate functions are given by

$$\bar{H}(x) = 1 - \frac{2 e^a}{e^{2a} - 1} \sinh \left(a \left(1 - e^{-\lambda_2(e^{\lambda_1 x} - 1)} \right) \right),$$

and the hazard rate function is given by

$$r(x) = \frac{h(x)}{\bar{H}(x)} = \frac{2a e^a \lambda_1 e^{\lambda_1 x} \lambda_2 e^{-\lambda_2(e^{\lambda_1 x} - 1)} \cosh \left(a \left(1 - e^{-\lambda_2(e^{\lambda_1 x} - 1)} \right) \right)}{e^{2a} - 1 - 2e^a \sinh \left(a \left(1 - e^{-\lambda_2(e^{\lambda_1 x} - 1)} \right) \right)}$$

3.1 Some Properties of the OHCEE Distribution

In this section, we obtain some properties of the OHCEE distribution, involving quantiles, moments, moment generating function and extreme order statistics. Some shapes of density function and hazard rate function are shown in Figs. 1 and 2.

3.2 Quantiles

For the OHCEE distribution, the p th quantile x_p is the solution of $H(x_p) = p$, hence

$$x_p = \frac{1}{\lambda_1} \log \left(1 - \frac{1}{\lambda_2} \log \left(1 - \frac{1}{a} \operatorname{arcsinh} \left(\frac{e^{2a} - 1}{2e^a} p \right) \right) \right)$$

$$= \frac{1}{\lambda_1} \log \left(1 - \frac{1}{\lambda_2} \log \left(1 - \frac{1}{a} \log \left(\frac{e^{2a} - 1}{2e^a} p + \sqrt{\left(\frac{e^{2a} - 1}{2e^a} p \right)^2 + 1} \right) \right) \right); \quad 0 \leq p \leq 1$$

which is the base of generating OHCEE random variates.

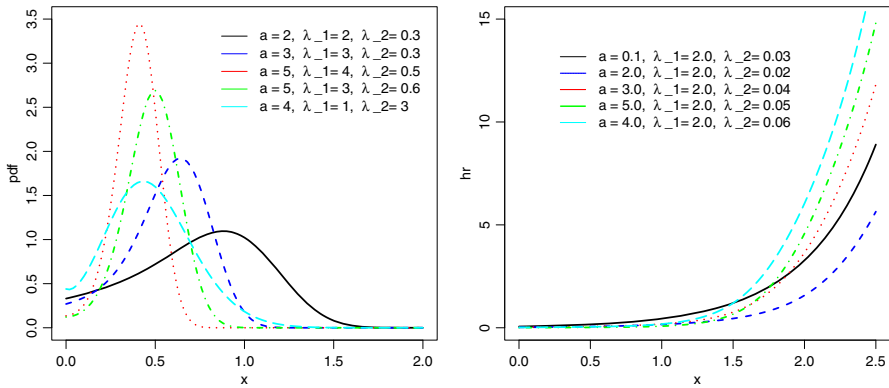


Fig. 1 plots of the OHC-EE(a, λ_1, λ_2) (left) and failure rate function (right) for selected values of a, λ_1, λ_2

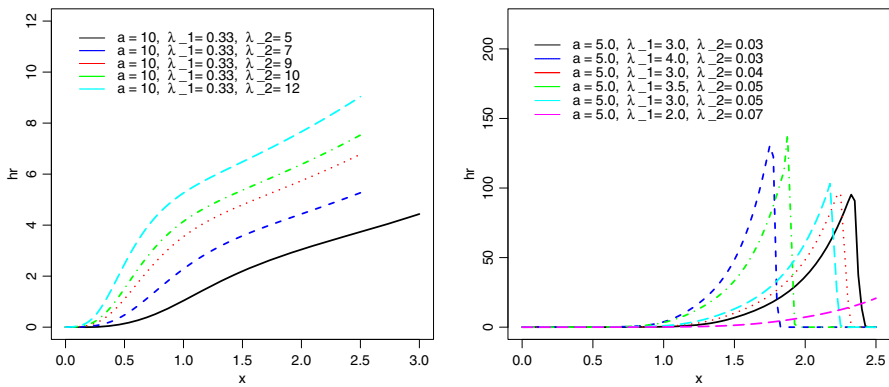


Fig. 2 failure rate function shapes for selected values of the parameters

3.3 Moments and Moment Generating Function

In this subsection, moments and related measures including coefficients of variation, skewness and kurtosis are presented. Tables of values for the first six moments, standard deviation (SD), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) are also presented. The r th moment of the OHCEE distribution, denoted by μ'_r is

$$\begin{aligned} \mu'_r &= E(X^r) = \frac{2a e^a}{e^{2a} - 1} \int_0^\infty x^r \lambda_1 e^{\lambda_1 x} \lambda_2 e^{-\lambda_2(e^{\lambda_1 x} - 1)} \sum_{n=0}^\infty \frac{a^{2n} (1 - e^{-\lambda_2(e^{\lambda_1 x} - 1)})^{2n}}{(2n)!} dx \\ &= \sum_{n=0}^\infty \frac{a^{2n+1}}{(2n)!} \frac{2 e^a}{e^{2a} - 1} \int_0^\infty x^r \lambda_1 e^{\lambda_1 x} \lambda_2 e^{-\lambda_2(e^{\lambda_1 x} - 1)} (1 - e^{-\lambda_2(e^{\lambda_1 x} - 1)})^{2n} dx \\ &= \sum_{n=0}^\infty \frac{a^{2n+1}}{(2n)!} \frac{2 e^a}{e^{2a} - 1} \int_0^\infty \left(\frac{1}{\lambda_1} \log(1 + y)\right)^r \lambda_2 e^{-\lambda_2 y} (1 - e^{-\lambda_2 y})^{2n} dy \end{aligned}$$

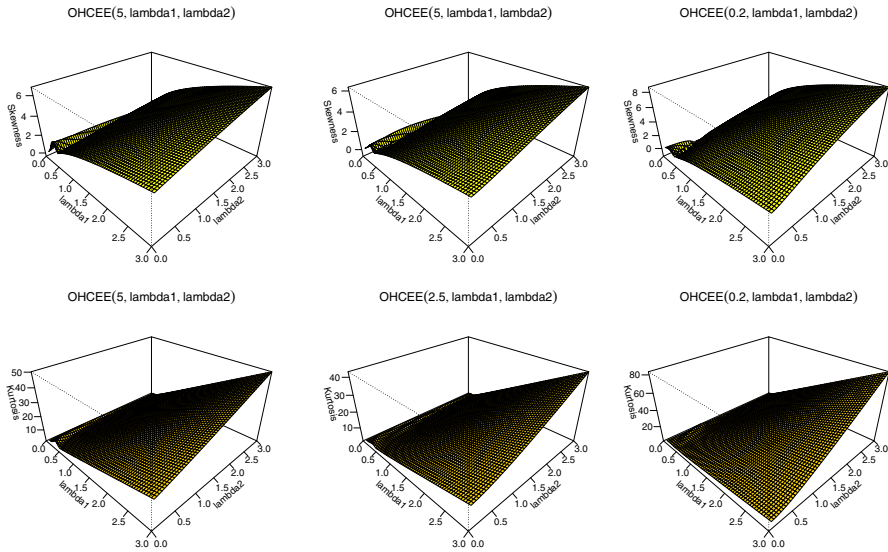


Fig. 3 3D plots of skewness and kurtosis of OHCEE distribution for some fixed values of parameter a

