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Oka's inequality for currents and applications

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1 Introduction

In a 1962 paper Oka [Ok] proved that given a family of varieties in an open set $\Omega \subset \mathbb{C}^2$ the set $G \subset \Omega$ where the family is normal is pseudoconvex in Ω. The proof is based on the following remarkable inequality.

Let

$$H = \{(|z| < 1, |w| < r) \cup (r_1 < |z| < 1, |w| < 1) r_1, r < 1\}$$

and

$$\Delta_\rho = \{|z| < \rho, |w| < \rho\} .$$

If V is a closed complex curve in Δ_1, then for any $\rho < 1$ there exists C_ρ such that

$$\mathrm{vol}(V \cap \Delta_\rho) \leqq C_\rho \mathrm{vol}(V \cap H) . \qquad (*)$$

Here C_ρ is independent of V.

The result was generalized to varieties of codimension 1 in $\Omega \subset \mathbb{C}^n$ by Fujita [Fu]. Riemenschneider [Ri] proved the analogue of inequality (*) for varieties of dimension p in $\Omega \subset \mathbb{C}^n$, in which case the standard Hartogs figure has to be replaced by the right analogue. It turns out that the domain G where a family of analytic varieties of dimension p, has locally bounded mass is p-pseudoconvex (see below for a precise definition) in Ω. This notion was introduced by Rothstein [Ro], see also [Siu2], and developed by Andreotti and Grauert [A-G]. Observe that when a sequence of analytic varieties has bounded volume, then any limit in the Hausdorff metric is an analytic variety, as follows from Bishop's theorem [Bi].

The second author has given in [Si] some estimates on currents from which it is easy to deduce an inequality generalizing (*) to closed positive currents of bidimension (p, p). The purpose in [Si] was to extend the domain of definition of the Monge Ampere operator $(dd^c)^k$ to some unbounded plurisubharmonic functions. It was not realized there that the given condition for defining $(dd^c u)^k$

was that the set where u is locally unbounded is in the $n-k$ pseudoconvex envelope of its complement.

We start here by proving a version of Oka's inequality for currents of the form uT, where T is a positive closed current of bidimension (p,p) in Ω and u is a negative plurisubharmonic function in Ω. It follows from this inequality that $G := \{z \in \Omega; uT$ has bounded mass in a neighborhood of $z\}$ is p-pseudoconvex in Ω (usual pseudoconvexity coincides with $n-1$ pseudoconvexity). If the current (uT) has locally bounded mass in Ω one can define $dd^c u \wedge T := dd^c(uT)$.

We then prove a convergence result for the operator $(u_1,\ldots,u_q) \mapsto dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T$. The main idea is that if we control the mass or convergence on an open set, then we have the same type of control in the envelope of l-pseudoconvexity for the right l. When the (u_j) are bounded, this operator was studied by Chern, Levine and Nirenberg [CLN] and Bedford and Taylor [BT]. The case where the (u_j) are unbounded has been considered in Griffiths [Gr], Siu [Siu1], Sibony [Si] and more recently by Demailly [De].

Under pseudoconvexity assumptions we can define $T \wedge R_1 \wedge \ldots \wedge R_q$ where T is a current of bidegree $(n-p, n-p)$ and R_j are currents of bidegree $(1,1)$. When the currents are in $\mathbb{P}^k$ we prove a Bezout type theorem i.e. express the mass of $T \wedge \ldots \wedge R_q$ in terms of the mass of the factors.

Let T be a positive closed current in $\mathbb{P}^k$ of bidegree $(1,1)$. So locally T can be written as $dd^c u$. Assuming that u is continuous, we show that support T^l is connected, provided $2l \leqq k$.

In the last paragraph we apply the results on currents to holomorphic dynamics in $\mathbb{P}^k$. This was the main motivation to try to develop some tools in order to understand closed currents which are not analytic varieties. To a holomorphic surjective map $f : \mathbb{P}^k \mapsto \mathbb{P}^k$ of degree $d > 1$, one associates a positive closed current T of bidegree $(1,1)$. The support of $T^l := T \wedge \ldots \wedge T$ (l factors) are of dynamical interest. In particular the support of T coincides with the Julia set J_0 of f: the sequence (f^n) is equicontinuous precisely on $\mathbb{P}^k \backslash J_0$. In this context we show that if $2l \leqq k$, then support T^l is connected.

