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Offset-free Model Predictive Control: A Study of Different Formulations with Further Results

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***Abstract*— This paper presents discussions on offset-free model predictive control (MPC) methods for linear discrete-time systems in the presence of deterministic system disturbances. The general approach is based on the use of a disturbance model and an observer to estimate the disturbance states. The recent development in offset-free MPC has established the equivalence of the velocity form (without output delay) to a specific choice of the disturbance model and observer. In this note, it was shown that this particular disturbance model and observer is not necessarily equivalent to the velocity form with output delay. Nevertheless, it was shown that the velocity form with output delay is equivalent to a different choice of the disturbance model and observer. An import of this result is that the velocity forms (with and without delayed output) belong to the same general approach - disturbance model and observer. Furthermore, areas that may be considered in future researches are also highlighted.**

***Index Terms*— Model predictive control, disturbance rejection, linear discrete-time systems, offset-free control.**

I. INTRODUCTION

Offset-free model predictive control (MPC) is designed to eliminate permanent misalignment between output and target that are caused by internal and/or external disturbances. This work focuses on deterministic unmeasured constant or slowly-varying disturbance rejection techniques. In MPC schemes based on state-space systems, two approaches are generally used [1] and they involve the use of disturbance model method and plant deviation model approach.

In general, the first method involves the use of a disturbance model along with an observer. A majority of the offset-free MPC schemes based on the disturbance model achieve zero tracking error by the introduction of a constant output disturbance in the plant model [2], [3]. The approach has been suggested in the control of a variety of systems/processes subjected to unmeasured disturbances [4], [5]. General formulations of disturbance models for MPC with observers in linear state-space systems have been widely studied [6], [7].

The velocity model approach can either be partial or complete [8]–[10]. In partial velocity form, only the change in control input is used and the augmented state contains the actual system state and control signal. On the other hand, the complete velocity form utilizes the increment of both the input and states and the augmented model contains the state increment and system output. The aforementioned categorization does not include the approach [11] commonly used in robust MPC designs, where input and output increments are used.

The use of disturbance models and increment form of MPC were considered to be completely different approaches to disturbance rejection until Pannocchia [12] presented an important result to show that the 'conventional' complete increment model is indeed a particular form of the disturbance model and observer gains. In this note, we aim to extend this result to the complete increment form where the augmented state contains a delayed output and it will highlight the areas that may be considered for future research. The work also demonstrates the relative practical benefit of two different architectures of offset-free MPC by considering measurement noise in the illustrative example. Please note that this work does not include discussions on the state disturbance method [13], which avoids the use of augmented states.

The remaining part of this article is organized as follows. In Section II, the widely used disturbance model with observer technique will be discussed and Section III presents MPC based on increment models. Next, a discussion and analysis of the increment form equivalent disturbance model will be given in Section IV and an illustrative example will be presented in Section V and finally, concluding remarks are given in Section VI.

II. DISTURBANCE MODEL AND OBSERVER

The disturbance model approach is the most widely used method to eliminate permanent offset in MPC. Consider a discrete-time system affected by disturbance w_k, as follows:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + w_k, \\ y_k &= Cx_k, \end{aligned} \tag{1}$$

where $x_k \in \mathbb{R}^{n_x}$ is the system state vector, $u_k \in \mathbb{R}^{n_u}$ is the control vector, $y_k \in \mathbb{R}^{n_y}$ is the output vector. $A \in \mathbb{R}^{n_x \times n_x}$ and $B \in \mathbb{R}^{n_x \times n_u}$ are system and control input matrices respectively and $C \in \mathbb{R}^{n_y \times n_x}$ is the output matrix. The pairs (A, B) and (C, A) are assumed to be respectively stabilisable and detectable. Furthermore, w_k is an unmeasured state disturbance vector of the class considered in this note, which result in a mismatch between the model (A, B, C) and the actual plant. In this note, the augmentation used by [7] is

