Off-Diagonal Bethe Ansatz and Exact Solution of a Topological Spin Ring

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A general method is proposed for constructing the Bethe ansatz equations of integrable models without
$U(1)$ symmetry. As an example, the exact spectrum of the XXZ spin ring with a Möbius-like topological
boundary condition is derived by constructing a modified $T - Q$ relation based on the functional
connection between the eigenvalues of the transfer matrix and the quantum determinant of the mono-
dromy matrix. With the exact solution, the elementary excitations of the topological XX spin ring are
discussed in detail. It is found that the excitation spectrum indeed shows a nontrivial topological nature.

Integrable models play important roles in statistical physics, quantum field theory, and condensed matter physics because those models provide some benchmarks for understanding the corresponding universal classes. Since Yang and Baxter's pioneering works [1–3], the Yang-Baxter relation has become a cornerstone for constructing and solving the integrable models. Especially, the $T - Q$ relation method [2,3] and the algebraic Bethe ansatz method [4–6] developed from the Yang-Baxter equation have become two very popular methods for dealing with the exact solutions of the known integrable models. Generally speaking, there are two classes of integrable models. One possesses $U(1)$ symmetry, and the other does not. Three well-known examples without $U(1)$ symmetry are the XYZ spin chain [5,7], the XXZ spin chain with an antiperiodic boundary condition [8–14], and the ones with unparallel boundary fields [15–19]. It has been demonstrated that the algebraic Bethe ansatz and $T - Q$ relation can successfully diagonalize the integrable models with $U(1)$ symmetry. However, for those without $U(1)$ symmetry, only some very special cases such as the XYZ spin chain with an even site number [5,7] and the XXZ spin chain with constrained unparallel boundary fields [15–17] can be dealt with because of the existence of a proper "local vacuum state" in these special cases. The main obstacle applying the algebraic Bethe ansatz and Baxter's method to general integrable models without $U(1)$ symmetry lies in the absence of such a "local vacuum." A promising method for approaching such a kind of problems is Sklyanin's separation of variables method [20,21], which has been recently applied to some integrable models [11–14,18,19]. However, a systematic method is still absent to derive the usual Bethe ansatz equations (BAEs) which are crucial for studying the physical properties in the thermodynamic limit.

In this Letter, we develop a general method for dealing with the integrable models without $U(1)$ symmetry. The central point lies in how to construct a $T - Q$ relation and

the usual BAEs for those models based on the connection between two basic invariants of the monodromy matrix: i.e., its trace (transfer matrix) and its quantum determinant which do not depend on the basis choice and whether there exists a reference state. As a concrete example, we study the spectrum of the XXZ spin ring with a Möbius-like topological boundary condition, as it is tightly related to the recent study on the topological states of matter. In fact, the topological boundary problem in many body systems has been rarely touched. With the inhomogeneous XXZ topological spin ring model, we elucidate how our method works to derive the exact spectrum and the BAEs by constructing and solving recursive functional equations. Particular attention is focused on the elementary excitations of the homogeneous XX spin ring with an antiperiodic boundary condition, as it is the simplest quantum realization of the Möbius stripe. Our exact solution shows that the elementary excitations of this simple model indeed exhibit a nontrivial topological nature.

We start from the following model Hamiltonian

$$H = -\sum_{j=1}^{N} [\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \cosh\eta \sigma_j^z \sigma_{j+1}^z], \quad (1)$$

with the antiperiodic boundary conditions $\sigma_{N+1}^{\alpha} = \sigma_1^x \sigma_1^\alpha \sigma_1^x$. N is the site number of the system, and σ_j^α ($\alpha = x, y, z$) is the Pauli matrix on the site j along the α direction. With such a topological boundary condition, the spin on the Nth site connects with that on the first site after rotating the π angle along the x direction [a kink on the $(N, 1)$ bond] and forms a torus in the spin space. With an unitary transformation $U_n H U_n^{-1}$, $U_n = \prod_{j=1}^n \sigma_j^x$, the kink can be shifted to the $(n, n+1)$ bond without changing the spectrum of the Hamiltonian. Notice here the braiding is in the quantum space rather than in the real space, and therefore the present model describes a quantum Möbius stripe. We define a Z_2 operator $U_N = \prod_{j=1}^N \sigma_j^x$. It can be easily checked that $U_N^2 = 1$ and

$[H, U_N] = 0$. Therefore, the present model possesses a global Z_2 invariance, indicating the double degeneracy of the eigenstates.