2 Oka's inequality

We first define the notion of k-pseudoconvexity, see [Ri]. Let $0 < r_1' < r'$ and $0 < r_1 < r$.

Definition 2.1 *An* **$(n-k, k)$ Hartogs figure** H *is defined as*

$$\begin{aligned} H = &\{(z,w); z \in \mathbb{C}^{n-k}, w \in \mathbb{C}^k, \|z\| < 1, \|w\| < r\} \\ &\cup \{(z,w); z \in \mathbb{C}^{n-k}, w \in \mathbb{C}^k, r_1 < \|z\| < 1, \|w\| < 1\} \end{aligned}$$

where $0 < r_1, r < 1$. *We set*

$$\hat{H} = \{(z,w); z \in \mathbb{C}^{n-k}, w \in \mathbb{C}^k, \|z\| < 1, \|w\| < 1\} = \Delta^n .$$

Here $\|z\| = \max |z_j|$, $\|w\| = \max |w_j|$.

Definition 2.2 (k-pseudoconvexity) *Let $\Omega_0 \subset \Omega$ be open subsets of $\mathbb{C}^n, 0 < k < n$. Then Ω_0 is* **k-pseudoconvex** *in Ω if it satisfies the Kontinuitätssatz with respect to $(n-k)$ polydiscs. More precisely, whenever H is an $(n-k, k)$ Hartogs figure, $\Phi : \hat{H} \mapsto \Omega$ is a $1-1$ holomorphic map and $\Phi(H) \subset \Omega_0$, then $\Phi(\hat{H}) \subset \Omega_0$.*

Usual pseudoconvexity is the same as $(n-1)$ pseudoconvexity.

For $\rho > 0, \Delta_\rho$ will denote the polydisc of radius ρ.

Theorem 2.3 [Ri] *Let X be a pure k-dimensional closed complex analytic subvariety of the unit polydisc $\Delta \subset \mathbb{C}^n, 0 < k < n$. Then if $0 < \rho < 1$, and H is an $(n-k, k)$ Hartogs figure, $\mathrm{vol}(X \cap \Delta_\rho) \leqq C_\rho \mathrm{vol}(X \cap H)$ for C_ρ independent of X.*

It follows that if (X_i) is a family of closed analytic varieties of pure dimension k in $\Omega \subset \mathbb{C}^n$, then if $\Omega' := \{z; \exists \delta > 0; \sup_i \mathrm{vol}(X_i \cap B(z;\delta)) < \infty\}$ then Ω' is k-pseudoconvex in Ω.

We want to prove a version of the previous theorem for positive closed currents or even for currents of the form uT where T is a positive closed current and u is a plurisubharmonic function.

For the fundamental results on currents we refer to [de Rh], [Le1] or [LG]. We recall a few facts.

Let Ω be an open set in $\mathbb{C}^n$. Denote by $D^{p,q}(\Omega)$ the space of smooth differential forms of bidegree (p,q) with compact support in Ω. The space of currents of bidimension (p,q), hence of bidegree $(n-p, n-q)$ is the dual space of $D^{p,q}(\Omega)$. A current T of bidimension (p,p) and bidegree $(n-p, n-p)$ is positive if for all $\alpha_1, \ldots, \alpha_p \in D^{1,0}(\Omega)$ the current $T \wedge i\alpha_1 \wedge \overline{\alpha_1} \wedge \ldots \wedge i\alpha_p \wedge \overline{\alpha_p}$ is a positive distribution. A current U is negative if $-U$ is positive.

If α is a k- covector in $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, let $|\alpha|$ denote the usual Hilbertian norm of α. If T is a current of bidimension (p,p) of order zero, i.e. with measure coefficients, we define the measure $M_V[T]$ on Ω as follows. If V is open in Ω, let $M_V[T] = \sup\{|T(\Phi)|; \Phi \in D^{p,p}(V), |\Phi(x)| \leqq 1, x \in V\}$. This is called the mass norm of T.

Let $d^c := (i/2\pi)(\bar{\partial} - \partial)$, then $dd^c = (i/\pi)\partial\bar{\partial}$. Let $\beta := dd^c|z|^2$. If T is a positive current of bidimension (p,p), then the trace measure σ_T is defined as $\sigma_T := T \wedge \beta^p/p!$. It is easy to show that there exists a constant C, depending only on n and p such that for every open set $V \subset \Omega$,

$$(1/C)\sigma_T(V) \leqq M_V[T] \leqq \sigma_T(V).$$

An upper semicontinuous function $u : \Omega \mapsto [-\infty, \infty)$ is plurisubharmonic if and only if $u \in L^1_{\mathrm{loc}}(\Omega)$ and $dd^c u \geqq 0$, for short we will say that u is p.s.h.

