Off-shell pion-nucleon scattering and dispersion relations

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S U M M A R Y

The problem of pion-nucleon scattering when one of the pions is virtual is discussed. By assuming that one dimensional dispersion relations in the energy variable can be written for suitably defined invariant amplitudes, and that also in the off-shell case the dispersive integrals are dominated by the T=J=3/2 amplitude, an Omnès-type integral equation is deduced for this amplitude. It is shown that the solution is particularly simple in the static limit and for squared four-momenta of the virtual pion less than $\sim$ 8 squared pion masses.

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I. Introduction

The analytic continuation of a scattering amplitude, when one of the incoming particles becomes virtual, has a certain interest in connection with the description of inelastic processes in terms of a single particle exchange.

It has been shown by Da Prato [1] that the most naïve formulation of this interaction picture which has been applied so far [3] works satisfactorily well in the process of single pion production in proton-proton collisions at 2.85 GeV [4]. At this energy the experimentally observed low-and high-energy peaks are theoretically reproduced: these peaks occur for low values of the squared four momentum Δ^2 of the virtual intermediate particle [5]. For higher values of Δ^2 the agreement with experiment becomes worse: in this region, however, the theoretical calculations cannot be trusted because the approximations involved are no longer justified. We do not know whether the discrepancies between the predictions of the peripheral model and the experiment at high Δ^2 are mainly caused by the physical occurrence of other mechanisms of interaction (many particles exchange) or are the consequence of the assumption that the off-shell 3.3 amplitude still dominates the off-shell scattering and is equal to its on-shell value. The hypothesis that the low-energy off-shell scattering is still dominated from the 3.3 resonance can be phenomenologically justified from the fact that in the quoted experiment at 2.85 GeV [4], there is evidence at the same time of a very important one-pion exchange contribution and of the formation of the 3.3 isobar. The purpose of this paper is to investigate the analytic continuation of the 3.3 partial wave to unphysical values of the mass of the incoming pion. By assuming one-dimensional dispersion relations for "off-shell" invariant amplitudes suitably chosen, we shall derive a linear integral equation for the "off-shell" partial amplitude of the $T=J=3/2$ state by closely following the procedure by Chew, Goldberger, Low and Nambu [6]. This equation can be reduced to an equation of the type already discussed and solved by Omnès [7]. Its solution will give us the correct energy and Δ^2 - dependence of the off-shell 3.3 amplitude. In Section II we discuss the so-called pole approximation, which has been used so far in the picture of the production processes through the exchange of a single particle, and we point out the assumptions which are implied in it. Section III contains the definition of the invariant and partial wave amplitudes for the off-shell scattering as well as certain relations similar to the unitarity condition in the on-shell case. In Section IV the dispersion relations for the off-shell amplitudes are written down and an equation is derived for the T=J=3/2 off-shell amplitude, by using the same procedure as in ref. [6]. Finally, in Section V it is shown that the "unitarity" condition allows this equation to be transformed into a linear integral equation of the Omnès type which in principle can be solved exactly. The solution is particularly simple in the static limit and for $\Delta^2 \lesssim 8\mu^2$ (μ = pion mass).

II. The one-particle exchange interaction picture

We briefly recall in this section the procedure which has been used so far [1), 2), 3)] in the calculation of single-particle exchange effects. If we consider in general the process

$$p_1+p_2 \rightarrow q_1+q_2+ \dots q_n \qquad (1)$$

we can define an invariant M matrix element from the S matrix element through

$$S_{fi} = \delta_{fi} - i(2\pi)^{4-\frac{3}{2}(n+2)_2 - b/2} \left[\prod_{i=1}^{f} m_i\right]^{\frac{1}{2}} \left[p_{1o}p_{2o}q_{1o} \dots q_{no}\right]^{-\frac{1}{2}} \cdot \qquad (2)$$

$$\cdot\ \delta^4(q_1+ \dots +q_n-p_1-p_2)M_{fi}$$

where $b(f)$ is the total number of bosons (fermions) present in the initial and final states, $p_{1o} \ldots q_{no}$ are the energy components of the 4-vectors $p_1 \ldots q_n$ respectively. For the process of single pion production in nucleon-nucleon collisions

$$p_1+p_2 \rightarrow q_1+q_2+q \tag{3}$$

(p_i and q_i nucleons, q pion) supposed to occur through the exchange of a single pion of 4-momentum $k = q_2-p_2$ (Fig. 1),