The integrability of the present model is associated with the following Lax operator

$$L_{0j}(\lambda) = \begin{pmatrix} \sinh\left[\tilde{\lambda}_j + \frac{\eta}{2}(1 + \sigma_j^z)\right] & \sinh\eta\sigma_j^- \\ \sinh\eta\sigma_j^+ & \sinh\left[\tilde{\lambda}_j + \frac{\eta}{2}(1 - \sigma_j^z)\right] \end{pmatrix}$$

and the monodromy matrix

$$T_0(\lambda) = L_{01}(\lambda)\cdots L_{0N}(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix},$$

where $\tilde{\lambda}_j = \lambda - \theta_j$, λ is the spectral parameter, θ_j are the site inhomogeneous constants, η is the crossing parameter, as usual, the index 0 indicates the auxiliary space, and j indicates the quantum space. Both the Lax operator and the monodromy matrix satisfy the Yang-Baxter relation

$$R_{12}(\lambda_1 - \lambda_2)L_{1j}(\lambda_1)L_{2j}(\lambda_2) = L_{2j}(\lambda_2)L_{1j}(\lambda_1)R_{12}(\lambda_1 - \lambda_2),$$

$$R_{12}(\lambda_1 - \lambda_2)T_1(\lambda_1)T_2(\lambda_2) = T_2(\lambda_2)T_1(\lambda_1)R_{12}(\lambda_1 - \lambda_2),$$

$$\tag{2}$$

with $R_{12}(\lambda) = L_{12}(\lambda|\theta_j = 0)$. The transfer matrix of the system is defined as

$$\tau(\lambda) = \mathrm{tr}_0\sigma_0^x T_0(\lambda) = B(\lambda) + C(\lambda), \tag{3}$$

where tr_0 means tracing the auxiliary space. From Eq. (2), one can prove that the transfer matrices with different spectral parameters are mutually commutative, i.e., $[\tau(\lambda), \tau(\mu)] = 0$. Therefore, $\tau(\lambda)$ serves as the generating functional of the conserved quantities of the corresponding system. The first order derivative of logarithm of the transfer matrix gives the Hamiltonian (1)

$$H = -2\sinh\eta\frac{\partial\ln\tau(\lambda)}{\partial\lambda}\bigg|_{\lambda=0,\theta_j=0} + N\cosh\eta. \tag{4}$$

Define the state $|0\rangle = \otimes|\uparrow\rangle_j$. From the definition of the Lax operator, we obtain

$$C(\lambda)|0\rangle = 0, \quad A(\lambda)|0\rangle = a(\lambda)|0\rangle, \quad D(\lambda)|0\rangle = d(\lambda)|0\rangle, \tag{5}$$

where $a(\lambda) = \prod_{n=1}^N \sinh(\lambda - \theta_n + \eta)$ and $d(\lambda) = \prod_{n=1}^N \sinh(\lambda - \theta_n)$. Before going further, we introduce the following useful formula [6]:

$$C(\lambda)\prod_{l=1}^n B(\mu_l)|0\rangle = \sum_{l=1}^n M_n^l(\lambda, \{\mu_j\})B_{n-1}^l|0\rangle$$
$$+ \sum_{k>l}\tilde{M}_n^{kl}(\lambda, \{\mu_j\})B_{n-1}^{kl}|0\rangle, \tag{6}$$

which can be obtained from the commutation relations derived from the Yang-Baxter relation (2), where

$$B_{n-1}^l = \prod_{j\neq l} B(\mu_j), \qquad B_{n-1}^{kl} = B(\lambda)\prod_{j\neq k,l} B(\mu_j),$$

and

$$M_n^l(\lambda, \{\mu_j\}) = g(\lambda, \mu_l)a(\lambda)d(\mu_l)\prod_{j\neq l}f(\lambda, \mu_j)f(\mu_j, \mu_l)$$
$$+ g(\mu_l, \lambda)a(\mu_l)d(\lambda)\prod_{j\neq l}f(\mu_j, \lambda)f(\mu_l, \mu_j), \tag{7}$$