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presented here and is given by

$$\overbrace{\begin{bmatrix} x_k \\ d_k \end{bmatrix}}^{\varepsilon_k} = \overbrace{\begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix}}^{\mathcal{A}} \begin{bmatrix} x_{k-1} \\ d_{k-1} \end{bmatrix} + \overbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}^{\mathcal{B}} u_{k-1}, \qquad (2)$$
$$y_k = \overbrace{\begin{bmatrix} C & D_d \end{bmatrix}}^{\mathcal{C}} \varepsilon_k,$$

where $d_k \in \mathbb{R}^{n_d}$, $B_d \in \mathbb{R}^{n_x \times n_d}$ and $D_d \in \mathbb{R}^{n_y \times n_d}$. In [7], it was shown that for the system (2) to be detectable, the system (1) must be detectable and the following condition must be satisfied

$$\text{rank} \begin{bmatrix} I - A & -B_d \\ C & D_d \end{bmatrix} = n_x + n_d \qquad (3)$$

and this guarantees the existence of an asymptotically stable observer for the augmented system. Moreover, the dimension of the disturbance that will guarantee that condition (3) holds is given as $n_d \leq n_y$. This also guarantees the existence of the pair (B_d, D_d), which is the disturbance model in principle. The satisfaction of (3) is sufficient to obtain an offset-free response in the presence of model mismatch and/or external disturbance provided that the number of integrating disturbances used in the plant augmentation is equal to the measurement outputs, that is, $n_d = n_y$.

The authors of [14] validated the results in [7] and demonstrated that if the disturbance model is added only to the desired output such that $n_d \neq n_y$, steady-state error and closed-loop instability could result when a mismatch exists between the plant and the model. Nevertheless, it is possible to obtain a gain (L_x, L_d) that eliminates steady-state offset in this condition but the gain would be dependent on the parameters of the penalty function [7], [15]. This is undesirable because a change in the cost function parameter leads to re-tuning of the observer.

In general, an observer is needed to obtain the disturbance state estimate as well as any unmeasured system state, if present. To implement an observer to obtain $\hat{\varepsilon}_k$, the filtered estimate of ε_k, let $L_o = \left[\begin{smallmatrix} L_x \\ L_d \end{smallmatrix}\right]$ be the gain of the observer. Where L_x is the gain associated with the state x_k, and L_d is the observer gain associated with the disturbance d_k. Then, the estimate $\hat{\varepsilon}_k$ of the augmented state can be obtained by implementing the observer

$$\hat{\varepsilon}_k = \mathcal{A}\hat{\varepsilon}_{k-1} + \mathcal{B}u_{k-1} + L_o(y_k - \hat{y}_k), \qquad (4)$$

where $(y_k - \hat{y}_k)$ is the output prediction error in time-step k. Although earlier studies [6], [7], [16] have alluded to the fact that different disturbance models (B_d, D_d) give different closed-loop performance when external disturbances are present, [17] presented a very interesting result that established the equivalence of different disturbance models.

Generally, the design of the disturbance model (B_d, D_d) is usually separated from the observer design in offset-free MPC. For the first time, an innovative procedure for the simultaneous design of the disturbance model and observer gain was proposed by [2] and this is referred to as 'combined offset-free MPC'. The effectiveness of the approach was validated by simulating the CSTR process. More recently, the approach was used in the control of a diesel engine [18] where experimental validation of the approach was presented.

III. INCREMENT MODEL-BASED MPC

Increment model-based MPC or simply increment form MPC involves the use of the deviation of the system variable(s) to introduce integral action in the closed-loop control.

A. Partial Increment Form

In the rejection of disturbance using this scheme, the disturbance w_k given in (1) is rejected via an 'indirect' means because it demands that the applied control signal be estimated. The augmented state is formed by augmenting the actual system state with $u_k = u_{k-1} + \mu_k$ to obtain

$$\begin{bmatrix} x_{k+1} \\ u_k \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & I \end{bmatrix} \begin{bmatrix} x_k \\ u_{k-1} \end{bmatrix} + \begin{bmatrix} B \\ I \end{bmatrix} \mu_k, \qquad (5a)$$

$$y_k = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x_k \\ u_{k-1} \end{bmatrix}. \qquad (5b)$$

where μ_k is the control increment. The optimisation problem for the system (5) is formed using the typical MPC objective function [19]:

$$J = \frac{1}{2} \parallel e_{t+N} \parallel_S^2 + \frac{1}{2} \sum_{k=0}^{N-1} \parallel e_{t+k} \parallel_Q^2 + \frac{1}{2} \sum_{k=0}^{Nu-1} \parallel \mu_{t+k} \parallel_R^2, \qquad (6)$$

where $e_k = r_k - y_k$ is the output error vector and $\parallel x \parallel_R^2 = x^T R x$. $Q \succeq 0$, $S \succeq 0$ and $R \succ 0$ are symmetric weighting matrices. $N \in \mathbb{N}$ and $N_u \in \mathbb{N}$ are the prediction and control horizon respectively. For a tracking problem, the quadratic problem (QP) seemed to be well-posed since at steady-state, the combination $y = r$ (set-point) and $\mu = 0$ are possible. To guarantee the elimination of the constant disturbance w_k affecting the system using this approach, the estimate of the control $\hat{u}_k$ must be used in the prediction equation instead of the actual control signal u_k [1].

B. Complete Increment Form

In the complete increment form of linear models described in this section, the increments in both the inputs and states are used [1], [8], [9], [14], [20] and the augmented state contains the state deviation and output of the same time step. This method without output delay in the augmented state will simply be referred to as the complete increment or velocity form in the rest of this note. For convenience, let the state increment for any time instant $k > 0$ be defined as $\sigma_k \triangleq x_k - x_{k-1}$. Then, the nominal augmented state-space model can be written as

$$\begin{bmatrix} \sigma_k \\ y_k \end{bmatrix} = \overbrace{\begin{bmatrix} A & 0 \\ CA & I \end{bmatrix}}^{\bar{A}} \begin{bmatrix} \sigma_{k-1} \\ y_{k-1} \end{bmatrix} + \overbrace{\begin{bmatrix} B \\ CB \end{bmatrix}}^{\bar{B}} \mu_{k-1} \qquad (7a)$$

$$y_k = C\sigma_k + y_{k-1} = CA\sigma_{k-1} + CB\mu_{k-1} + y_{k-1} \qquad (7b)$$

As in the previous case, the QP is formed using the objective function (6) with $e_k = r_k - \left[\begin{smallmatrix} 0 & I \end{smallmatrix}\right] \left[\begin{smallmatrix} \sigma_k \\ y_k \end{smallmatrix}\right]$. The formulation guarantees zero tracking error in the presence of disturbances provided that $\mu_k = 0$ and $\sigma_k = 0$ holds at steady-state. It is important to assertively state that this method eliminates offset even when no estimator is used in the presence of disturbance w_k unlike the partial increment form where the use of an estimator to obtain the estimate $\hat{u}_k$ is a requirement to achieve offset-free steady-state when w_k is present.

It will be good to expatiate on the conditions that establish the offset-free property of the conventional approach to aid our discussion of the form with output delay. To proceed, it should be noted that y_k is measured but the state increment σ_k may not be measurable. Hence, an observer is generally needed to obtain the estimate of the unmeasured components of σ_k. Then, the observer to be designed for the model (7) has a gain matrix defined as $L_v \triangleq \begin{bmatrix} L_\sigma \\ L_y \end{bmatrix}$, which makes it convenient to construct a general steady-state observer equation as

$$\begin{bmatrix} \hat{\sigma}_k \\ \hat{y}_k \end{bmatrix} = \begin{bmatrix} A & 0 \\ CA & I \end{bmatrix} \begin{bmatrix} \hat{\sigma}_{k-1} \\ \hat{y}_{k-1} \end{bmatrix} + \begin{bmatrix} B \\ CB \end{bmatrix} \mu_{k-1} + \begin{bmatrix} L_\sigma \\ L_y \end{bmatrix} [y_k - \hat{y}_k] . \tag{8}$$

At steady-state, $y_k = y_{k-1}\ \forall\ k$ for a tracking problem. From equation (8), if at steady-state $\mu_k = 0$ and $\sigma_k = 0\ \ \forall\ k$, it has been shown [1] that $y_k = \hat{y}_k$ is achieved as long as either L_σ or L_y is of full-rank. The prediction of the system output by the controller will depend on the estimated output. Since one can guarantee that the estimate reaches the actual output at steady-state, the elimination of steady-state error by this scheme is guaranteed in the presence of model mismatch and/or external disturbances.