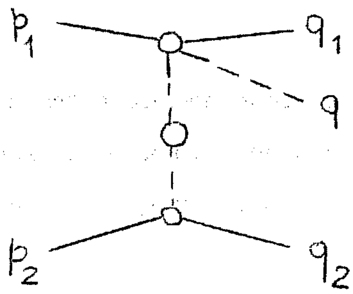

Fig. 1

we have

$$M_{fi} = \frac{\bar{u}(q_2)G_r\gamma_5 u(p_2)}{\Delta^2+\mu^2} K(\Delta^2)K'(\Delta^2)M_{\pi N}(\omega^2,t^2,\Delta^2) \tag{4}$$

In Eq. (4) $\Delta^2=k^2=(q_2-p_2)^2$, $\omega^2 = -(q_1+q)^2$, $t^2 = (q_1-p_1)^2$; G_r is the renormalized and rationalized pion-nucleon coupling constant: $K(\Delta^2)$ is the "pionic form factor" of the nucleon, defined by

$$\langle q_2 | j_\alpha(0) | p_2 \rangle = (2\pi)^{-3}\left[\frac{m^2}{p_{2o}q_{2o}}\right]^{\frac{1}{2}} \bar{u}(q_2)G_r\gamma_5 \tau_\alpha u(p_2)K(\Delta^2) \tag{5}$$

(where $j_\alpha(0)$ is the current of the pion field and τ_α are the isotopic spin matrices for the nucleons); $K'(\Delta^2)(\Delta^2+\mu^2)^{-1}$ is the complete pion propagator; $M_{\pi N}$ is defined through

$$\langle q_1, q\beta | j_\alpha(0) | p_1 \rangle = (2\pi)^{-9/2}\left[\frac{m^2}{2q_o p_{1o} q_{1o}}\right]^{\frac{1}{2}} M_{\pi N}^{\alpha\beta}(\omega^2, t^2, \Delta^2) \qquad (6)$$

(here we include explicitly the isotopic spin indices α, β). The latter quantity for $\Delta^2 = -\mu^2$ reduces to the M matrix element for the physical πN scattering.

$$k+p_1 \rightarrow q+q_1 \qquad (7)$$

at total c.m. energy ω and nucleon momentum transfer t^2. At the "pole" value, $\Delta^2 = -\mu^2$, which corresponds to a real exchanged pion, we have $K(-\mu^2) = K'(-\mu^2) = 1$. Therefore, near the pole everything in (4) is known, and we get

$$M_{fi}^{(pole)} = \frac{\bar{u}(q_2)\, G_r \gamma_5 u(p_2)}{\Delta^2+\mu^2} M_{\pi N}(\omega^2, t^2) \qquad (8)$$

We might expect that the use of Eq. (8) in the physical region for small $\Delta^2 (\Delta^2 \simeq \mu^2)$ could give a good approximation, since the pole is not far and the Δ^2-dependence of all the off-shell quantities is supposed to be smooth. This approximation (pole approximation) would be in fact reasonable if $M_{\pi N}$ contained only an s wave interaction. For higher order waves, however, the situation is not free from ambiguities.

In fact the t^2 dependence of $M_{\pi N}$ occurs only through the scattering angle $\epsilon = \widehat{\vec{q}_1 \vec{p}_1}$ in the c.m. system of q_1 and q. This angle is defined by the relation (m = nucleon mass)

$$\cos \epsilon = \frac{2p_{1o}q_{1o}-t^2-2m^2}{2\,|\vec{p}_1|\,|\vec{q}_1|} \tag{9}$$

and turns out to be Δ^2-dependent, since p_{1o} and $|\vec{p}_1|$ depend on Δ^2. Its "on-shell" value is given by performing the limit $\Delta^2 \to -\mu^2$

$$(\cos\epsilon)_{on} = 1- \frac{t^2}{2\,|\vec{q}_1|^2} \tag{10}$$

Now, if one considers the allowed phase space of ω^2 and t^2 for fixed Δ^2, one finds that, also for Δ^2 small, that is in the region where the pole approximation should be expected to hold, expression (9) is always limited between -1 and +1, while expression (10) reaches values considerably outside this interval, and diverges at threshold. We then see that one cannot keep the whole element $M_{\pi N}(\omega^2, t^2\Delta^2)$ on-shell, since the angular part contained in it depends very strongly on Δ^2, also for small Δ^2. The most reasonable application of the "pole approximation" consists in expliciting the angular dependence of $M_{\pi N}$, using for $\cos\epsilon$ the Δ^2-dependent expression (9) and treating the partial wave amplitudes as Δ^2-independent. This approximation has been used in all the calculation used so far [3], also if its meaning has not yet been fully discussed.