$$\tilde{M}_n^{kl}(\lambda, \{\mu_j\}) = g(\lambda, \mu_k)g(\mu_l, \lambda)f(\mu_l, \mu_k)a(\mu_l)d(\mu_k)$$
$$\times \prod_{j\neq k,l}f(\mu_j, \mu_k)f(\mu_l, \mu_j) + g(\lambda, \mu_l)$$
$$\times g(\mu_k, \lambda)f(\mu_k, \mu_l)a(\mu_k)d(\mu_l)$$
$$\times \prod_{j\neq k,l}f(\mu_j, \mu_l)f(\mu_k, \mu_j),$$

$$g(\lambda, \mu) = \frac{\sinh\eta}{\sinh(\mu - \lambda)}, \quad f(\lambda, \mu) = \frac{\sinh(\lambda - \mu - \eta)}{\sinh(\lambda - \mu)}. \tag{8}$$

We adopt the procedure introduced in Ref. [10]. Suppose $|\Psi\rangle$ is an eigenstate of $\tau(\lambda)$ and independent of λ. We have $\tau(\lambda)|\Psi\rangle = \Lambda(\lambda)|\Psi\rangle$. In addition, we define $F_n(\{\mu_j\}) = \langle\Psi|\prod_{j=1}^n B(\mu_j)|0\rangle$ and put $F_0 = \langle\Psi|0\rangle = 1$. Consider the quantity $\langle\Psi|\tau(\lambda)\prod_{j=1}^n B(\mu_j)|0\rangle$. By acting $\tau(\lambda)$ right and left alternatively, we have the following functional relations:

$$\Lambda(\lambda)F_n = \sum_l M_n^l(\lambda)F_{n-1}^l + \sum_{k>l}\tilde{M}_n^{kl}(\lambda)F_{n-1}^{kl} + F_{n+1},$$

$$F_1(\lambda) = \Lambda(\lambda), \qquad F_{N+1} \equiv 0, \tag{9}$$

where $F_n = F_n(\{\mu_j\})$, $F_{n-1}^l = F_{n-1}(\{\mu_j\}_{j\neq l})$, $F_{n-1}^{kl} = F_{n-1}(\lambda, \{\mu_j\}_{j\neq k,l})$, and $\{\mu_j\}$, indicating the parameter set $\{\mu_1, \ldots, \mu_n\}$ for $n = 1, \ldots, N$. Notice that we have $N + 2$ equations and $N + 2$ unknown functions Λ and F_n. The function $F_n(\{\mu_j\})$ is symmetric by exchanging the variables μ_j because of $[B(\mu_j), B(\mu_l)] = 0$ and is a degree $N - 1$ trigonometrical polynomial. The eigenvalue $\Lambda(\lambda)$ therefore can be parametrized as

$$\Lambda(\lambda) = \Lambda_0\prod_{j=1}^{N-1}e^{z_j}\sinh(\lambda - z_j), \tag{10}$$

where Λ_0 is a constant and $\{z_1, \ldots, z_{N-1}\}$ is a set of roots of $\Lambda(\lambda)$ with $\Lambda(z_j) = 0$. The recursion equations (9) determine the eigenvalue $\Lambda(\lambda)$. From Eq. (4), we can easily derive the eigenvalue of the Hamiltonian as

$$E = -2\sinh\eta\frac{\partial\ln\Lambda(\lambda)}{\partial\lambda}\bigg|_{\lambda=0,\theta_j=0} + N\cosh\eta$$

$$= -2\sinh\eta\sum_{j=1}^{N-1}\coth z_j + N\cosh\eta. \tag{11}$$

Since $d(\theta_j) = 0$, all the functions M_n^j and $\tilde{M}_n^{jk}$ are zero as long as their variables belong to the parameter set $\{\theta_1, \ldots, \theta_N\}$ and $\theta_j \neq \theta_k \neq \theta_l \pm \eta$. Therefore, the following relations hold:

$$F_n(\theta_1, \ldots, \theta_n) = \prod_{j=1}^{n} \Lambda(\theta_j). \qquad (12)$$