C. Complete Increment Form with Output Delay

In opposition to the conventional approach of formulating MPC using complete increment models, [21] proposed a scheme where the current state deviation is augmented with the previous output to obtain the augmented system state. Let the augmented state be defined as $\varsigma_k \triangleq \left[\begin{smallmatrix} \sigma_k \\ y_{k-1} \end{smallmatrix} \right]$. Then, one can write the equation of the augmented model in the compact form

$$\varsigma_k = \tilde{A}\varsigma_{k-1} + \tilde{B}\mu_{k-1}, \tag{9a}$$

$$y_k = \tilde{C}\varsigma_k. \tag{9b}$$

The corresponding augmented system matrices are given as follows:

$$\tilde{A} \triangleq \begin{bmatrix} A & 0 \\ C & I \end{bmatrix}, \quad \tilde{B} \triangleq \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \tilde{C} \triangleq \begin{bmatrix} C & I \end{bmatrix}.$$

The proof used to show that conventional increment form achieves offset-free control can readily be extended to this scheme. If an observer where to be designed for the model (9), it must also be shown that the integral mode introduced by the increment model into the observer guarantees that the output estimate $\hat{y}_k$ attains the actual output of the system. To see this, let the observer gain be $L_w = \begin{bmatrix} L_s \\ L_y \end{bmatrix}$ and it becomes convenient to write a general observer equation as

$$\hat{\varsigma}_k = \tilde{A}\hat{\varsigma}_{k-1} + \tilde{B}\mu_{k-1} + L_w\left(y_k - \hat{y}_k\right), \tag{10a}$$

$$\hat{y}_k = C\hat{\sigma}_k + \hat{y}_{k-1}. \tag{10b}$$

Given the condition that at steady-state $\sigma_k = 0$ and $\mu_k = 0$, and recalling that for a tracking problem, $y_k = y_{ss}$ and $\hat{y}_k = \hat{y}_{ss}\ \forall\ k$, where y_{ss} and $\hat{y}_{ss}$ are respectively the true plant output and the estimated output at steady-state. The stationary observer relation (10a) can be written explicitly as

$$\begin{bmatrix} 0 \\ \hat{y}_{ss} \end{bmatrix} = \begin{bmatrix} 0 \\ \hat{y}_{ss} \end{bmatrix} + \begin{bmatrix} L_s \\ L_y \end{bmatrix} \left[y_{ss} - \begin{bmatrix} C & I \end{bmatrix} \begin{bmatrix} 0 \\ \hat{y}_{ss} \end{bmatrix} \right],$$

which implies $0 = L_s[y_{ss} - \hat{y}_{ss}]$ and $\hat{y}_{ss} = \hat{y}_{ss} + L_y[y_{ss} - \hat{y}_{ss}]$. Hence, if either L_s or L_y has a full rank, the following holds

$$y_{ss} = \hat{y}_{ss}. \tag{11}$$

Therefore, an offset-free control is also ensured provided that the conditions $\sigma_k = 0$ and $\mu_k = 0$ are satisfied at steady-state.

At this point, it is pertinent to mention that in practical applications, the integral modes introduced by the increment forms that were described all cause the controlled system to loose its open-loop stability [1] and the usual way of achieving stability in MPC, which is by taking $N = \infty$ [22] leads to an unbounded objective function. However, this problem can be solved [23], [24] by introducing some constraint conditions and panic variables into the online optimization, which ensures that the integrating mode goes to zero at the end of the control horizon, N_u. By using this approach, closed-loop stability can be shown [1] to be guaranteed by the objective function (i.e the objective function can be shown to be a Liapunov function) and the offset-free property of the increment forms of MPC are preserved. Alternatively, one can also guarantee [21] nominal stability of the closed-loop system by choosing a weighting matrix for the terminal state that is the solution of the Discrete-time Ricatti equation.

IV. INCREMENT FORM EQUIVALENT DISTURBANCE MODELS

The results presented by [12] showed that the conventional velocity form is indeed a particular case of the disturbance model and observer approach. This made it clear that it is no longer appropriate to consider the methods as alternative techniques but 'simply as particular choices of the general approach' [12].