For the partial wave amplitudes at low Δ^2 the approximation used seems rather reasonable. In any case, we think it useful to investigate their Δ^2- dependence as carefully as possible, as a way either of getting some insight on the consistency of the approximations used at low Δ^2, or of approaching the problem of the high Δ^2 behaviour, where further difficulties arise at least from the presence of the form factors. This investigation will be carried out in the following sections.

III. The off-shell amplitude

The off-shell amplitude $M_{\pi N}^{\alpha\beta}(\omega^2, t^2, \Delta^2)$ defined in Sect. 2 can be written by invariance requirements as

$$M_{\pi N}^{\alpha\beta}(\omega^2, t^2, \Delta^2) = \bar{u}(p_1)\left[-A_{\alpha\beta} + i\underline{q}B_{\alpha\beta}\right]u(q_1) \tag{11}$$

where $A_{\alpha\beta}$ and $B_{\alpha\beta}$ depend on t^2, ω^2 and Δ^2 and can be written

$$A_{\alpha\beta} = \delta_{\alpha\beta} A^{(+)} + \tfrac{1}{2}\left[\tau_\beta, \tau_\alpha\right] A^{(-)} \tag{12}$$

$$B_{\alpha\beta} = \delta_{\alpha\beta} B^{(+)} + \tfrac{1}{2}\left[\tau_\beta, \tau_\alpha\right] B^{(-)} \tag{13}$$

$A^{(\pm)}$ are related to the analogous off-shell amplitudes for fixed total isotopic spin through the relations

$$\begin{aligned} A^{(+)} &= \tfrac{1}{3}\left(A^{(\frac{1}{2})} + 2A^{(3/2)}\right) \\ A^{(-)} &= \tfrac{1}{3}\left(A^{(\frac{1}{2})} - A^{(3/2)}\right) \end{aligned} \tag{14}$$

and similarly for $B^{(\pm)}$.

By introducing two-dimensional spinors χ we get

$$M_{\pi N}^{(\pm)} = 4\pi\omega\, m^{-1}\chi_f^\dagger\left[f_1^{(\pm)} + \frac{(\vec{\sigma}\,\vec{q}_1)(\vec{\sigma}\,\vec{p}_1)}{|\vec{q}_1|\,|\vec{p}_1|}\, f_2^{(\pm)}\right]\chi_i \tag{15}$$

Here $\vec{q}_1$, $\vec{p}_1$ are the 3-momenta of the nucleons calculated in the c.m. system of q_1 and q, and $\sigma_i (i=1, 2, 3)$ are the Pauli matrices. We have

$$f_1^{(\pm)} = \frac{1}{4\pi} \frac{R_1}{2\omega} \left[A^{(\pm)} + (\omega - m) B^{(\pm)} \right] \tag{16}$$

$$f_2^{(\pm)} = \frac{1}{4\pi} \frac{R_2}{2\omega} \left[-A^{(\pm)} + (\omega + m) B^{(\pm)} \right] \tag{17}$$

with

$$\begin{aligned} R_1 &= \left[(p_{1o}+m)(q_{1o}+m) \right]^{\frac{1}{2}} ; \\ R_2 &= \left[(p_{1o}-m)(q_{1o}-m) \right]^{\frac{1}{2}} \end{aligned} \tag{18}$$

Next we expand f_1 and f_2 in terms of off-shell scattering amplitudes with given parity and angular momentum. This expansion gives [6)]

$$f_1^{(\pm)} = \sum_{\ell=0}^{\infty} f_{\ell+}^{(\pm)}(\omega, \Delta^2) P'_{\ell+1}(\cos \epsilon) - \sum_{\ell=2}^{\infty} f_{\ell-}^{(\pm)}(\omega, \Delta^2) P'_{\ell-1}(\cos \epsilon) \tag{19}$$

$$f_2^{(\pm)} = \sum_{\ell=0}^{\infty} \left[f_{\ell-}^{(\pm)}(\omega, \Delta^2) - f_{\ell+}^{(\pm)}(\omega, \Delta^2) \right] P'_{\ell}(\cos \epsilon) \tag{20}$$

where $\cos \epsilon$ is the scattering angle defined by (6).