Theorem 2.4 *Let H be an $(n-l, l)$ Hartogs figure. Assume $\hat{H} = \Delta$. For every $\rho < 1$, there exists a constant C_ρ such that for every negative current U in a neighborhood of $\bar{\Delta}$, of bidimension (l,l), such that $dd^c U \geqq 0$ we have*

$$M_{\Delta_\rho}[U] + M_{\Delta_\rho}[dd^c U] \leqq C_\rho M_H[U].$$

Proof. Let Ω be an open set in $\mathbb{C}^n$. Assume $\omega \Subset \tilde{\omega} \Subset \Omega$, and that $\omega = \{z; z \in \tilde{\omega}, \psi(z) < 0, \nabla\psi|\partial\omega \neq 0\}$ for some smooth function ψ defined on $\tilde{\omega}$. Let M be a closed subset in Ω. Let $x \in M \cap \omega$ and let $\phi \in C^\infty(\Omega)$ such that
(i) $\phi(x) > 0$
(ii) $\phi \equiv 0$ on a neighborhood V of $\partial\omega \cap M$.

Let h be a smooth function in Ω, $0 \leqq h \leqq 1$, $h = 1$ in a neighborhood of ω.

We will need the following lemma. See also Lemma 4.1 in [Si].

Lemma 2.5 *Let ω, Ω, ϕ, h be as defined above. Let S and U be currents with smooth coefficients in Ω. Assume S is closed of bidegree (p, p) and U is of bidegree (q,q) with $p + q = n - 1$. Then we have*

$$(1) \quad \int_\omega \phi(1-h)dd^cU \wedge S + \int_\omega (-U) \wedge dd^c\phi \wedge S = \int_\omega (-U) \wedge dd^c(\phi h) \wedge S .$$

If $U \leqq 0$, $dd^cU \geqq 0$ then relation (1) *holds without smoothness assumption on U.*

Proof. When χ is a smooth function with compact support then since S is closed

$$\int \chi dd^cU \wedge S = \int dd^c\chi \wedge U \wedge S .$$

So we just apply the above formula with $\chi = \phi(1-h)$ which is of compact support since $h = 1$ near $\partial\omega$.

If U is not smooth, let U_ε be regularized currents, U_ε is just convolution of U with an approximation of identity. Relation (1) holds for U_ε. We can just let $\varepsilon \mapsto 0$.

The formula of integration by part is written in the form (1) to emphasize that if $h = 0$ near x and if $dd^c\phi \wedge S$ is positive then it is posible to control the mass of dd^cU near x and of $(-U) \wedge dd^c\phi \wedge S$ by what happens near the support of $dd^c(\phi h)$.

We continue the proof of the Theorem. Let Ω be a neighborhood of $\bar{\Delta}$ where U is defined. Recall that $H = \{(z,w), z \in \mathbb{C}^{n-l}, w \in \mathbb{C}^l, \|z\| < 1, \|w\| < r_1\} \cup \{r_1' < \|z\| < 1, \|w\| < 1\}$. Let $M = \Delta \backslash H$. Let M_1 be the set of points in $\Delta_{\rho_1} \backslash H$ for some $\rho < \rho_1 < 1$, such that exactly one of the w-coordinates is $\geqq r_1$. Let $x \in M_1$. For simplicity assume that $|x_{n-l+1}| = |w_1| \geqq r_1$ and $|x_{n-l+2}| = |w_2| < r_1, \ldots, |x_n| = |w_l| < r_1$. Let $\phi = 1/|w_1|^2$. Let H_1 be the $(n-l, 1)$ Hartogs figure obtained by fixing the coordinates $w_2 = x_{n-l+2}, \ldots, w_l = x_n$, with the same numbers r_1, r_1'. We can construct an open set ω_1 in $\mathbb{C}^{n-l+1}$ such that $\phi(x) > \max_{\partial\omega_1 \cap [\hat{H}_1 \backslash H_1]} \phi$. Fatten ω_1 to obtain an open set ω in $\mathbb{C}^n$ and extend ϕ to Φ_1 such that $\Phi_1(x) > \max_{\partial\omega \cap [\hat{H} \backslash H]} \Phi_1$, by adding to the trivial extension a function θ of $w_2, \ldots, w_n$ only.