The need to show that the results presented by [12] can also be extended to the increment form with output delay is one of the motivations behind this note. This is particularly important to establish that the increment form with output delay is part of the 'so-called' general approach. The increment forms and their equivalent disturbance model will be presented in the following subsections.

A. Complete Increment form

In this subsection, we will summarise the results presented in [12], [14] that showed that a particular choice of disturbance model and observer gain is equivalent to the conventional increment form without output delay.

Theorem 1 ([14]): The increment model (7) and the observer (8) with a stable output deadbeat observer gain such that $L_v = \left[\begin{smallmatrix} L_\sigma \\ I \end{smallmatrix}\right]$, is equivalent to a specific form of the disturbance model and observer gains given as

$$B_d = L_\sigma, \quad D_d = I - CL_\sigma, \quad L_x = L_\sigma, \quad L_d = I. \quad (12)$$

In choosing a disturbance model and observer gains, it is pertinent to ensure that the detectability of the original system is preserved *i.e* the condition (3) is fulfilled. Then, it becomes essential to show that the disturbance model and observer gains (12) ensures that the condition holds.

Proposition 2 ([12]): The choice of the disturbance model, $B_d = L_\sigma, D_d = I - CL_\sigma$ ensures that the detectability condition (3) holds, provided that L_σ is chosen such that $(A - L_\sigma CA)$ is Hurwitz.

To proceed, it is necessary to establish that the choice of the disturbance model and observer gains (12) does not lead to loss of assymptotic stability of the augmented system.

Proposition 3 ([12]): Consider the augmented system (2) and observer (4) with matrices given by (12), and the gain $L_x = L_\sigma$ is selected such that $(A - L_\sigma CA)$ is stable. Then, the augmented matrix $(\mathcal{A} - L_o\mathcal{C}\mathcal{A})$ of the designed observer is stable.

Proposition 4 ([12]): Consider the augmented system (7) and observer (8) with matrices given by (12), and the gain L_σ is selected such that $(A - L_\sigma CA)$ is stable. Then, the augmented observer matrix $(\bar{A} - L_v\bar{C}\bar{A})$ is stable.

Remark 1: The above results are very important as they guarantee that the stability of the augmented observer system matrices, $(\mathcal{A} - L_o\mathcal{C}\mathcal{A})$ and $(\bar{A} - L_v\bar{C}\bar{A})$, are solely dependent on the stability of unaugmented system gain matrix $(A - L_\sigma CA)$. Hence, the choice of the disturbance model and observer gain does not impose eigenvalues that may cause the system to become unstable.

B. Increment Form with Output Delay

This subsection investigates and presents some results on the relationship between the velocity form with output delay and the disturbance model and observer gains given by (12).

To proceed, using the gain matrix $L_w = \left[\begin{smallmatrix} L_s \\ I \end{smallmatrix}\right]$, expand (10a) to get $\hat{\sigma}_k$ and substitute (10b) into the result to obtain

$$\hat{\sigma}_k = A\hat{\sigma}_{k-1} + B\mu_{k-1} + L_s(y_k - C\hat{\sigma}_k - \hat{y}_{k-1}). \quad (13)$$

By noting that $y_{k-1} = \hat{y}_{k-1}$ because of the deadbeat output observer and substituting the uncorrected estimate $\hat{\sigma}_k = A\hat{\sigma}_{k-1} + B\mu_{k-1}$ into the right hand side of (13) one obtains

$$\begin{aligned}\hat{\sigma}_k = (I - L_sC)A\hat{\sigma}_{k-1} + (I - L_sC)B\mu_{k-1} \\ + L_s(y_k - y_{k-1}).\end{aligned} \quad (14)$$

Based on (14), it is easy to hastily conclude that this form is also equivalent to (12) provided that $L_s = L_\sigma$. However, this conclusion cannot be fully substantiated without showing that the augmented observer matrix (13) is asymptotically stable. This can quickly be investigated as follows:

$$\begin{aligned}(\tilde{A} - L_w\tilde{C}\tilde{A}) &= \begin{bmatrix} A & 0 \\ C & I \end{bmatrix} - \begin{bmatrix} L_s \\ I \end{bmatrix}\begin{bmatrix} C & I \end{bmatrix}\begin{bmatrix} A & 0 \\ C & I \end{bmatrix}, \\ &= \begin{bmatrix} A & 0 \\ C & I \end{bmatrix} - \begin{bmatrix} L_\sigma CA + L_\sigma C & L_\sigma \\ CA + C & I \end{bmatrix}, \\ &= \begin{bmatrix} A - L_\sigma CA - L_\sigma C & -L_\sigma \\ -CA & 0 \end{bmatrix}.\end{aligned} \quad (15)$$