$$\begin{aligned}
 &= \sum_{n=0}^{\infty} \frac{a^{2n}}{\lambda_1^r (2n+1)!} \frac{2e^a}{e^{2a}-1} \int_0^{\infty} (\log(1+y))^r (2n+1)\lambda_2 e^{-\lambda_2 y} (1-e^{-\lambda_2 y})^{2n} dy \\
 &= \sum_{n=0}^{\infty} \frac{a^{2n}}{\lambda_1^r (2n+1)!} \frac{2e^a}{e^{2a}-1} \int_0^{\infty} (\log(1+y))^r f_{GE(2n+1, \lambda_2)}(y) dy \\
 &= \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} \frac{d_{r,j} (-1)^j a^{2n}}{\lambda_1^r (2n+1)!} \frac{2e^a}{e^{2a}-1} \int_0^{\infty} y^j f_{GE(2n+1, \lambda_2)}(y) dy \\
 &= \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} \frac{d_{r,j} (-1)^j a^{2n}}{\lambda_1^r (2n+1)!} \frac{2e^a}{e^{2a}-1} E_{X_{GE(2n+1, \lambda_2)}}[Y^j] \tag{5}
 \end{aligned}$$

where GE is Generalized Exponential distribution. The above integrate follows by using the series

$$\log^r(1+y) = \left(-\sum_{k=1}^{\infty} \frac{(-1)^k}{k} y^k \right)^r$$

see the website <https://www.wolframalpha.com>, and by using the identity

$$\left(\sum_{j=0}^{\infty} a_j x^j \right)^t = \sum_{j=0}^{\infty} d_{t,j} x^j$$

where $d_{t,j} = (ja_0)^{-1} \sum_{m=1}^j [m(t+1) - j] a_m d_{t,j-m}$ and $d_{t,0} = a_0^t$.

Table 1 Moments of the OHCEE distribution for some parameter values $a = 2$

μ'_r	$\lambda_1 = 0.5, \lambda_2 = 0.5$	$\lambda_1 = 0.5, \lambda_2 = 1.5$	$\lambda_1 = 1.5, \lambda_2 = 0.5$	$\lambda_1 = 1.5, \lambda_2 = 1.5$
μ'_1	2.333524	1.185911	0.7778413	0.3953037
μ'_2	6.861136	1.943436	0.7623485	0.2159373
μ'_3	22.67089	3.760179	0.8396628	0.1392659
μ'_4	81.05204	8.144519	1.000642	0.1005496
μ'_5	307.6889	19.22541	1.266209	0.0791169
μ'_6	1226.292	48.65189	1.682157	0.0667378
SD	1.415802	0.537051	0.1573114	0.0596723
CV	0.6067227	0.4528594	0.3953037	0.1509531
CS	0.0312526	0.4615376	0.0312529	0.4615372
CK	2.320697	2.680524	2.320698	2.680524

Table 2 Moments of the OHCEE distribution for some parameter values $\lambda_2 = 0.5$

μ'_r	$a = 0.3, \lambda_1 = 0.5$	$a = 0.5, \lambda_1 = 1$	$a = 0.8, \lambda_1 = 1.2$	$a = 1, \lambda_1 = 1.8$
μ'_1	1.860842	0.943458	0.8113764	0.5553856
μ'_2	4.789929	1.225457	0.8961562	0.4156781
μ'_3	14.49713	1.866463	1.153372	0.360613
μ'_4	48.74443	3.152546	1.639693	0.344453
μ'_5	176.9999	5.743568	2.507746	0.3532041
μ'_6	682.5668	11.10332	4.062111	0.3830325
SD	1.327196	0.335344	0.2378245	0.1072249
CV	0.7132234	0.3554414	0.2931124	0.1930639
CS	0.4214693	0.39924	0.3476608	0.303294
CK	2.488223	2.460495	2.404912	2.366611

The variance, CV, CS, and CK are given by

$$\sigma^2 = \mu'_2 - \mu^2, \quad CV = \frac{\sigma}{\mu} = \frac{\sqrt{\mu'_2 - \mu^2}}{\mu} = \sqrt{\frac{\mu'_2}{\mu^2} - 1}, \tag{6}$$

$$CS = \frac{E[(X - \mu)^3]}{[E(X - \mu)^2]^{3/2}} = \frac{\mu'_3 - 3\mu\mu'_2 + 2\mu^3}{(\mu'_2 - \mu^2)^{3/2}}, \tag{7}$$

and

$$CK = \frac{E[(X - \mu)^4]}{[E(X - \mu)^2]^2} = \frac{\mu'_4 - 4\mu\mu'_3 + 6\mu^2\mu'_2 - 3\mu^4}{(\mu'_2 - \mu^2)^2} \tag{8}$$

respectively. Table 1 lists the first six moments of the OHCEE distribution for selected values of the parameters, by fixing $a = 2$. And Table 2 lists the first six moments of

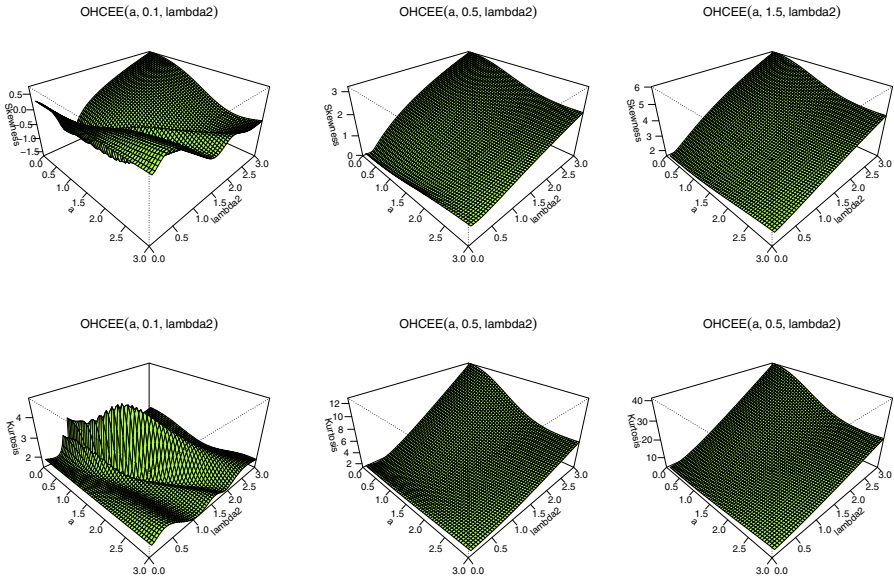


Fig. 4 3D plots of skewness and kurtosis of OHCEE distribution for some fixed values of parameter λ_1

the OHCEE distribution for selected values of the parameters, by fixing $\lambda_1 = 0.5$. These values can be determined numerically using R.

The moment generating function of the OHCEE distribution is given by

$$E(e^{tX}) = \sum_{n=0}^{\infty} \sum_{k=0}^{2n} \frac{2e^a a^{2n+1} \binom{2n}{k} (-1)^k \lambda_2}{(e^{2a} - 1)(2n)!} \frac{e^{\lambda_2(k+1)}}{(\lambda_2(k+1))^{t/\lambda_1+1}} \Gamma\left(\frac{t}{\lambda_1} + 1, \lambda_2(k+1)\right)$$

In order to investigate and analyze amount of skewness and kurtosis of the new model under the four parameters a , λ_1 and λ_2 , 3D diagrams are presented in 3, 4 and 5. Analysis of these graphs shows that all four parameter are effective in variation of skewness and kurtosis.