It is obvious that all quantities defined by (11) to (19) reduce to the corresponding physical quantities defined in ref. [6)] as $\Delta^2 = -\mu^2$.

Let us consider now the "unitarity" condition, which can be derived from general principles when off-shell states are implied. For the case of matrix element (4) it can be shown [8] that the relation obtained is similar to the one valid on-shell. In particular, for the partial amplitude $f_{\ell\pm}(\omega,\Delta^2)$ one gets the following relation

$$f^*_{\ell\pm}(\omega,\Delta^2) = e^{-2i\delta_{\ell\pm}(\omega)} f_{\ell\pm}(\omega,\Delta^2) \qquad (21)$$

where $\delta_{\ell\pm}(\omega)$ is the physical phase shift for π N scattering in the channel of angular momentum ℓ and total angular momentum $\ell \pm \frac{1}{2}$. By introducing the physical transition amplitude $f_{\ell\pm}(\omega) = \frac{1}{q} e^{i\delta_{\ell\pm}(\omega)}$, (21) transforms into

$$\mathrm{Im} f_{\ell\pm}(\omega,\Delta^2) = q f^*_{\ell\pm}(\omega) f_{\ell\pm}(\omega,\Delta^2) \qquad (22)$$

This relation will be used in the following developments.

IV. Dispersion relations

We start with the assumption that the invariant amplitudes $A^{(\pm)}$ and $B^{(\pm)}$ obey dispersion relations of the form [9]

$$\mathrm{Re}A^{(\pm)}(\nu,t^2,\Delta^2) = \frac{\rho}{\pi}\int_{\nu_{min}}^{\infty} \mathrm{Im}A^{(\pm)}(\nu',t^2,\Delta^2)\left(\frac{1}{\nu'-\nu} \pm \frac{1}{\nu'+\nu}\right) d\nu'$$

$$(23)$$

$$\mathrm{Re}B^{(\pm)}(\nu,t^2,\Delta^2) = \frac{G_r^2}{2m} K(\Delta^2)\left(\frac{1}{\nu_B-\nu} \mp \frac{1}{\nu_B+\nu}\right) + \frac{\rho}{\pi}\int_{\nu_{min}}^{\infty} \mathrm{Im}B^{(\pm)}(\nu',t^2,\Delta^2) \cdot$$

$$\cdot \left(\frac{1}{\nu'-\nu} \mp \frac{1}{\nu'+\nu}\right) d\nu'$$

ν is the invariant

$$\nu = -\frac{1}{4m}(k+q)(p_1+q_1) \tag{24}$$

Furthermore

$$\nu_B = \frac{\Delta^2-\mu^2}{4m} - \frac{t^2}{4m} \tag{25}$$

$$\nu_{min} = \mu + \frac{\Delta^2+\mu^2}{4m} - \frac{t^2}{4m} \tag{26}$$

The dispersion relations written in the form (23) take already into account the crossing invariance, which is maintained when the incident pion k goes off-shell.

For the following development it is convenient to change the variable of integration. Let us define

$$\nu_L = \frac{\omega^2-m^2-\mu^2}{2m} \tag{27}$$

$$b = \frac{\Delta^2+\mu^2}{4m} - \frac{t^2}{4m} \tag{28}$$

$$\nu_o = -\frac{\mu^2}{2m} \tag{29}$$

we have

$$\begin{aligned} \nu &= \nu_L+b \\ \nu_B &= \nu_o+b \\ \nu_{min} &= \mu+b \end{aligned} \tag{30}$$

$$\mathrm{Ref}^{(\pm)}_{p3/2} = -\frac{p_1 q_1}{3\omega} R_1 \left\{ \pm \frac{G_r^2}{4\pi} \frac{\omega - m}{4m^2} \frac{K(\Delta^2)}{(\nu_o + \nu_L + \frac{\beta}{2m})^2} \left[1 + \frac{2(p_{1o}p_{2o} - m^2)}{m(\nu_o + \nu_L + \beta/2m)} \right] \right.$$