From the $n = N$ case of Eq. (9), we obtain

$$\Lambda(\lambda) = \sum_{j=1}^{N} \frac{a(\theta_j)d(\lambda)}{\Lambda(\theta_j)} g(\theta_j, \lambda) \prod_{l \neq j}^{N} f(\theta_j, \theta_l) f(\theta_l, \lambda)$$

$$= -\sum_{j=1}^{N} \frac{a(\theta_j)d(\theta_j - \eta)}{\Lambda(\theta_j)d_j(\theta_j)\sinh(\lambda - \theta_j + \eta)} a(\lambda), \qquad (13)$$

with $d_j(\theta_j) = \prod_{l \neq j}^{N} \sinh(\theta_j - \theta_l)$. This equation gives the closed recursive solution of $\Lambda(\lambda)$. Putting $\lambda \to \theta_j - \eta$, we readily have

$$\Lambda(\theta_j)\Lambda(\theta_j - \eta) = \Delta_q(\theta_j), \qquad j = 1, \ldots, N, \qquad (14)$$

where $\Delta_q(\theta_j) = -a(\theta_j)d(\theta_j - \eta)$ is the quantum determinant [6]. Similar relations were also derived in Refs. [13,14,18] with the separation of variables method. The above equations determine the $N - 1$ roots $\{z_j\}$ and Λ_0 in Eq. (10). In fact, the operator identity $B(\theta_j)B(\theta_j - \eta) = 0$ can be demonstrated with the definition of the monodromy matrix. With this operator identity and considering the quantity $\langle \Psi | \tau(\theta_j)\tau(\theta_j - \eta) | 0 \rangle$, one can easily deduce Eq. (14). Taking the limit of Eq. (14) with $\theta_j \to 0$ leads to the following equations which completely determine the spectrum $\Lambda(\lambda)$ of the homogeneous model

$$\frac{\partial^l}{\partial u^l} \ln[-\sinh^N(u + \eta)\sinh^N(u - \eta)]|_{u=0}$$

$$= \frac{\partial^l}{\partial u^l} \ln[\Lambda(u)\Lambda(u - \eta)]|_{u=0}, \qquad l = 0, \ldots, N - 1. \qquad (15)$$

However, these relations are quite hard to be used to study the physical properties, especially in the thermodynamic limit. Thus, a proper set of BAEs in the usual form is still crucial. As $\Lambda(\lambda)$ is a trigonometrical polynomial of degree $N - 1$ with the very periodicity $\Lambda(\lambda + i\pi) = (-1)^{N-1}\Lambda(\lambda)$, we conjecture the following modified $T - Q$ relation [2,3]:

$$\Lambda(\lambda) = e^{\lambda} a(\lambda) \frac{Q_1(\lambda - \eta)}{Q_2(\lambda)} - e^{-\lambda - \eta} d(\lambda) \frac{Q_2(\lambda + \eta)}{Q_1(\lambda)}$$

$$- b(\lambda) \frac{a(\lambda)d(\lambda)}{Q_1(\lambda)Q_2(\lambda)}, \qquad (16)$$

where

$$Q_1(\lambda) = \prod_{j=1}^{M} \sinh(\lambda - \mu_j), \qquad Q_2(\lambda) = \prod_{j=1}^{M} \sinh(\lambda - \nu_j), \qquad (17)$$

and $b(\lambda)$ is an adjust function. For N even, $M = N/2$,

$$b(\lambda) = e^{i\phi_1 + \lambda} - e^{i\phi_2 - \lambda - \eta}, \qquad (18)$$

with

$$i\phi_1 = \sum_{j=1}^{N} \theta_j - M\eta - 2\sum_{j=1}^{M} \mu_j,$$

$$-i\phi_2 = \sum_{j=1}^{N} \theta_j - M\eta - 2\sum_{j=1}^{M} \nu_j \qquad (19)$$

to cancel the leading terms in Eq. (16) when $\lambda \to \pm\infty$. Obviously, the conjectured $\Lambda(\lambda)$ satisfies Eq. (14) automatically. The BAEs determined by the regularity of $\Lambda(\lambda)$ [which ensures $\Lambda(\lambda)$ to be a trigonometrical polynomial of degree $N - 1$] read

$$d(\nu_j) = \frac{e^{\nu_j}}{b(\nu_j)} Q_1(\nu_j - \eta)Q_1(\nu_j),$$

$$a(\mu_j) = -\frac{e^{-\mu_j - \eta}}{b(\mu_j)} Q_2(\mu_j + \eta)Q_2(\mu_j), \qquad j = 1, \ldots, \frac{N}{2}. \qquad (20)$$