3.4 Order Statistics

Order statistics play an important role in probability and statistics. In this section, we present the distribution of the i th order statistic from the OHCEE distribution. The pdf of the i th order statistic from the OHCEE pdf $f_{OHCEE}(x)$ is given by

$$\begin{aligned} f_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!} f_{OHCEE}(x) [F_{OHCEE}(x)]^{i-1} [1 - F_{OHCEE}(x)]^{n-i} \\ &= \frac{n!}{(i-1)!(n-i)!} f_{OHCEE}(x) \sum_{m=0}^{n-i} \binom{n-i}{m} (-1)^m [F_{OHCEE}(x)]^{m+i-1} \end{aligned}$$

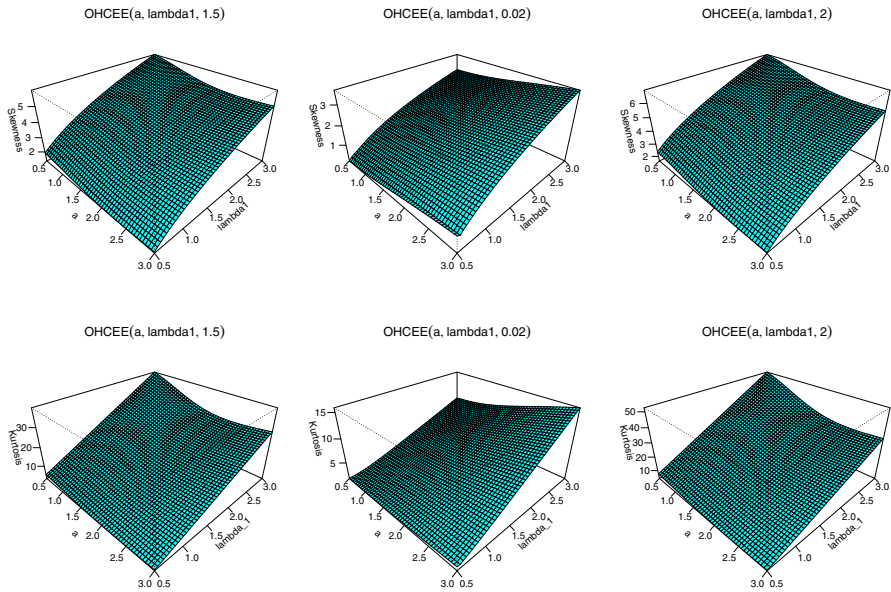
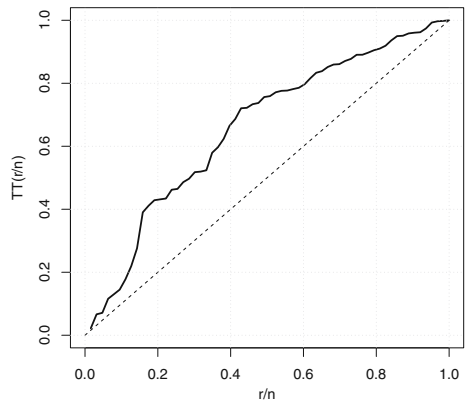


Fig. 5 3D plots of skewness and kurtosis of OHCEE distribution for some fixed values of parameter λ_2

Fig. 6 Scaled-*TTT* plot of the data set



by using the binomial expansion $[1 - F_{OHCEE}(x)]^{n-i} = \sum_{m=0}^{n-i} \binom{n-i}{m} (-1)^m [F_{OHCEE}(x)]^m$, so that

$$f_{i:n}(x) = \frac{1}{B(i, n - i + 1)} \sum_{m=0}^{n-i} \binom{n-i}{m} (-1)^m [F_{OHCEE}(x)]^{m+i-1} f_{OHCEE}(x)$$

4 Inference Procedure

In this section, we consider estimation of unknown parameters of the $OHCEE(a, \lambda_1, \lambda_2)$ distribution by maximum likelihood method, Bayesian and boot-strap estimation. Also, we get the stress–strength parameter under this distribution.

4.1 Maximum Likelihood Estimation

Let x_1, \dots, x_n be a random sample from the OHCEE distribution and $\Delta = (a, \lambda_1, \lambda_2)$ the vector of parameters. The log-likelihood function is given by

$$L = L(\Delta) = n \log \left(\frac{2a e^a}{e^{2a} - 1} \right) + n \log(\lambda_1) + \lambda_1 \sum_{i=1}^n x_i + n \log(\lambda_2) - \lambda_2 \sum_{i=1}^n (e^{\lambda_1 x_i} - 1) + \sum_{i=1}^n \log \left(\cosh \left(a \left(1 - e^{-\lambda_2 (e^{\lambda_1 x_i} - 1)} \right) \right) \right) \quad (9)$$

The elements of the score vector are given by

$$\begin{aligned} \frac{dL}{da} &= n \frac{2e^{3a}(1-a) - 2e^a(1+a)}{2a e^a (e^{2a} - 1)} \\ &\quad + \sum_{i=1}^n (1 - e^{-\lambda_2 (e^{\lambda_1 x_i} - 1)}) \tanh \left(a \left(1 - e^{-\lambda_2 (e^{\lambda_1 x_i} - 1)} \right) \right) = 0, \\ \frac{dL}{d\lambda_1} &= \frac{n}{\lambda_1} + \sum_{i=1}^n x_i - \lambda_2 \sum_{i=1}^n x_i e^{\lambda_1 x_i} \\ &\quad + a \lambda_2 \sum_{i=1}^n x_i e^{\lambda_1 x_i} e^{-\lambda_2 (e^{\lambda_1 x_i} - 1)} \tanh \left(a \left(1 - e^{-\lambda_2 (e^{\lambda_1 x_i} - 1)} \right) \right) = 0, \end{aligned}$$

and

$$\begin{aligned} \frac{dL}{d\lambda_2} &= \frac{n}{\lambda_2} - \sum_{i=1}^n (e^{\lambda_1 x_i} - 1) \\ &\quad + a \sum_{i=1}^n (e^{\lambda_1 x_i} - 1) e^{-\lambda_2 (e^{\lambda_1 x_i} - 1)} \tanh \left(a \left(1 - e^{-\lambda_2 (e^{\lambda_1 x_i} - 1)} \right) \right) = 0 \end{aligned}$$

respectively.

The maximum likelihood estimates, $\hat{\Delta}$ of $\Delta = (a, \lambda_1, \lambda_2)$ are obtained by solving the nonlinear equations $\frac{dL}{da} = 0$, $\frac{dL}{d\lambda_1} = 0$, $\frac{dL}{d\lambda_2} = 0$. these equations are not in closed form and the values of the parameters a, λ_1 and λ_2 must be found by using iterative methods. Therefore, the maximum likelihood estimates, $\hat{\Delta}$ of $\Delta = (a, \lambda_1, \lambda_2)$ can be determined using an iterative method such as the NewtonRaphson procedure.

4.2 Stress–Strength Parameter Estimation

In reliability, the stress–strength model describes the life of a component which has a random strength X subjected to a random stress Y . In this section, we consider the problem of estimating $R = P(X > Y)$, under the assumption that $X \sim OHCEE(a_1, \lambda_{11}, \lambda_{21})$, $Y \sim OHCEE(a_2, \lambda_{12}, \lambda_{22})$, and X and Y are independently distributed. Then, we can get

$$\begin{aligned} R = P(X > Y) &= \int_0^{\infty} [1 - H_X(y)] h_Y(y) dy \\ &= \int_0^{\infty} \left[1 - \frac{2e_1^a}{e^{2a_1} - 1} \sinh \left(a_1 \left(1 - e^{-\lambda_{21}(e^{\lambda_{11}y} - 1)} \right) \right) \right] \\ &\quad \times \frac{2a_2 e^{a_2}}{e^{2a_2} - 1} \lambda_{12} e^{\lambda_{12}y} \lambda_{22} e^{-\lambda_{22}(e^{\lambda_{12}y} - 1)} \cosh \left(a_2 \left(1 - e^{-\lambda_{22}(e^{\lambda_{12}y} - 1)} \right) \right) dy \end{aligned} \quad (10)$$