$$+ \frac{P}{\pi} \int_\mu^\infty d\nu_L' \frac{\mathrm{Imf}^{(\pm)}_{p3/2}(\nu_L', \Delta^2)}{p_1' q_1'} \left\{ -\frac{3}{2} \frac{\omega' + \omega}{R_1'} \frac{1}{\nu_L' - \nu_L} \pm \right.$$

$$\pm \left(-\frac{3}{2}\right) \frac{\omega' - \omega + 2m}{R_1'} \left[1 + \frac{2(p_{1o}p_{2o} - m^2)}{m(\nu_L' + \nu_L + \beta/2m)} \right] \frac{1}{\nu_L' + \nu_L + \beta/2m} \pm \qquad (37)$$

$$\pm \frac{1}{2m} \left[\frac{\omega' - \omega + 2m}{R_1'} \cdot 3(p_{1o}' p_{2o}' - m^2) + R_1'(\omega' + \omega - 2m) \right] \cdot$$

$$\left. \left. \cdot \left[1 + \frac{2(p_{1o}p_{2o} - m^2)}{m(\nu_L' + \nu_L + \beta/2m)} \right] \frac{1}{\nu_L' + \nu_L + \beta/2m} \right\} \right\}$$

We have put $\beta = \Delta^2 + \mu^2$, and made use of the identity $R_1' R_2' = p_1' q_1'$. This equation, in the limit $\beta \to 0$ (which implies $p_{1o}, p_1 \to q_{1o}, q_1$; $R_1 \to q_{1o} + m$) reduces to Eq. (3.32) of CGLN.

Since from (14) it can be deduced that

$$\mathrm{Ref}_{33} = \mathrm{Ref}^{(+)}_{p3/2} - \mathrm{Ref}^{(-)}_{p3/2} \qquad (38)$$

we form the combination (38) of Eqs. (37), use the relations

$$\mathrm{Im}f^{(+)}_{p3/2} \simeq \frac{2}{3}\,\mathrm{Im}f_{33}$$
$$\mathrm{Im}f^{(-)}_{p3/2} \simeq -\frac{1}{3}\,\mathrm{Im}f_{33} \qquad (39)$$

and finally get

$$\mathrm{Re}g_{33}(\omega,\Delta^2) = B + \frac{\mathrm{P}}{\pi}\int_{\mu}^{\infty} d\nu'_L\, \mathrm{Im}g_{33}(\omega',\Delta^2)\left[\frac{1}{\nu'_L-\nu_L}\,\frac{\omega'+\omega}{2\omega'} + \frac{\rho}{3}\,\frac{1}{\nu'_L+\nu_L+\beta/2m}\right] \qquad (40)$$

where

$$g_{33}(\omega,\Delta^2) = \frac{1}{p_1 q_1}\,\frac{2\omega}{R_1}\,f_{33}(\omega,\Delta^2) \qquad (41)$$

$$B(\omega,\Delta^2) = \frac{4}{3}f^2(\omega-m)\,\frac{K(\Delta^2)}{(\nu_o+\nu_L+\beta/2m)^2}\left[1+\frac{\frac{2}{m}(q_{1o}p_{1o}-m^2)}{\nu_o+\nu_L+\beta/2m}\right] \qquad (42)$$

$$\rho(\omega,\omega',\Delta^2) = \frac{\omega'-\omega+2m}{2\omega'} - \frac{1/m}{\nu'_L+\nu_L+\beta/2m}\left[\frac{\omega'-\omega+2m}{2\omega'}(q'_{1o}p'_{1o}-m^2) + \right.$$
$$\left. + \frac{(\omega'-\omega+2m)R_1'^2}{3\omega'}\right] + \frac{\frac{2}{m}(q_{1o}p_{1o}-m^2)}{\nu'_L+\nu_L+\beta/2m}\,\frac{\omega'-\omega+2m}{2\omega'} - \qquad (43)$$
$$- \frac{\frac{2}{m^2}(q_{1o}p_{1o}-m^2)}{(\nu'_L+\nu_L+\beta/2m)^2}\left[\frac{\omega'-\omega+2m}{2\omega'}(q'_{1o}p'_{1o}-m^2) + \frac{(\omega'-\omega+2m)R_1'^2}{3\omega'}\right]$$