Let $\Phi_2 := \Phi_1 + \delta|(z,w)|^2$. For $\delta > 0$ small enough, $\Phi_2(x) > \sup_{\partial\omega \cap M} \Phi_2 := c$. But also $dd^c\Phi_2 \wedge (dd^c|w'|^2)^{l-1} > 0$ where $w' = (w_2, \ldots, w_l)$. This is because the negativity in $dd^c\Phi_2$ introduced by the function $\theta(w')$ is cancelled by $(dd^c|w'|^2)^{l-1}$.

Let λ be a smooth increasing convex function on $[0,\infty)$, vanishing on a neighborhood of $[0,c]$ and then strictly increasing, we can assume that $\lambda(\Phi_2(x)) > 0$. Let $\Phi = \lambda \circ \Phi_2$. We apply Lemma 2.5 with $S = (dd^c|w'|^2)^{l-1}$ and $\phi = \Phi$ and h a function vanishing near x, h has value 1 near $\partial\omega$, but $M \cap \text{support}\, h$ is contained in $(\phi = 0)$.

Relation (1) in Lemma 2.5 gives an estimate of $dd^cU \wedge (dd^c|w'|^2)^{l-1}$ and of $(-U) \wedge dd^c\phi \wedge S$ near x in terms of the mass of U on a compact set of H. If we apply the same argument to suitable perturbations of the coordinates w, we finally get an estimate of the mass of $(-U)$ and dd^cU near x in terms of quantities supported in a compact set of H. We have proved the theorem for M_1 instead of Δ_ρ. Let M_j be the set of points in $\Delta_{\rho j}\backslash H$ such that exactly j of the w-coordinates are $\geqq r_1$ where we have chosen $1 > \rho_1 > \cdots > \rho_l = \rho$. We will prove the theorem by induction in j. For simplicity of notation we prove it for $j = 2$. Let $x \in M_2$. We can assume that $|x_{n-l+1}| = |w_1| \geqq r_1$ and $|x_{n-l+2}| = |w_2| \geqq r_1$ while $|w_3| < r_1, \ldots, |w_l| < r_1$.

Let $\phi_2 = 1/|w_2|^2$. We can construct an open set ω_2 in $\mathbb{C}^{n-l+1}$ such that $\phi_2(x) > \max_{\partial\omega_2 \cap [\hat{H}_2 \backslash H_2]} \phi$. We fatten ω_2 to obtain an open set ω in $\mathbb{C}^n$ and proceed as above. We will obtain an estimate of the mass of U and dd^cU near x in term of the mass in $H \cup M_1$ which in turn is controlled by the mass in H. So by induction we get the estimate in the theorem.

Corollary 2.6 *Let (U_i) be a sequence of negative currents of bidimension (p,p) in a complex manifold Ω. Assume for all i, $dd^cU_i \geqq 0$. Then the domain Ω_0 on which the family has locally bounded mass is p-pseudoconvex. In particular, if U is a negative current of bidimension (p,p) in Ω, such that $dd^cU \geqq 0$, then $\Omega\backslash(\operatorname{supp} U)$ is p-pseudoconvex in Ω. If T is a positive closed current of bidimension (p,p) on Ω, then $\Omega\backslash\operatorname{supp} T$ is p-pseudoconvex.*

Proof. Let H be an $(n-p,p)$ Hartogs figure and $f : \hat{H} \mapsto \Omega$ be an injective holomorphic map. If $f(H) \subset \Omega_0$, Theorem 2.4 shows that $f(\hat{H}) \subset \Omega_0$. The estimate in the theorem is just given with $f = \text{Id}$. So Ω_0 is p-pseudoconvex in Ω. To show that $\Omega\backslash\operatorname{supp} U$ is p-pseudoconvex we just apply the first part of the corollary to the sequence of currents $U_k = kU$, $k \in \mathbb{N}$. If T is positive and closed, we apply the previous result to $U = -T$.

Let T be a positive closed current of bidimension (p,p) in Ω. Let u be a plurisubharmonic function on Ω. We consider the closed set $M(T,u) = \{q; q \in \Omega, uT$ is not of finite mass in any neighborhood of $q\}$. For simplicity, since the results are of semi local nature (in neighborhoods of compacts) we will assume that $u < 0$.