To compute the MLE of R , let us first obtain the MLEs of a_1 , a_2 , λ_{11} , λ_{12} , λ_{21} and λ_{22} . Suppose x_1, \dots, x_n be the observed values of a random sample of size n from $OHCEE(a_1, \lambda_{11}, \lambda_{12})$ and y_1, \dots, y_m be the observed values of a random sample of size m from $OHCEE(a_2, \lambda_{12}, \lambda_{22})$. Therefore, the log-likelihood function of a_1 , a_2 , λ_{11} , λ_{12} , λ_{21} and λ_{22} is given by

$$\begin{aligned} L(a_1, a_2, \lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}) &= n \log \left(\frac{2a_1 e_1^a}{e^{2a_1} - 1} \right) \\ &\quad + n \log(\lambda_{11}) + \lambda_{11} \sum_{i=1}^n x_i + n \log(\lambda_{21}) \\ &\quad - \lambda_{21} \sum_{i=1}^n (e^{\lambda_{11} x_i} - 1) \\ &\quad + \sum_{i=1}^n \log \left(\cosh \left(a_1 \left(1 - e^{-\lambda_{21}(e^{\lambda_{11} x_i} - 1)} \right) \right) \right) \\ &\quad + m \log \left(\frac{2a_2 e_2^a}{e^{2a_2} - 1} \right) \\ &\quad + m \log(\lambda_{12}) + \lambda_{12} \sum_{i=1}^m y_i + m \log(\lambda_{22}) \\ &\quad - \lambda_{22} \sum_{i=1}^m (e^{\lambda_{12} y_i} - 1) \\ &\quad + \sum_{i=1}^m \log \left(\cosh \left(a_2 \left(1 - e^{-\lambda_{22}(e^{\lambda_{12} y_i} - 1)} \right) \right) \right) \end{aligned} \quad (11)$$

It follows that the MLEs of a_1 , a_2 , λ_{11} , λ_{12} , λ_{21} and λ_{22} , say \hat{a}_1 , \hat{a}_2 , $\hat{\lambda}_{11}$, $\hat{\lambda}_{12}$, $\hat{\lambda}_{21}$ and $\hat{\lambda}_{22}$, are the simultaneous solutions of the following equations:

$$\begin{aligned}
 & n \frac{2e^{3a_1}(1-a_1) - 2e_1^a(1+a_1)}{2a_1 e_1^a(e^{2a_1} - 1)} + \sum_{i=1}^n (1 - e^{-\lambda_{12}(e^{\lambda_{11}x_i} - 1)}) \tanh\left(a_1 \left(1 - e^{-\lambda_{12}(e^{\lambda_{11}x_i} - 1)}\right)\right) = 0, \\
 & m \frac{2e^{3a_2}(1-a_2) - 2e_2^a(1+a_2)}{2a_2 e_2^a(e^{2a_2} - 1)} + \sum_{i=1}^m (1 - e^{-\lambda_{22}(e^{\lambda_{12}y_i} - 1)}) \tanh\left(a_2 \left(1 - e^{-\lambda_{22}(e^{\lambda_{12}y_i} - 1)}\right)\right) = 0, \\
 & \frac{n}{\lambda_{11}} + \sum_{i=1}^n x_i - \lambda_{21} \sum_{i=1}^n x_i e^{\lambda_{11}x_i} + a_1 \lambda_{21} \sum_{i=1}^n x_i e^{\lambda_{11}x_i} e^{-\lambda_{21}(e^{\lambda_{11}x_i} - 1)} \tanh\left(a_1 \left(1 - e^{-\lambda_{21}(e^{\lambda_{11}x_i} - 1)}\right)\right) = 0, \\
 & \frac{m}{\lambda_{12}} + \sum_{i=1}^m y_i - \lambda_{22} \sum_{i=1}^m y_i e^{\lambda_{12}y_i} + a_2 \lambda_{22} \sum_{i=1}^m y_i e^{\lambda_{12}y_i} e^{-\lambda_{22}(e^{\lambda_{12}y_i} - 1)} \tanh\left(a_2 \left(1 - e^{-\lambda_{22}(e^{\lambda_{12}y_i} - 1)}\right)\right) = 0, \\
 & \frac{n}{\lambda_{21}} - \sum_{i=1}^n (e^{\lambda_{11}x_i} - 1) + a_1 \sum_{i=1}^n (e^{\lambda_{11}x_i} - 1) e^{-\lambda_{21}(e^{\lambda_{11}x_i} - 1)} \tanh\left(a_1 \left(1 - e^{-\lambda_{21}(e^{\lambda_{11}x_i} - 1)}\right)\right) = 0 \\
 & \frac{m}{\lambda_{22}} - \sum_{i=1}^m (e^{\lambda_{12}y_i} - 1) + a_2 \sum_{i=1}^m (e^{\lambda_{12}y_i} - 1) e^{-\lambda_{22}(e^{\lambda_{12}y_i} - 1)} \tanh\left(a_2 \left(1 - e^{-\lambda_{22}(e^{\lambda_{12}y_i} - 1)}\right)\right) = 0
 \end{aligned}$$

Once we obtain $\hat{a}_1, \hat{a}_2, \hat{\lambda}_{11}, \hat{\lambda}_{12}, \hat{\lambda}_{21}$ and $\hat{\lambda}_{22}$ therefore, we compute the MLE of R as \hat{R} .

Here, the maximum likelihood approach does not give an explicit estimator for the MLEs of parameters and hence the MLE of R. In practice, one has to use numerical methods to find the MLEs, such methods are well implemented in R software packages.

4.3 Bootstrap Estimation

The parameters of the fitted distribution can be estimated by parametric (resampling from the fitted distribution) or non-parametric (resampling with replacement from the original data set) bootstraps resampling (see [5]). These two parametric and nonparametric bootstrap procedures are described as follows.

Parametric bootstrap procedure:

1. Estimate θ (vector of unknown parameters), say $\hat{\theta}$, by using the *MLE* procedure based on a random sample.
2. Generate a bootstrap sample $\{X_1^*, \dots, X_m^*\}$ using $\hat{\theta}$ and obtain the bootstrap estimate of θ , say $\hat{\theta}^*$, from the bootstrap sample based on the *MLE* procedure.
3. Repeat Step 2 *NBOOT* times.
4. Order $\hat{\theta}^*_1, \dots, \hat{\theta}^*_{NBOOT}$ as $\hat{\theta}^*_{(1)}, \dots, \hat{\theta}^*_{(NBOOT)}$. Then obtain γ -quantiles and $100(1 - \alpha)\%$ confidence intervals of parameters.

In case of the *OHCEE* distribution, the parametric bootstrap estimators (PBs) of α, β, λ and ξ , say $\hat{\alpha}_{PB}, \hat{\beta}_{PB}, \hat{\lambda}_{PB}$ and $\hat{\xi}_{PB}$, respectively.

Nonparametric bootstrap procedure

1. Generate a bootstrap sample $\{X_1^*, \dots, X_m^*\}$, with replacement from original data set.
2. Obtain the bootstrap estimate of θ with MLE procedure, say $\hat{\theta}^*$, by using the bootstrap sample.
3. Repeat Step 2 *NBOOT* times.

4. Order $\hat{\theta}^*_1, \dots, \hat{\theta}^*_{NBOOT}$ as $\hat{\theta}^*_{(1)}, \dots, \hat{\theta}^*_{(NBOOT)}$. Then obtain γ -quantiles and $100(1 - \alpha)\%$ confidence intervals of parameters.

In case of the *OHCEE* distribution, the nonparametric bootstrap estimators (NPBs) of α, β, λ and ξ , say $\hat{\alpha}_{NPB}, \hat{\beta}_{NPB}, \hat{\lambda}_{NPB}$ and $\hat{\xi}_{NPB}$, respectively.