V. The integral equation

We will show in this section how Eq. (40) can be transformed into an integral equation for g_{33}. From (22) and (41) it follows

$$\mathrm{Im} g_{33}(\omega, \Delta^2) = e^{i\delta_{33}} \sin \delta_{33} g_{33}(\omega, \Delta^2) \tag{44}$$

where δ_{33} is the physical $T=J=3/2$ πN phase shift.
Eq. (40) can thus be written

$$g_{33}(\nu_L, \Delta^2) = B + \frac{1}{\pi} \int_{\mu}^{\infty} d\nu_L' \, e^{i\delta_{33}'} \sin \delta_{33}' \left[\frac{1}{\nu_L' - \nu_L - i\epsilon} \frac{\omega' + \omega}{2\omega'} + \right.$$

$$\left. + \frac{\varrho}{3} \frac{1}{\nu_L' + \nu_L + \beta/2m} \right] g_{33}(\nu_L', \Delta^2) \tag{45}$$

This equation is of the general type discussed by Omnès [7] (one can also take the factor $\frac{\omega' + \omega}{2\omega'}$ into account, by slightly modifying Omnès' procedure). Therefore (45) can be solved exactly. The condition $\delta_{33}(\nu_L = \mu) = 0$ ensures the uniqueness of the solution. We can then obtain in this way the expression of the off-shell low energy 3.3 amplitude for arbitrary Δ^2. We will write down its expression only in the static limit and for $\Delta^2/m^2 \ll 1$.

Let us first go to the static limit by leaving Δ^2 arbitrary, but not too large (let's say $\Delta^2 \lesssim 2m^2$). If we put

$$u = \frac{\omega - m}{m} \tag{46}$$

and consider values of u small with respect to unity, we can prove that

$$|\rho| \lesssim \frac{1}{3} \tag{47}$$

The second term in the integral of (40) or (45) can therefore be neglected, as CGLN did. We note that in our case this approximation is even better due to the presence of the term $\beta/2m$ in the denominator. Furthermore

$$\frac{\omega'+\omega}{2\omega'} \simeq 1 \; ; \qquad \frac{2\omega}{R_1} \simeq 1 \tag{48}$$

$$\frac{V_L}{m} \simeq u \tag{49}$$

and

$$B \simeq \frac{4f^2 u}{3m} \frac{K(\beta)}{(u+\beta/2m^2)^2} \left(1 + \frac{\beta/m^2}{u+\beta/2m^2}\right) \sqrt{1 + \frac{\beta}{4m^2}} \tag{50}$$

With these approximations (45) simplifies greatly and reduces to

$$g_{33}(u,\Delta^2) = B + \frac{1}{\pi} \int_{\mu/m}^{\infty} du' \frac{e^{i\delta'_{33}} \sin \delta'_{33}}{u'-u-i\epsilon} g_{33}(u',\Delta^2) \tag{51}$$

whose solution can be obtained from formula (2.11) of ref. [7]. The integral on the r.h.s. of (51) is already in the simplest possible form. The B term however, is still not very simple. In order to write down the solution in a particular case, let us consider also the limit $\beta \sim \mu^2$. We can then develop (50) in powers of β, by retaining terms up to the order β/m^2u, which is about 10^{-1} when $\beta \sim \mu^2$. We obtain

$$B \simeq \frac{4f^2u}{3m} K(\beta) \tag{52}$$

We see that all terms in β/m^2u cancel exactly in (50) and the corrections are of the second order in this parameter [10]. Therefore, we can reasonably extend the solution of (51) with B given by (52) to higher values of β. We will get a 10% accuracy for $\beta/m^2u \lesssim \frac{1}{3}$ (or $\Delta^2 \lesssim 5\mu^2$) and a 25% accuracy for $\Delta^2 \lesssim 8\mu^2$. The approximate solution valid in this range can simply be obtained as follows. By reinserting (44) into (51) and expressing B through (52) we get

$$g_{33}(u,\Delta^2) = \frac{4f^2u}{3m} K(\Delta^2) + \frac{1}{\pi}\int_{\mu/m}^{\infty} du' \frac{\mathrm{Im}\, g_{33}(u',\Delta^2)}{u'-u-i\epsilon} \tag{53}$$