Proposition 2.7 *Let T, u, Ω be as above. Then $\Omega\backslash M(T,u)$ is p-pseudoconvex. And on $\Omega\backslash M(T,u)$ the current uT is negative and satisfies $dd^c(uT) \geqq 0$.*

Proof. Let H be an $(n-p,p)$ Hartogs figure, assume $\hat{H} = \Delta \Subset \Omega$. Let $u_j \searrow u$ be a decreasing sequence of smooth plurisubharmonic functions in a neighborhood of $\bar{\Delta}$. If $U_j := u_j T$ we clearly have $U_j \leqq 0$ for large j and $dd^c U_j \geqq 0$. Theorem 2.4 implies that

$$\int_{\Delta_\rho} |u_j|\sigma_T \leqq C_\rho \int_H |u_j|\sigma_T \leqq C_\rho \int_H |u|\sigma_T .$$

Here σ_T denotes the trace measure for the current T. Hence if $\int_H |u|\sigma_T < \infty$ we get that

$$\int_{\Delta_\rho} |u|\sigma_T \leqq C_\rho \int_H |u|\sigma_T ,$$

by Lebesgue's dominated convergence theorem. The p-pseudoconvexity of $\Omega \backslash M(T,u)$ follows easily from the above estimate. Since locally in $\Omega \backslash M(T,u)$, $u_j T \mapsto uT$ in the sense of currents and since $dd^c(u_j T) = dd^c u_j \wedge T \geqq 0$ we get that $dd^c(uT) \geqq 0$.

3 The operator $(u_1, \ldots, u_q) \mapsto dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T$

It is very useful, in many questions in Algebraic Geometry and Complex Analysis, to define an expression such that $dd^c u \wedge T$ where T is a closed (p,p) current and u is an unbounded plurisubharmonic function. The case where u is bounded is studied in [BT].

Here we want to extend the approach in [Si] to define

$$dd^c u_1 \wedge \ldots \wedge dd^c u_k \wedge T$$

under quite general assumptions on $u_1, \ldots, u_k$ and T.

Let Ω be an open subset in $\mathbb{C}^n$. If u is a plurisubharmonic function in Ω, define $M(u) := \{q; q \in \Omega, u$ is unbounded in any neighborhood of $q\}$. Recall that if T is a positive closed (p,p) current we define $M(T,u) := \{z; z \in \Omega, u$ is not σ_T integrable in a neighborhood of $q\}$. In $\Omega \backslash M(T,u)$ we will define $dd^c u \wedge T := dd^c(uT)$.

If X is a closed set in Ω, we say that X is in the envelope of p-pseudoconvexity of $\Omega \backslash X$ with respect to Ω if all points in X can be reached by pushing polydiscs of dimension $(n-p)$ using biholomorphic images of $(n-p,p)$ Hartogs figures with hulls in Ω. So we use the same procedure as for obtaining the envelope of holomorphy by pushing one dimensional discs.

Proposition 3.1 *Let u be a plurisubharmonic function in Ω. And let T be a positive closed bidimension (p,p) current on Ω. If $M(u) \cap \operatorname{supp} T := X$ is in the envelope of p-pseudoconvexity of $\Omega \backslash X$ with respect to Ω, then u is locally σ_T integrable.*

Proof. Without loss of generality we can assume $u < 0$ on Ω. It is clear that $M(T,u) \subset M(u) \cap \operatorname{supp} T$. Since as proved in Proposition 2.7 $\Omega \backslash M(T,u)$ is p-pseudoconvex it follows that $M(T,u)$ is empty.

Proposition 3.2 *Let u_j, u be nonpositive plurisubharmonic functions in Ω. Let T_j, T be positive closed currents of bidimension (p, p) in Ω. Assume*

(1) $T_j \mapsto T$,

(2) $(u_j T_j)$ *has uniformly bounded mass on every compact,*

(3) $u_j \mapsto u$ *in* L^1_{loc}.

If L is any weak limit of $u_j T_j$, then $L \leqq uT$. If $u_j \geqq u$ and $T_j \leqq T$, then $u_j T_j \mapsto uT$.