4.4 Bayesian Estimation

In this section we consider Bayesian inference of the unknown parameters of the *OHCEE*(a, λ_1, λ_2). It is assumed that a is known and both of the parameters of λ_1 and λ_2 are unknown and have the independent gamma prior distributions with probability density functions

$$g(\lambda_1) \propto e^{-c\lambda_1} \lambda_1^{b-1}, \quad \lambda_1 > 0, \quad b, c > 0,$$

and

$$g(\lambda_2) \propto e^{-s\lambda_2} \lambda_2^{d-1}, \quad \lambda_2 > 0, \quad s, d > 0,$$

Thus, the joint prior distribution for λ_1 and λ_2 is

$$g(\lambda_1, \lambda_2) \propto \lambda_1^{b-1} \lambda_2^{d-1} e^{-(c\lambda_1 + s\lambda_2)}, \quad \lambda_1 > 0, \quad \lambda_2 > 0, \quad b, c, s, d > 0.$$

Now, the posterior density function of λ_1 and λ_2 given the data, denoted by $\pi(\lambda_1, \lambda_2 | \underline{x})$, can be obtained the following as

$$\begin{aligned} \pi(\lambda_1, \lambda_2 | \underline{x}) &= \frac{1}{R(\underline{x})} \lambda_1^{n+b-1} \lambda_2^{n+d-1} e^{-c\lambda_1} e^{-\left(s + \sum_{i=1}^n (e^{\lambda_1 x_i} - 1)\right)} e^{\lambda_1 \sum_{i=1}^n x_i} \\ &\times \prod_{i=1}^n \cosh\left(a\left(1 - e^{-\lambda_2(e^{\lambda_1 x_i} - 1)}\right)\right), \quad \lambda_1 > 0, \quad \lambda_2 > 0. \end{aligned} \quad (12)$$

where

$$\begin{aligned} R(\underline{x}) &= \int_0^\infty \int_0^\infty \lambda_1^{n+b-1} \lambda_2^{n+d-1} e^{-c\lambda_1} e^{-\left(s + \sum_{i=1}^n (e^{\lambda_1 x_i} - 1)\right)} e^{\lambda_1 \sum_{i=1}^n x_i} \\ &\times \prod_{i=1}^n \cosh\left(a\left(1 - e^{-\lambda_2(e^{\lambda_1 x_i} - 1)}\right)\right) d\lambda_1 d\lambda_2. \end{aligned}$$

Therefore, the Bayes estimate of any function of α and β , say $\theta(\alpha, \beta)$ under squared error loss function is

Table 3 Bayes estimator and posterior risk under different loss functions

loss function	Bayes estimator	Posterior risk
$L_1 = SELF = (\theta - d)^2$	$E(\theta x)$	$Var(\theta x)$
$L_2 = WSELF = \frac{(\theta-d)^2}{\theta}$	$(E(\theta^{-1} x))^{-1}$	$E(\theta x) - (E(\theta^{-1} x))^{-1}$
$L_3 = MSELF = \left(1 - \frac{d}{\theta}\right)^2$	$\frac{E(\theta^{-1} x)}{E(\theta^{-2} x)}$	$1 - \frac{E(\theta^{-1} x)^2}{E(\theta^{-2} x)}$
$L_4 = PLF = \frac{(\theta-d)^2}{d}$	$\sqrt{E(\theta^2 x)}$	$2\left(\sqrt{E(\theta^2 x)} - E(\theta x)\right)$
$L_5 = KLF = \left(\sqrt{\frac{d}{\theta}} - \sqrt{\frac{\theta}{d}}\right)$	$\sqrt{\frac{E(\theta x)}{E(\theta^{-1} x)}}$	$2\left(\sqrt{E(\theta x)E(\theta^{-1} x)} - 1\right)$

$$\hat{\theta}_{Bayes} = \frac{1}{R(x)} \int_0^\infty \int_0^\infty \theta(\alpha, \beta) \lambda_1^{n+b-1} \lambda_2^{n+d-1} e^{-c\lambda_1} e^{-\left(s+\sum_{i=1}^n (e^{\lambda_1 x_i} - 1)\right)} e^{\lambda_1 \sum_{i=1}^n x_i} \times \prod_{i=1}^n \cosh\left(a\left(1 - e^{-\lambda_2(e^{\lambda_1 x_i} - 1)}\right)\right) d\lambda_1 d\lambda_2 \tag{13}$$

4.4.1 Lindleys Approximation

It is impossible to compute Eq. (2) analytically. Lindleys approximation is used to compute the ratio of integrals of the form (1). Based on Lindleys approximation, the approximate Bayes estimates of λ_1 and λ_2 under squared error loss function are

$$\hat{\lambda}_{1Lindley} = \hat{\lambda}_1 + \frac{1}{2} \left[2\rho_{\lambda_1} \sigma_{\lambda_1 \lambda_1} + 2\rho_{\lambda_2} \sigma_{\lambda_1 \lambda_2} + \sigma_{\lambda_1 \lambda_1} (L_{\lambda_1 \lambda_1 \lambda_1} \sigma_{\lambda_1 \lambda_1} + L_{\lambda_1 \lambda_2 \lambda_1} \sigma_{\lambda_1 \lambda_2} + L_{\lambda_2 \lambda_1 \lambda_1} \sigma_{\lambda_2 \lambda_1} + L_{\lambda_2 \lambda_2 \lambda_1} \sigma_{\lambda_2 \lambda_1}) + \sigma_{\lambda_2 \lambda_1} (L_{\lambda_1 \lambda_1 \lambda_2} \sigma_{\lambda_1 \lambda_1} + L_{\lambda_1 \lambda_1 \lambda_2} \sigma_{\lambda_1 \lambda_2} + L_{\lambda_2 \lambda_1 \lambda_2} \sigma_{\lambda_2 \lambda_1} + L_{\lambda_2 \lambda_2 \lambda_2} \sigma_{\lambda_2 \lambda_2}) \right], \tag{14}$$

and

$$\hat{\lambda}_{2Lindley} = \hat{\lambda}_2 + \frac{1}{2} \left[2\rho_{\lambda_1} \sigma_{\lambda_2 \lambda_1} + 2\rho_{\lambda_2} \sigma_{\lambda_2 \lambda_2} + \sigma_{\lambda_1 \lambda_2} (L_{\lambda_1 \lambda_1 \lambda_1} \sigma_{\lambda_1 \lambda_1} + L_{\lambda_1 \lambda_2 \lambda_1} \sigma_{\lambda_1 \lambda_2} + L_{\lambda_2 \lambda_1 \lambda_1} \sigma_{\lambda_2 \lambda_1} + L_{\lambda_2 \lambda_2 \lambda_1} \sigma_{\lambda_2 \lambda_2}) + \sigma_{\lambda_2 \lambda_2} (L_{\lambda_1 \lambda_1 \lambda_2} \sigma_{\lambda_1 \lambda_1} + L_{\lambda_1 \lambda_2 \lambda_2} \sigma_{\lambda_1 \lambda_2} + L_{\lambda_2 \lambda_1 \lambda_2} \sigma_{\lambda_2 \lambda_1} + L_{\lambda_2 \lambda_2 \lambda_2} \sigma_{\lambda_2 \lambda_2}) \right], \tag{15}$$

respectively. Here $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are the MLEs of λ_1 and λ_2 , respectively. The details involved in the derivations of Eqs. (3) and (4) are placed in the ‘‘Appendix’’ by a function of the parameter and corresponding estimator. Five well-known loss functions and associated Bayesian estimators and corresponding posterior risk are presented in Table 3.

5 Algorithm and a Simulation Study

In this section, we give one algorithm for generating the random data x_1, \dots, x_n from the OHCEE distribution and hence a simulation study is obtained to evaluate the performance of MLEs.

5.1 Algorithm

Here, we obtain a algorithm for generating the random data x_1, \dots, x_n from the OHCEE distribution as follows. The algorithm is based on generating random data from the inverse cdf of the OHCEE distribution.