From the above equation (15), the eigenvalues of the observer matrix $(\tilde{A} - L_w\tilde{C}\tilde{A})$ is not necessarily the same as that of the unaugmented matrix $(A - L_\sigma CA)$. This implies that one cannot guarantee the stability of the augmented observer matrix $(\tilde{A} - L_w\tilde{C}\tilde{A})$ by simply ensuring that the unaugmented system $(A - L_sCA)$ is stable. Hence, it would be inaccurate to conclude that the increment form with output delay is equivalent to the disturbance model (12). Nonetheless, it is possible to show that an alternative choice of the disturbance model and observer is equivalent to this form of complete increment model.

Theorem 5: The increment model (9) and the observer (10) with a stable output deadbeat observer gain such that $L_w = \left[\begin{smallmatrix} L_s \\ I \end{smallmatrix}\right]$, is equivalent to the following choice of the disturbance model and observer gains:

$$B_d = L_s, \quad D_d = I, \quad L_x = L_s, \quad L_d = I. \quad (16)$$

Proof 6: To proceed, let (10a) be re-written in the form

$$\hat{\varsigma}_k = \tilde{A}\hat{\varsigma}_{k-1} + \tilde{B}\mu_{k-1} + L_w\left(y_{k-1} - \hat{y}_{k-1}\right). \quad (17)$$

By substituting $\hat{y}_{k-1} = \tilde{C}\hat{\sigma}_{k-1} + \hat{y}_{k-2}$ into (17), one can obtain the equation of the estimated state increment as

$$\begin{aligned}\hat{\sigma}_k &= A\hat{\sigma}_{k-1} + B\mu_{k-1} + L_s\left(y_{k-1} - \tilde{C}\hat{\sigma}_{k-1} - \hat{y}_{k-2}\right) \\ &= (A - L_sC)\hat{\sigma}_{k-1} + B\mu_{k-1} + L_s(y_{k-1} - y_{k-2}).\end{aligned} \quad (18)$$

Note that $\hat{y}_{k-2} = y_{k-2}$ because of the use of a deadbeat output observer. Following similar procedure, one can conveniently write equation (4) as

$$\hat{\varepsilon}_k = \mathcal{A}\hat{\varepsilon}_{k-1} + \mathcal{B}u_{k-1} + L_o(y_{k-1} - \hat{y}_{k-1}), \quad (19)$$

By expanding (19), the estimated state equation is given by

$$\hat{x}_k = A\hat{x}_{k-1} + Bu_{k-1} + B_d\hat{d}_{k-1} + L_s(y_{k-1} - C\hat{x}_{k-1}). \quad (20)$$

If the above equation is re-written for the time step $k-1$ and the resulting equation is then subtracted from (20), the following can be obtained

$$\hat{\sigma}_k = (A - L_sC)\hat{\sigma}_{k-1} + B\mu_{k-1} + L_s(y_{k-1} - y_{k-2}) \quad (21)$$

The comparison of (21) and (18) completes the proof.

It is pertinent to ensure that the detectability of the original system is preserved *i.e* the condition (3) is fulfilled by the choice of disturbance model and observer gains. Hence, the authors will now show that the disturbance model in (16) ensures that the condition holds.

Proposition 7: The choice of the disturbance model, $B_d = L_s, D_d = I$ ensures that the detectability condition

(3) holds provided that L_s is chosen such that $(A - L_sC)$ is Hurwitz.