Proof. Let $\gamma \geqq 0$ be a smooth form with compact support $X \subset \Omega$ of bidegree (p, p). Let f be a continuous function on X such that $f \geqq u$ on X. By Hartogs lemma, see [H],

$$\overline{\lim_j} \sup_X (u_j - f) \leqq \sup_X (u - f) \leqq 0 .$$

Hence given $\varepsilon > 0$, there is a j_0 so that for $j \geqq j_0, u_j \leqq f + \varepsilon$. Since $\gamma \geqq 0$ and $T_j \geqq 0$ we have

$$u_j T_j \wedge \gamma \leqq (f + \varepsilon) T_j \wedge \gamma .$$

Hence

$$L \wedge \gamma \leqq (f + \varepsilon) T \wedge \gamma .$$

Since ε and f are arbitrary we get that $L \leqq uT$. If $u_j \geqq u$ and $T_j \leqq T$, we can assume using again Hartogs lemma that $u_j \leqq 0$ on support γ, we have

$$\int uT \wedge \gamma \leqq \int u_j T \wedge \gamma \leqq \int u_j T_j \wedge \gamma \mapsto L \wedge \gamma .$$

So $L = uT$.

Corollary 3.3 *Let $u_n \leqq 0$ be plurisubharmonic in Ω and assume that $u_n \mapsto u$ in L^1_{loc} and that $u_n \geqq u$. Then for any positive closed current of bidimension (p, p), we have in $\Omega \backslash M(T, u)$ that $u_n T \mapsto uT$ and that $dd^c u_n \wedge T \mapsto dd^c u \wedge T$.*

Proof. Since $u_j \leqq 0, |u_j| \leqq |u|$ so given a compact K in $\Omega \backslash M(T, u)$,

$$\int_K |u_j| d\sigma_T \leqq \int_K |u| d\sigma_T .$$

Hence we can apply Proposition 3.2.

Let Λ^α denote the Hausdorff measure of dimension α. We have the following convergence result.

Corollary 3.4 *Let (u_j) be a sequence of plurisubharmonic functions in Ω. Assume $u_j \mapsto u$ in L^1_{loc} and $u_j \geqq u$. Let T be a positive, closed current of bidimension (p, p) in Ω. If $\Lambda^{2p}(M(u) \cap \operatorname{supp} T) = 0$, then $u_j T \mapsto uT$ in Ω.*

Proof. We just have to check that $M(T, u)$ is empty. It is clear that $M(T, u) \subset M(u) \cap \operatorname{Supp} T := E$. The idea is to show that E is in the p envelope of the complement. By the results of Federer [Fe], for almost all $n - p$ planes say parallel to a given one the intersection with E is empty, hence we can construct Hartogs figures of the right type. More precisely, fix $x \in E$. Since

$\Lambda^{2p}(E\setminus\{x\})=0$, almost every $n-p$ complex plane through x does not intersect $E\setminus\{x\}$. Assume that $x=0$ and let $L=\{w_1=\dots w_p=0\}$ be such a plane. Let $(z_1,\dots,z_{n-p})$ be the coordinates in L. Fix a polydisc $\Delta^{n-p}\Subset\Omega$ centered at the origin. Since $\partial\Delta^{n-p}\cap E=\emptyset$, we can find a polydisc Δ^p in the orthogonal complement $L^\perp$ centered at the origin such that $(\partial\Delta^{n-p}\times\Delta^p)\cap E=\emptyset$. We can then complete the Hartogs figure $H\subset\Omega\setminus E$ such that $0\in\hat{H}$.

Theorem 3.5 *Let T be a closed positive current of bidimension (p,p) in $\Omega, 0<p<n$. Let $u_1,\dots,u_q$ be plurisubharmonic $\leqq 0$ functions in Ω. If for all $j_1,\dots,j_m$, $M(u_{j_1})\cap\dots\cap M(u_{j_m})\cap\operatorname{supp}T$ is in the $p-m+1$ envelope of pseudoconvexity of the complement, then $u_1dd^cu_2\wedge\dots\wedge dd^cu_q\wedge T$ has locally bounded mass in Ω and similarly for $dd^cu_1\wedge\dots\wedge dd^cu_q\wedge T$. Moreover the mass on a compact set of Ω is majorized by the mass on a compact where all the u_j are bounded. We also have $u_1^jdd^cu_2^j\wedge\dots\wedge dd^cu_q^j\wedge T\mapsto u_1dd^cu_2\wedge\dots\wedge dd^cu_q\wedge T$ provided $u_k^j\mapsto u_k$ in L^1_{loc} and $u_k^j\geqq u_k$, moreover $dd^cu_1^j\wedge\dots\wedge dd^cu_q^j\wedge T$ converge weakly to $dd^cu_1\wedge\dots\wedge dd^cu_q\wedge T$.*