Algorithm

1. Generate $U_i \sim Uniform(0, 1)$; $i = 1, \dots, n$,
2. set

$$X_p = \frac{1}{\lambda_1} \log \left(1 - \frac{1}{\lambda_2} \log \left(1 - \frac{1}{a} \log \left(\frac{e^{2a} - 1}{2e^a} p + \sqrt{\left(\frac{e^{2a} - 1}{2e^a} p \right)^2 + 1} \right) \right) \right); \quad 0 \leq p \leq 1$$

5.2 Monte Carlo Simulation Study

In this section, we assess the performance of the MLE's of the parameters with respect to sample size n for the $OHCEE(a, \lambda_1, \lambda_2)$ distribution. We used above Algorithm to generate data from the OHCEE distribution. The assessment of performance is based on a simulation study by using the Monte Carlo method. Let \hat{a} , $\hat{\lambda}_1$ and $\hat{\lambda}_2$ be the MLEs of the parameters a , λ_1 and λ_2 , respectively. We compute the mean square error (MSE) and bias of the MLEs of the parameters a , λ_1 and λ_2 , based on the simulation results of $N = 2000$ independence replications. results are summarized in Table 4 for different values of n , a , λ_1 and λ_2 . From Table 4 the results verify that MSE and bias of the MLEs of the parameters decrease with respect to sample size n increases. Hence, we can see the MLEs of a , λ_1 and λ_2 , are consistent estimators.

6 Practical Data Applications

In this section, we present the application of the OHCEE model to one practical data set to illustrate its flexibility among a set of competitive models (Table 6).

The windshield on a large aircraft is a complex piece of equipment, comprised basically of several layers of material, including a very strong outer skin with a heated layer just beneath it, all laminated under high temperature and pressure. Failures of these items are not structural failures. Instead, they typically involve damage or delamination of the nonstructural outer ply or failure of the heating system. These failures do not result in damage to the aircraft but do result in replacement of the

Table 4 MSEs and Average biases (values in parentheses) of the simulated estimates

		a = 0.3	$\lambda_1 = 0.5$	$\lambda_2 = 1.5$
n	30	0.8189 (0.1738)	0.0865 (0.0544)	132.3321 (2.3249)
	50	0.7216 (0.1095)	0.0519 (0.0187)	40.7226 (1.3271)
	100	0.6768 (0.1396)	0.0275 (−0.0182)	15.1627 (0.9064)
	200	0.5902 (0.1135)	0.0172 (−0.0303)	4.4592 (0.6561)
		a = 0.5	$\lambda_1 = 1$	$\lambda_2 = 2$
n	30	1.0717 (0.1182)	0.4595 (0.1103)	353.7897 (4.8048)
	50	0.8683 (0.0302)	0.2869 (0.0377)	165.9069 (3.2168)
	100	0.7678 (0.0574)	0.1643 (−0.0512)	65.0995 (2.2592)
	200	0.6099 (0.0405)	0.1014 (−0.0595)	17.2459 (1.2489)
		a = 0.8	$\lambda_1 = 0.5$	$\lambda_2 = 1.5$
n	30	1.2207 (−0.0169)	0.0873 (0.0515)	106.8476 (2.3648)
	50	1.0508 (−0.0300)	0.0538 (0.0134)	54.7730 (1.7901)
	100	0.8062 (−0.0321)	0.0304 (−0.0151)	17.3773 (1.1130)
	200	0.6423 (−0.0515)	0.0196 (−0.0180)	4.9051 (0.6426)
		a = 1	$\lambda_1 = 1.5$	$\lambda_2 = 2$
n	30	1.4442 (−0.0288)	1.0211 (0.1839)	297.3460 (4.7064)
	50	1.1301 (−0.0819)	0.7082 (0.0773)	163.7686 (3.5522)
	100	0.8579 (−0.0866)	0.4011 (−0.0087)	64.1250 (2.0630)
	200	0.6941 (−0.1346)	0.2789 (−0.0210)	26.7817 (1.2582)
		a = 2	$\lambda_1 = 2$	$\lambda_2 = 1.5$
n	30	2.3294 (−0.1836)	2.1399 (0.5090)	223.8817 (3.3442)
	50	1.6476 (−0.2089)	1.4322 (0.3452)	57.4308 (1.7158)
	100	1.1889 (−0.2213)	0.8673 (0.2137)	12.2516 (0.6404)
	200	0.9679 (−0.2788)	0.6181 (0.2014)	3.7672 (0.2618)

windshield. We consider the data on service times for a particular model windshield given in Table 16.11 of Murthy et al. [8]. These data were recently studied by Ramos et al. [9]. These data are:

0.046 1.436 2.592 0.140 1.492 2.600 0.150 1.580 2.670 0.248 1.719 2.717 0.280 1.794 2.819 0.313 1.915 2.820 0.389 1.920 2.878 0.487 1.963 2.950 0.622 1.978 3.003 0.900 2.053 3.102 0.952 2.065 3.304 0.996 2.117 3.483 1.003 2.137 3.500 1.010 2.141 3.622 1.085 2.163 3.665 1.092 2.183 3.695 1.152 2.240 4.015 1.183 2.341 4.628 1.244 2.435 4.806 1.249 2.464 4.881 1.262 2.543 5.140.

Graphical measure The total time test (TTT) plot due to Aarset [3] is an important graphical approach to verify whether the data can be applied to a specific distribution or not. According to Aarset [3], the empirical version of the TTT plot is given by plotting $T(r/n) = [\sum_{i=1}^r y_{i:n} + (n - r)y_{r:n}] / \sum_{i=1}^n y_{i:n}$ against r/n , where $r = 1, \dots, n$

Table 5 Bootstrap point and interval estimation of the parameters a , λ_1 and λ_2

	Parametric bootstrap		Non-parametric bootstrap	
	Point estimation	CI	Point estimation	CI
a	2.735	(0.464, 4.430)	2.641	(0.672, 4.063)
λ_1	0.501	(0.248, 0.967)	0.497	(0.324, 0.847)
λ_2	0.910	(0.136, 3.224)	0.927	(0.188, 2.080)

and $y_{i:n}$ ($i = 1, \dots, n$) are the order statistics of the sample. Aarset [3] showed that the hazard function is constant if the TTT plot is graphically presented as a straight diagonal, the hazard function is increasing (or decreasing) if the TTT plot is concave (or convex). The hazard function is U-shaped if the TTT plot is convex and then concave, if not, the hazard function is unimodal. The TTT plots for data set is presented in Fig. 6. These plots indicate that the empirical hazard rate functions of the data set is increasing. Therefore, the OHCEE distribution is appropriate to fit this data set. parametric and non-parametric bootstrap methods for the real data set. We provide results of bootstrap estimation based on 10,000 bootstrap replicates in Table 5.

We fit the OHCEE distribution to the one data set and compare it with the HCE, gamma, generalized exponential and Weibull densities. Table 6 shows the MLEs of parameters, log-likelihood, Akaike information criterion (AIC), Cramrvon Mises (W^*), AndersonDarling (A^*) and p-value(P) statistics for the data set. The OHCEE distribution provides the best fit for the data set as it shows the lowest AIC, A^* and W^* than other considered models. The relative histograms, fitted OHCEE, HCE, gamma, generalized exponential and Weibull PDFs for data are plotted in Fig. 7. The plots of empirical and fitted survival functions, P–P plots and Q–Q plots for the OHCEE and other fitted distributions are displayed in Figs. 7 and 8 respectively. These plots also support the results in Table 6. We compare the OHCEE model with a set of competitive models, namely:

- (i) Hyperbolic Cosine-Exponential distribution (HCE) [6]. The two-parameter HCE density function is given by

$$f(x; a, \lambda) = \frac{2a e^a}{e^{2a} - 1} \lambda e^{-\lambda x} \cosh(a(1 - e^{-\lambda x})); \quad x > 0$$

where $a > 0$ and $\lambda > 0$.