The BAEs for the homogeneous model are exactly the above equations by putting all $\theta_j = 0$. The eigenvalues of Hamiltonian (1) take the following form

$$E(\{\mu_j, \nu_j\}) = 2\sinh\eta \sum_{j=1}^{M} \left\{ \frac{\cosh(\mu_j + \eta)}{\sinh(\mu_j + \eta)} - \frac{\cosh(\nu_j)}{\sinh(\nu_j)} \right\}$$

$$+ N\cosh\eta - 2\sinh\eta. \qquad (21)$$

For odd N, we put $M = (N + 1)/2$ and

$$b(\lambda) = \frac{1}{2}[e^{i\phi_1 + 2\lambda} + e^{i\phi_2 - 2\lambda - 2\eta}], \qquad (22)$$

where ϕ_1 and ϕ_2 take the same form as Eq. (19) with $M = (N + 1)/2$ and $\theta_j = 0$. In this case, the BAEs and the eigenvalue of the Hamiltonian are still given by Eqs. (20) and (21), respectively. The nested nature of the BAEs is due to the topological boundary and broken $U(1)$ symmetry.

Generally, the Bethe roots distribute in the whole complex plane with the selection rules $\mu_j \neq \mu_l$, $\mu_j \neq \nu_l$, and $\mu_j \neq \nu_l - \eta$ which ensure the simplicity of "poles" in our $T - Q$ ansatz. Numerical solutions of the BAEs for small size (up to $N = 6$) with a random choice of η indicate that the BAEs indeed give the complete solutions of the model (namely, the eigenvalues calculated from the BAEs coincide exactly to those obtained from exact diagonalization). Numerical results for $N = 3$ and $\eta = \ln 2$ are shown in Table I. For imaginary η, the numerical simulations for $N = 8, 10$ indicate that the distribution of the Bethe roots in the ground state is almost on a straight line $\text{Im}\{\mu_j\} \sim \text{Im}\{\nu_j\} \sim (i\pi - \eta/2) \equiv -(\bar{\eta}/2)$ and $\{\text{Re}\mu_j\} \sim \{-\text{Re}\nu_j\}$. This strongly suggests that in the thermodynamic limit $N \to \infty$, the BAEs for the ground state can be rewritten as [Eq. (20) over its complex conjugate]

TABLE I. Numerical solutions of the BAEs for $\eta = \ln 2$, $N = 3$, and $M = 2$. E_0 is the eigenenergy, and elv indicates the number of the energy levels. The eigenvalues are exactly the same as those of the exact diagonalization.

μ_1	μ_2	ν_1	ν_2	E_0	elv
$-0.822\,76 - 0.500\,00i\pi$	$0.646\,39 - 0.500\,00i\pi$	$-1.339\,54 - 0.500\,00i\pi$	$0.129\,62 - 0.500\,00i\pi$	$-3.022\,00$	1
$-0.484\,93 - 0.260\,54i\pi$	$-0.484\,93 + 0.260\,54i\pi$	$-0.208\,21 - 0.260\,54i\pi$	$-0.208\,21 + 0.260\,54i\pi$	$-3.022\,00$	1
$-1.108\,39 + 0.195\,25i\pi$	$-0.212\,37 + 0.002\,07i\pi$	$-0.480\,78 + 0.002\,07i\pi$	$0.415\,24 + 0.195\,25i\pi$	$-1.250\,00$	2
$-0.247\,63 - 0.201\,57i\pi$	$-0.231\,24 - 0.495\,75i\pi$	$-0.461\,90 - 0.495\,75i\pi$	$-0.445\,51 - 0.201\,57i\pi$	$-1.250\,00$	2
$-0.247\,63 + 0.201\,57i\pi$	$-0.231\,24 + 0.495\,75i\pi$	$-0.461\,90 + 0.495\,75i\pi$	$-0.445\,51 + 0.201\,57i\pi$	$-1.250\,00$	2
$-1.108\,39 - 0.195\,25i\pi$	$-0.212\,37 - 0.002\,07i\pi$	$-0.480\,78 - 0.002\,07i\pi$	$0.415\,24 + 0.195\,25i\pi$	$-1.250\,00$	2
$-0.355\,06 + 0.000\,00i\pi$	$-0.115\,41 + 0.500\,00i\pi$	$-0.577\,73 - 0.500\,00i\pi$	$-0.338\,09 + 0.000\,00i\pi$	$5.522\,00$	3
$-0.444\,82 - 0.091\,94i\pi$	$-0.444\,82 + 0.091\,94i\pi$	$-0.248\,33 - 0.091\,94i\pi$	$-0.248\,33 + 0.091\,94i\pi$	$5.522\,00$	3