If $K(\Delta^2) = 1$, Eq. (53) satisfied by $g_{33}(u,\Delta^2)$ has the same form as Eq. (4.5) of CGLN satisfied by $f_{33}(u)/q_1^2 = g_{33}(u,-\mu^2)$ in the static limit approximation. Since the quantity $g_{33}(u,\Delta^2)$ satisfies for low Δ^2 the same equation as its limit $g_{33}(u,-\mu^2)$, we can reasonably conclude that in the range of Δ^2 where (53) is valid we have $g_{33}(u,\Delta^2) = g_{33}(u,-\mu^2)$. The additional presence of $K(\Delta^2)$ simply gives this term as a multiplicative factor. We finally get

$$g_{33}(u,\Delta^2) = K(\Delta^2)g_{33}(u,-\mu^2) \tag{54}$$

or

$$f_{33}(u,\Delta^2) = \frac{p_1}{q_1} K(\Delta^2) f_{33}(u,-\mu^2) \tag{55}$$

The Δ^2- dependence is then contained in K and in the factor p_1/q_1 which is an increasing function of this variable.

VI. Concluding remarks

The main result of this paper is Eq. (45), which, as already stated, can be solved exactly with standard methods. The two assumptions used in order to derive it are the validity of the dispersion relations (23) and the dominance of the off-shell T=J=3/2 amplitude under the dispersive integrals. We think that the first assumption may be proved by modifying the well-known methods for deriving the usual dispersion relations. In the proof of the usual dispersion relations, both pions are considered off-shell and analytic continuation is made from unphysical values of the pion mass ($\Delta^2 > 0$, as in our case) to the physical value $-\mu^2$. However, the two masses are kept equal in this procedure.

In our case one should continue only one of the two masses and keep the other in the unphysical region. Though this procedure has not yet been considered in detail, we think that it should not introduce further difficulties [12].

The second assumption is difficult to justify from a purely theoretical standpoint. Consistency arguments [6] have indicated that in the on-shell case it is probably a good assumption [11]. In the off-shell case it should still be rather good unless dramatic changes are induced by the Δ^2- dependence. The experimental indications, as discussed in the introduction, are so far against such a strong Δ^2- dependence.

An interesting feature of the static limit and low Δ^2 solution (54) is the presence of $K(\Delta^2)$ as a multiplicative factor. It is easy to see that this feature is preserved also in the complete solution of (45), since K appears only as a multiplicative factor in the Born term. When the analytic continuation of $M_{\pi N}(\omega^2, t^2, \Delta^2)$ is put into (4), we see that the only unknown function of Δ^2 entering in this formula is the product $K^2(\Delta^2)K'(\Delta^2)$. It is not completely out of question as to whether an experimental estimate of this quantity can be made: in fact the many pions exchanges in process (3) probably give contributions which depend on the energy in a different way than the one pion exchange diagram. If the predictions of the one pion exchange theory (calculated with $K^2K' = 1$) diverged from the experimental data at various energies only by a Δ^2 (and not energy) dependence, one could reasonably attribute the error to the effect of the neglected form factors, and try to determine K^2K' from experiment.

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7) R. Omnès, Nuovo Cimento 8, 316 (1959).

8) The proof of formula (21) can be found in S. Fubini, Y. Nambu and Y. Wataghin, Phys. Rev. 111, 329 (195). Appendix 2, where invariance of the S matrix element under strong reflections is used. This procedure has been pointed out to us independently by Prof. V. Glaser.

9) For the discussion of this point see Sect. VI.

10) It is interesting to note that this cancellation occurs also in the neglected ϱ term. The only terms linear in β which survive in it are of the order $\beta/2m^2$ and can be neglected in the present approximation.

11) The main criticism to the work by CGLN, which concerns the neglect of the T=J=1 $\pi\pi$ resonance, does not regard the T=J=3/2 equation, which is very little influenced by this system. See J. Bowcock, N.W. Cottingham and D. Lurié, Nuovo Cimento 16, 918 (1960).

12) After completion of this work we were kindly informed by Dr. R. Omnès that the rigorous proof of the dispersion relations (23) we assumed is contained in his lecture notes about the 1960 Les Houches Summer Course on Dispersion Relations, edited by Hermann (Paris).