- (ii) The two-parameter Weibull distribution is given by

$$f(x; \alpha, \beta) = \frac{\alpha}{\beta} \left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^\alpha}; \quad x > 0$$

where $\alpha > 0$ and $\beta > 0$.

- (iii) The two-parameter Gamma distribution is given by

$$f(x; \alpha, \theta) = \frac{1}{\theta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-(x/\theta)}; \quad x > 0$$

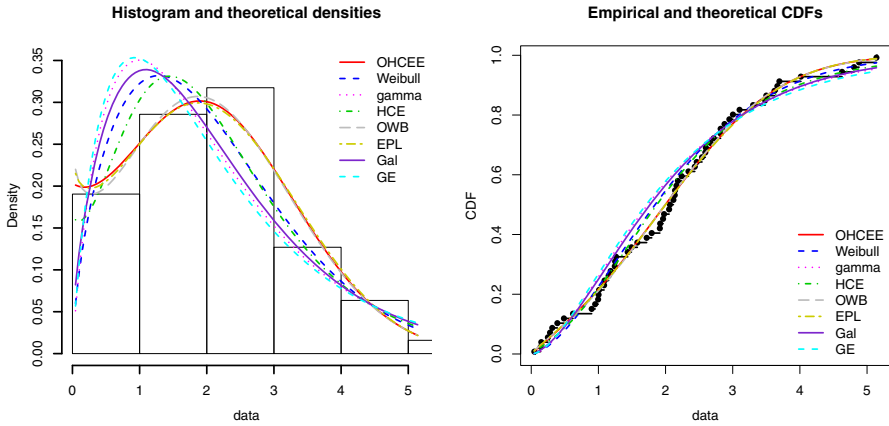


Fig. 7 Estimated densities and Empirical and Estimated cdf for the data set

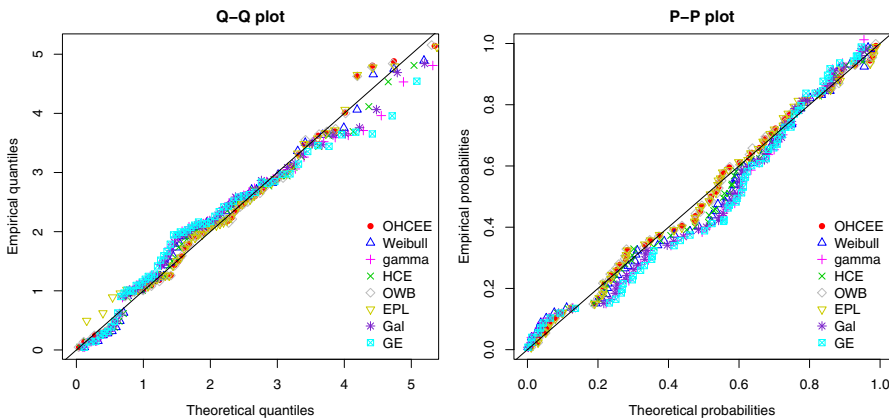


Fig. 8 Q–Q and P–P plots for the data set

where $\alpha > 0$ and $\theta > 0$ and $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$.

(iv) The two-parameter generalized exponential (GE) distribution is given by

$$f(x; \alpha, \lambda) = \alpha\lambda e^{-\lambda x} (1 - e^{-\lambda x})^{\alpha-1}; \quad x > 0$$

where $\alpha > 0$ and $\lambda > 0$.

As mentioned in inference section, there are not closed expression for MLE estimation of parameters a , λ_1 and λ_2 . We use numerical methods to obtain MLE estimation of these parameters. To evaluate the results of MLE estimation (Table 6), We provide profile-likelihood plots of *OHCEE* distribution for each parameter in Fig. 9.

The corresponding Bayesian point estimation and posterior risk provided in Table 7.

Table 6 Parameter estimates (standard errors), log-likelihood values and goodness of fit measures

Model	MLEs of parameters (s.e)	Log-likelihood	AIC	BIC	A*	W*	K.S	P
OHCEE	$\hat{a} = 2.58 (0.96)$	−97.91	201.83	208.26	0.30	0.04	0.06	0.95
	$\hat{\lambda}_1 = 0.24 (0.14)$							
	$\hat{\lambda}_2 = 2.08 (2.02)$							
HCE	$\hat{a} = 3.69 (0.67)$	−99.81	203.63	207.92	0.45	0.07	0.10	0.51
	$\hat{\lambda} = 0.89 (0.09)$							
Weibull	$\hat{\alpha} = 1.62 (0.16)$	−100.31	204.63	208.92	0.64	0.09	0.10	0.41
	$\hat{\beta} = 2.30 (0.18)$							
Gamma	$\hat{\alpha} = 1.9 (0.31)$	−102.83	209.66	213.95	1.16	0.2	0.58	0.84
	$\hat{\theta} = 0.91 (0.17)$							
GE	$\hat{\alpha} = 1.89 (0.34)$	−103.54	211.09	215.37	1.31	0.23	0.14	0.13
	$\hat{\lambda} = 0.69 (0.09)$							

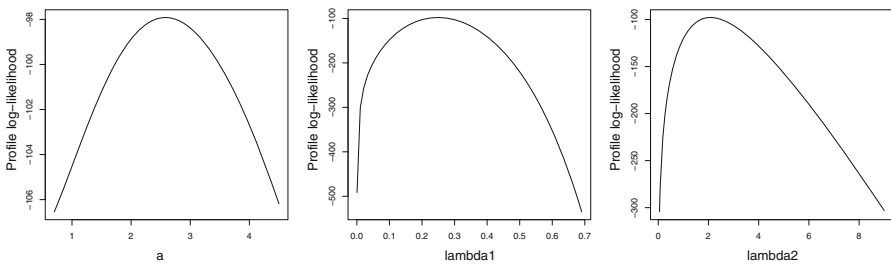


Fig. 9 profile-likelihood plots of OHCEE distribution

Table 7 Bayesian estimates and their posterior risks of parameters under different loss functions based on real data

Data	Service times			
	$\hat{\lambda}_1$		$\hat{\lambda}_2$	
Bayes	Estimate	Risk	Estimate	Risk
SELF	0.242722	0.00022	2.140396	0.04963
WSELF	0.243575	0.00085	2.165938	0.02554
MSELF	0.244380	0.00330	2.195038	0.01343
PLF	0.242265	0.00091	2.128769	0.02325
KLF	0.243148	0.00350	2.153129	0.01182

7 Conclusion

In this article, a new model for the lifetime distributions is introduced and its main properties are discussed. A special submodel of this family is taken up by considering exponential distributions in place of the parent distribution F and instead of the parent distribution G. We also show that the proposed distribution has variability of hazard

rate shapes such as increasing and upside-down bathtub shapes. Numerical results of maximum likelihood, Bayesian and bootstrap procedures for a set of real data are presented in separate tables. From a practical point of view, we show that the proposed distribution is more flexible than some common statistical distributions.

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Appendix

Lindley developed an asymptotic expansion for evaluating the ratio of integrals of the form

$$u^* = E[u(\alpha, \beta)|x] = \frac{\int \int u(\alpha, \beta) \exp[l(\alpha, \beta|x) + \rho(\alpha, \beta)] d\alpha d\beta}{\int \int \exp[l(\alpha, \beta|x) + \rho(\alpha, \beta)] d\alpha d\beta},$$

where $u(\alpha, \beta)$ is a function of α and β only, and $l(\alpha, \beta|x)$ is the log-likelihood function and $\rho(\alpha, \beta) = \log[\pi(\alpha, \beta)]$.