$$\frac{\sinh^N\left(\bar{\mu}_j - \frac{\bar{\eta}}{2}\right)}{\sinh^N\left(\bar{\mu}_j + \frac{\bar{\eta}}{2}\right)} = e^{i\chi_j} \prod_{l=1}^{M} \frac{\sinh(\bar{\mu}_j + \bar{\mu}_l - \bar{\eta})}{\sinh(\bar{\mu}_j + \bar{\mu}_l + \bar{\eta})}, \quad (23)$$

where $\bar{\mu}_j = \text{Re}(\mu_j)$ and χ_j accounts for the small deviation of μ_j from $\bar{\mu}_j - (\bar{\eta}/2)$. Based on the above equation, the energy can be derived with the ordinary method [6]. $\{\chi_j\}$ contribute a finite boundary energy and N^{-1} small correction to the excitations. To get the excitations, we need to either exchange two modes $\bar{\mu}_j$ and $\bar{\nu}_l = -\bar{\mu}_j$ or put them onto the complex plane because of the selection rules, indicating the topological nature of the system.

To show the physical effect of the topological boundary clearly, let us focus on the $\eta = i\pi/2$ case, i.e., the topological XX spin ring or equivalently the topological free fermion ring (via a Jordan-Wigner transformation). This model is tightly related to the problem of a Josephson junction embedded in a Luttinger liquid (see, for example, Refs. [22–24]). We note the BAEs about the roots $\{z_j\}$ can also be derived from the recursive equations [Eq. (9)]. From the $n = N$ case of Eq. (9), we obtain that

$$F_N(z, \{\mu_l\}) = \frac{1}{\Lambda_0} \sum_{j=1,z}^{N-1} M_N^{\prime j} F_{N-1}, \quad (24)$$

where $\mu_1, \ldots, \mu_{N-1} = \{\mu_l\}$ are free parameters, z is one of the roots of the eigenvalue $\Lambda(\lambda)$, and $M_N^{\prime j} = \lim_{\mu \to \infty} e^{-(N-1)\mu} M_N^j(\mu, z, \{\mu_l\})$. Meanwhile, $\Lambda(z)$ acting on $F_{N-1}(\{\mu_l\})$ gives

$$F_N(z, \{\mu_l\}) = -\sum_{j=1}^{N-1} M_{N-1}^j(z, \{\mu_l\}) F_{N-2}^j$$
$$- \tilde{M}_{N-1}^{jk}(z, \{\mu_l\}) F_{N-2}^{jk}(z, \{\mu_l\}). \quad (25)$$

From Eqs. (24) and (25), we have

$$F_{N-2}^j = -\frac{1}{M_{N-1}^j(z, \mu_j)} \Bigg[\sum_l M_N^{\prime l} F_{N-1} + \sum_{l \neq j} M_{N-1}^l F_{N-2}^l$$
$$- \sum_{k>l} \tilde{M}_{N-1}^{kl} F_{N-2}^{kl} \Bigg]. \quad (26)$$

The singular point of Eq. (26) is $\mu_j = z \pm i\pi/2$. Since F_{N-1} is a polynomial for all the variables μ_j, the residue of

the right-hand side of Eq. (26) must be zero at the singular point. Notice the fact that $M_n^j(z, \{\mu_j\}) = 0$ if $\mu_j \neq z \pm i\pi/2$ and $\tilde{M}_n^{kl}(z, \{\mu_j\}) = 0$ as long as μ_k or $\mu_l = z \pm i\pi/2$. We readily have the conclusion that $F_{N-1}(z, \{\mu_l\})$ is proportional to $F_2(z, z \pm i\pi/2)$ when one of the μ_l's is equal to $z \pm i\pi/2$, which must be zero according to the above analysis. This gives the constraint condition of the root z as $a(z)d(z + i\pi/2) = d(z)a(z + i\pi/2)$. Therefore, the roots z_n of $\Lambda(\lambda)$ satisfy the following Bethe ansatz equation:

$$\coth^{2N}(z_n) = 1, \qquad z_j \neq z_k \pm \frac{\pi}{2}i. \quad (27)$$

Equivalently, we have

$$\coth(z_n) = e^{i\pi n/N} \equiv e^{ik_n}, \quad n = \pm 1, \ldots, \pm(N-1). \quad (28)$$