Proof 8: To show that condition (3) is satisfied, consider the system

$$\begin{bmatrix} I-A & -L_\sigma \\ C & I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{22}$$

which is equivalent to the equations

$$(I - A)x - L_s y = 0, \tag{23a}$$

$$Cx + y = 0. \tag{23b}$$

By solving (23b) for y and substituting the result into (23a), one obtains

$$(A - L_sC - I)x = 0 \implies x = 0 \tag{24}$$

Equation (24) holds since $(A - L_sC)$ is assumed to be stable, which guarantees that $(A - L_sC - I)$ is invertible. Lastly, by substituting $x = 0$ into (23b), one readily obtains $y = 0$. Therefore, the system (22) has a unique solution $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$, which completes the proof.

It is also essential to show the conditions under which the augmented observers (17) and (19) are asymptotically stable given the disturbance model and observer (16).

Proposition 9: Consider the augmented system observer (17) with the gains $L_y = I$ and L_s that is selected such that $(A - L_sC)$ is stable. Then, the augmented observer matrix $(\tilde{A} - L_w\tilde{C})$ is stable.

Proof 10: This can be shown by direct substitution and simplification as follows:

$$\begin{aligned} (\tilde{A} - L_w\tilde{C}) &= \begin{bmatrix} A & 0 \\ C & I \end{bmatrix} - \begin{bmatrix} L_s \\ I \end{bmatrix} \begin{bmatrix} C & I \end{bmatrix}, \\ &= \begin{bmatrix} A - L_sC & -L_\sigma \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{25}$$

It cab be seen from (25) that the eigenvalues of the augmented observer matrix $(\tilde{A} - L_w\tilde{C})$ has the same eigenvalues as $(A - L_sC)$ and n_y zero eigenvalues at the origin. This, therefore, completes the proof since $(A - L_sC)$ is assumed to be stable.

Proposition 11: Consider the augmented system observer (19) with matrices given by (16), and the gain $L_x = L_s$ is selected such that $(A - L_sC)$ is stable. Then, the augmented matrix $(\mathcal{A} - L_o\mathcal{C})$ of the designed observer is stable.

Proof 12: This can directly be shown as follows:

$$(\mathcal{A} - L_o\mathcal{C}) = \begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix} - \begin{bmatrix} L_x \\ L_d \end{bmatrix} \begin{bmatrix} C & D_d \end{bmatrix}. \tag{26}$$

Based on (16), the above becomes

$$\begin{aligned} (\mathcal{A} - L_o\mathcal{C}) &= \begin{bmatrix} A & L_s \\ 0 & I \end{bmatrix} - \begin{bmatrix} L_s \\ I \end{bmatrix} \begin{bmatrix} C & I \end{bmatrix}, \\ &= \begin{bmatrix} A - L_sC & 0 \\ -C & 0 \end{bmatrix}. \end{aligned} \tag{27}$$

From (27), it can be seen that the eigenvalues of the augmented observer matrix $(\mathcal{A} - L_o\mathcal{C})$ are the same as those of the unaugmented system $(A - L_sC)$ along with n_y eigenvalues at the origin. This completes the proof since $(A - L_sC)$ is assumed to be Hurwitz.

The next section presents a simulation study that compares the performances of the complete increment form with output delay and its equivalent disturbance model and observer (16).

V. ILLUSTRATIVE EXAMPLE

As a means to effectively illustrate and summarise the findings of this note, the simulation of a multi-input multi-output system is presented. Consider the discrete-time state-space model obtained by sampling the stirred tank reactor system at 1.8s [13]:

$$A = \begin{bmatrix} 0.958 & 0 & 0 & \\ 0 & 0.9418 & 0 & 0 \\ 0 & 0 & 0.9048 & 0 \\ 0 & 0 & 0 & 0.9277 \end{bmatrix} \quad B = \begin{bmatrix} 0.25 & 0 \\ 0.25 & 0 \\ 0 & 0.5 \\ 0 & 0.5 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.1678 & 0 & 0.9516 & 0 \\ 0 & 0.2329 & 0 & 0.289 \end{bmatrix}.$$