Utilizing Lindleys method, u^* can be approximated as

$$\begin{aligned} u^* = u + \frac{1}{2} & \left[(u_{\alpha\alpha} + 2u_{\alpha}\rho_{\alpha})\sigma_{\alpha\alpha} + (u_{\alpha\beta} + 2u_{\alpha}\rho_{\beta})\sigma_{\alpha\beta} \right. \\ & + (u_{\beta\alpha} + 2u_{\beta}\rho_{\alpha})\sigma_{\beta\alpha} + (u_{\beta\beta} + 2u_{\beta}\rho_{\beta})\sigma_{\beta\beta} \\ & + (u_{\alpha}\sigma_{\alpha\alpha} + u_{\beta}\sigma_{\alpha\beta})(L_{\alpha\alpha\alpha}\sigma_{\alpha\alpha} + L_{\alpha\beta\alpha}\sigma_{\alpha\beta} + L_{\beta\alpha\alpha}\sigma_{\beta\alpha} + L_{\beta\beta\alpha}\sigma_{\beta\beta}) \\ & \left. + (u_{\alpha}\sigma_{\beta\alpha} + u_{\beta}\sigma_{\beta\beta})(L_{\alpha\alpha\beta}\sigma_{\alpha\alpha} + L_{\alpha\beta\beta}\sigma_{\alpha\beta} + L_{\beta\alpha\beta}\sigma_{\beta\alpha} + L_{\beta\beta\beta}\sigma_{\beta\beta}) \right] \end{aligned} \quad (16)$$

All functions of the right-hand side of Equation (5) are to be evaluated at the MLE \hat{u} .

For our setup, the joint prior distribution is $\pi(\lambda_1, \lambda_2) = \lambda_1^{b-1} \lambda_2^{d-1} e^{-(c\lambda_1 + s\lambda_2)}$ and

$$\rho(\lambda_1, \lambda_2) = \log[\pi(\lambda_1, \lambda_2)] = (b-1) \log \lambda_1 + (d-1) \log \lambda_2 - (c\lambda_1 + s\lambda_2)$$

which give out

$$\begin{aligned} \rho_{\lambda_1} &= \frac{d\rho}{d\lambda_1} = \frac{b-1}{\lambda_1} - c, & \rho_{\lambda_2} &= \frac{d\rho}{d\lambda_2} = \frac{d-1}{\lambda_2} - s, \\ u_{\alpha} &= \frac{du}{d\alpha}, & u_{\beta} &= \frac{du}{d\beta}, & u_{\alpha\beta} &= \frac{d^2u}{d\alpha d\beta}, & u_{\beta\alpha} &= \frac{d^2u}{d\beta d\alpha}, \\ u_{\alpha\alpha} &= \frac{d^2u}{d\alpha d\alpha}, & u_{\beta\beta} &= \frac{d^2u}{d\beta d\beta} \end{aligned}$$

When

$$\begin{aligned}
 u &= \lambda_1, \quad u_{\lambda_1} = 1, \quad u_{\lambda_1\lambda_1} = 0 = u_{\lambda_2} = u_{\lambda_1\lambda_2} = u_{\lambda_2\lambda_1} = u_{\lambda_2\lambda_2}, \\
 \hat{\lambda}_{1Lindley} &= \hat{\lambda}_1 + \frac{1}{2} \left[2\rho_{\lambda_1} \sigma_{\lambda_1\lambda_1} + 2\rho_{\lambda_2} \sigma_{\lambda_1\lambda_2} + \sigma_{\lambda_1\lambda_1} (L_{\lambda_1\lambda_1\lambda_1} \sigma_{\lambda_1\lambda_1} \right. \\
 &\quad + L_{\lambda_1\lambda_2\lambda_1} \sigma_{\lambda_1\lambda_2} + L_{\lambda_2\lambda_1\lambda_1} \sigma_{\lambda_2\lambda_1} + L_{\lambda_2\lambda_2\lambda_1} \sigma_{\lambda_2\lambda_1}) \\
 &\quad + \sigma_{\lambda_2\lambda_1} (L_{\lambda_1\lambda_1\lambda_2} \sigma_{\lambda_1\lambda_1} + L_{\lambda_1\lambda_1\lambda_2} \sigma_{\lambda_1\lambda_2} \\
 &\quad \left. + L_{\lambda_2\lambda_1\lambda_2} \sigma_{\lambda_2\lambda_1} + L_{\lambda_2\lambda_2\lambda_2} \sigma_{\lambda_2\lambda_2}) \right],
 \end{aligned}$$

and when

$$\begin{aligned}
 u &= \lambda_2, \quad u_{\lambda_2} = 1, \quad u_{\lambda_2\lambda_2} = 0 = u_{\lambda_1} = u_{\lambda_1\lambda_1} = u_{\lambda_2\lambda_1} = u_{\lambda_1\lambda_2}, \\
 \hat{\lambda}_{2Lindley} &= \hat{\lambda}_2 + \frac{1}{2} \left[2\rho_{\lambda_1} \sigma_{\lambda_2\lambda_1} + 2\rho_{\lambda_2} \sigma_{\lambda_2\lambda_2} + \sigma_{\lambda_1\lambda_2} (L_{\lambda_1\lambda_1\lambda_1} \sigma_{\lambda_1\lambda_1} \right. \\
 &\quad + L_{\lambda_1\lambda_2\lambda_1} \sigma_{\lambda_1\lambda_2} + L_{\lambda_2\lambda_1\lambda_1} \sigma_{\lambda_2\lambda_1} + L_{\lambda_2\lambda_2\lambda_1} \sigma_{\lambda_2\lambda_2}) \\
 &\quad + \sigma_{\lambda_2\lambda_2} (L_{\lambda_1\lambda_1\lambda_2} \sigma_{\lambda_1\lambda_1} + L_{\lambda_1\lambda_2\lambda_2} \sigma_{\lambda_1\lambda_2} \\
 &\quad \left. + L_{\lambda_2\lambda_1\lambda_2} \sigma_{\lambda_2\lambda_1} + L_{\lambda_2\lambda_2\lambda_2} \sigma_{\lambda_2\lambda_2}) \right],
 \end{aligned}$$

In the above expressions $\sigma_{ij} = (i, j)$ th element in the inverse of the negative Hessian matrix, $i, j = \alpha, \beta$, and L_{ijk} implies the term obtained from differentiating log L with respect to i, j and k .

References

1. Alizadeh M, Altun E, Cordeiro GM, Rasekhi M (2018) The odd power cauchy family of distributions: properties, regression models and applications. *J Stat Comput Simul* 88(4):785–807
2. Alzaatreh A, Lee C, Famoye F (2013) A new method for generating families of continuous distributions. *Metron* 71(1):63–79
3. Aarset MV (1987) How to identify a bathtub hazard rate. *IEEE Trans Reliab* 36(1):106–108
4. Cordeiro GM, Alizadeh M, Ozel G, Hosseini B, Ortega EMM, Altun E (2017) The generalized odd log-logistic family of distributions: properties, regression models and applications. *J Stat Comput Simul* 87(5):908–932
5. Efron B, Tibshirani RJ (1994) An introduction to the bootstrap. CRC Press, Boca Raton
6. Kharazmi O, Saadatinik A (2016) Hyperbolic cosine-F families of distributions with an application to exponential distribution. *Gazi Univ J Sci* 29(4):811829
7. Kharazmi O, Saadatinik A, Alizadeh M, Hamedani GG (2018) Odd hyperbolic cosine-FG (OHC-FG) family of lifetime distributions. *J Stat Theory Appl (JSTA)* (accepted)
8. Murthy DP, Xie M, Jiang R (2004) Weibull models, vol 505. Wiley, Hoboken
9. Ramos MW, Marinho PR, Silva RV, Cordeiro GM (2013) The exponentiated Lomax Poisson distribution with an application to lifetime data. *Adv Appl Stat* 34(2):107–135

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