The $N - 1$ pair solutions $\{z_j, z_j + (\pi/2)i\}$ mod $(i\pi)$ are located on two lines with an imaginary part $\pm i\pi/4$. The root sets are formed by choosing one and only one in each pair. This selection rule comes from that the poles of the right-hand side of Eq. (26) do not enter into the set of roots $\{z_j\}$ because the poles and the zeros satisfy the same equation (27). Therefore, there are 2^{N-1} possible choices to form a solution of $\Lambda(\lambda)$. With the Z_2 symmetry of the system, we demonstrate that the solutions are complete.
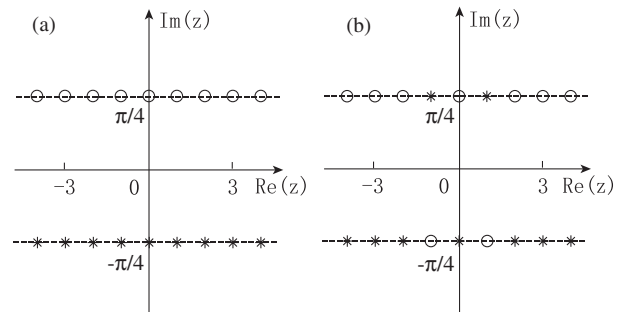

FIG. 1. (a) Schematic diagram of the ground state of the XX topological spin ring. The states in the lower solution line are all filled, and the upper solution line is unoccupied. (b) The elementary excitation of the XX topological spin ring. The "particle" in the upper solution line must correspond to a "hole" in the lower solution line with exactly the same real part.
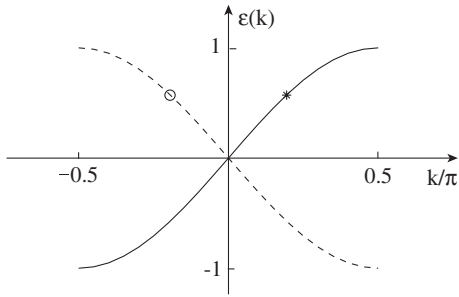
FIG. 2. Schematic diagram of the excitation spectrum of the *XX* topological spin ring. The ''particle'' and ''hole'' carry exactly the same energy and opposite momenta.

The ground state is formed by filling all roots along the $-i\pi/4$ line [as shown in Fig. 1(a)], and the ground state energy reads $E_g = -2\cot(\pi/2N)$ which is slightly different from that of the periodic boundary condition case. The elementary excitations of the system can be constructed by digging some holes in the lower solution line and putting the same number of particles in the upper solution line [as shown in Fig. 1(b)]. However, the positions of the holes and the particles are not arbitrary but obey the selection rules of $z_j \neq z_k \pm i\pi/2$. That means if there is a hole at $-k$, there must be a particle at k (as shown in Fig. 2). The energy of a particle-hole excitation is thus $\epsilon(k) = 4\sin|k|$. Such an excitation character is quite unlike that in the usual Luttinger liquids, where both the forward scattering and backward scattering are allowed and there is no constraint for the particle-hole excitations besides the Pauli principle in the charge neutral sector. In the present topological boundary case, each particle with momentum k must lock a hole with momentum $-k$ to form a virtual bound state, indicating the topological nature of the excitations.

In conclusion, we developed a general method for diagonalizing the integrable models without $U(1)$ symmetry. As an example, we constructed the exact solution of the *XXZ* spin ring with a topological boundary condition. We remark that the present method could be used to other integrable models without $U(1)$ symmetry. For those models, some off-diagonal elements of the monodromy matrix enter into the transfer matrix expression $\tau(\lambda)$. With the commutation relations derived from the corresponding Yang-Baxter equation, a similar relation of $\Lambda(\theta_j)\Lambda(\theta_j - \eta) \sim \Delta_q(\theta_j)$ can be obtained from some operator identities, with which a modified $T - Q$ relation as well as the usual BAEs can be constructed. Details will be given elsewhere.

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