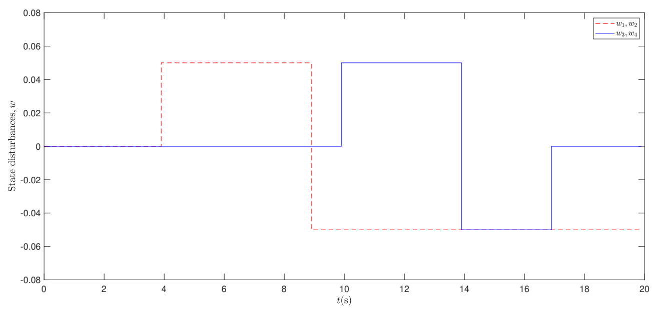

Fig. 1. Unmeasured exogenous system disturbance used in the simulation study

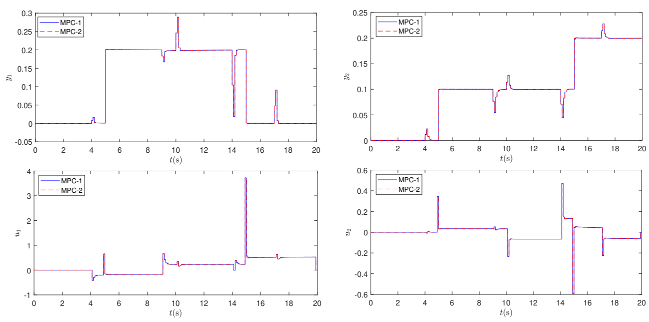

Fig. 2. Closed-loop output response of MPC-1 and MPC-2 in the presence of system disturbance w_k. System outputs (top) and controls (bottom).

The outputs of the model are required to track the desired reference in the presence of unmeasured exogenous system disturbance w_k, shown in Figure 1. The following MPC algorithms are compared:

- MPC-1 is the complete increment form with output delay. The observer gain matrix $L_s = \left[\begin{smallmatrix} 0.25 & 0 \\ 0 & 1.13 \\ 0.82 & 0 \\ 0 & 1.23 \end{smallmatrix}\right]$.
- MPC-2 is the MPC algorithm based on the disturbance model and observer (16).

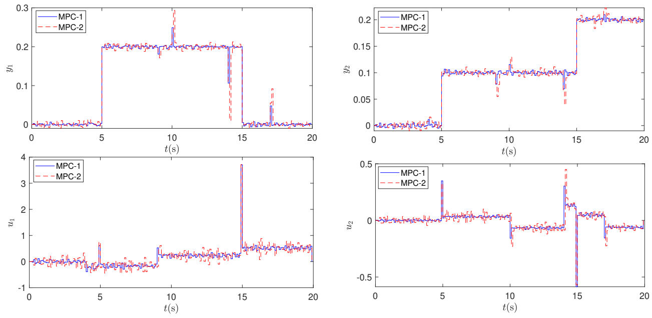

Fig. 3. Closed-loop output response of MPC-1 and MPC-2 in the presence of system disturbance w_k and measurement noise. System outputs (top) and controls (bottom)

The prediction horizon is chosen to be $N = 10$, control horizon $N_u = 2$ and the weighting matrices of the controllers are chosen as follows: $Q = I$, $R = 0.1I$. An input constraint is defined as $|u_k| \leq 4 \ \forall \ k$. The result of the comparative study is presented in Figure 2, where MPC-1 and MPC-2 both ensured the removal of permanent offset in the presence of the varying disturbances w_k. However, MPC-2 demonstrated greater sensitivity to measurement noise when compared to MPC-1 as shown in Figure 3.

VI. CONCLUSIONS AND FUTURE WORKS

A discussion of offset-free MPC schemes for the rejection of constant/slowly-varying deterministic disturbances in discrete-time linear systems has been presented. The paper described the recent advances in offset-free MPC that established the equivalence of the velocity form without output delay to a particular choice of disturbance model and observer, which is considered the general approach. The results were then extended to the velocity form with output delay and it was shown that it leads to a different choice of disturbance model. The issue of which form of complete increment form is more superior does not arise since different disturbance models are equivalent. Hence, complete velocity forms (with and without output delay) can no longer be correctly referred to as alternative approaches to disturbance rejection but as particular cases of the general approach. Furthermore, a simulation example showed that the disturbance model approach provided greater sensitivity to measurement noise compared to the increment form with output delay. On future research direction, in the area of varying disturbances, the increment of the disturbance may readily be left in the increment or velocity model and it may be useful to investigate means to utilise this disturbance increment information to obtain better performance over conventional approaches discussed in this note.

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