Proof. Case $q=1$. We assume $M(u)\cap\operatorname{supp}T$ is in the p-convex envelope of the complement relative to Ω. Then uT is well defined in Ω and if L is a compact in Ω, Theorem 2.4 shows that

$$\int_L|u|d\sigma_T\leqq c\int_K|u|d\sigma_T$$

where K is a compact of $\Omega\setminus M(u)$. Let χ be a nonnegative test function with value 1 in a neighborhood of L. If $\beta=dd^c|z|^2$, then

$$M_L[dd^cu\wedge T]\leqq\int\chi dd^cu\wedge T\wedge\beta^s=\int uT\wedge\beta^s\wedge dd^c\chi\,.$$

So we also get that $M_L[dd^cu\wedge T]\leqq c'M_{K'}[uT]$. For $q=1$ the convergence result is just Corollary 3.3.

So assume the theorem has been proved for $(q-1)$ functions (v_j). Let $S=dd^cu_2\wedge\dots\wedge dd^cu_q\wedge T$ which is, by induction, a well defined current on Ω of bidimension $(p-q+1,p-q+1)$. We want to show that $\Omega\setminus M(u_1,S)$ contains the complement of $M(u_1)\cap\dots\cap M(u_q)\cap\operatorname{supp}T$. Since $\Omega\setminus M(u_1,S)$ is $p-q+1$ pseudoconvex, the hypotheses of the theorem imply that $M(u_1,S)$ is empty.

Fix $z_0\notin M(u_1)\cap\dots\cap M(u_q)\cap\operatorname{supp}T$. If u_1 is bounded near z_0 we are done. Assume $z_0=0$ and that u_2 is bounded on a neighborhood of $\bar{B}:=\bar{B}(0,r)$. Replacing u_2 by $\max(u(z),A(|z|^2-r^2))$ we can assume that u_2 is unchanged on $B_1=B(0,r/4)$, and is equal to $A(|z|^2-r^2)$ in a neighborhood of ∂B. Let $T'=dd^cu_3\wedge\dots\wedge dd^cu_q\wedge T$. We will show that

$$\int_B-u_1dd^cu_2\wedge T'\wedge\beta^s \qquad (*)$$

is bounded by the mass of u_1T' on B.

Let h be a smooth function supported near ∂B with value 1 near ∂B. Let $v_1^j \searrow u_1, v_1^j$ smooth plurisubharmonic functions. We can apply Lemma 2.5 with $U = v_1^j dd^c u_2 \wedge T'$ and $S = \beta^{s-1}$ (in this case $\omega = B, \phi = |z|^2$ and M is empty). We have

$$\int_B \phi(1-h) dd^c v_1^j \wedge dd^c u_2 \wedge T' \wedge \beta^{s-1}$$
$$- \int_B v_1^j dd^c \phi \wedge dd^c u_2 \wedge T' \wedge \beta^{s-1}$$
$$= - \int_B v_1^j dd^c(\phi h) \wedge dd^c u_2 \wedge T' \wedge \beta^{s-1} .$$

Observe that $-v_1^j \geqq 0$ and $-A'\beta \leqq dd^c(\phi h) \leqq A'\beta$ and that on the support of h, $dd^c u_2 = A\beta$. So the last integral is smaller in absolute value than

$$-AA' \int_B v_1^j T' \wedge \beta^{s+1} .$$

Using Fatou's lemma and the induction hypothesis we get

$$\int_B |u_1| dd^c \phi \wedge dd^c u_2 \wedge T' \wedge \beta^{s-1} \leqq AA' \int_B |u_1| T' \wedge \beta^{s+1} .$$

Hence we have shown (*). As a consequence, also the mass of $dd^c u_1 \wedge dd^c u_2 \wedge T'$ on B_1 is bounded by the mass of $u_1 T'$ on B.

We now turn to the convergence question. Let $u_1^j \geqq u_1, \dots, u_q^j \geqq u_q$. We know that if R is any limit point of $u_1^j dd^c u_2^j \wedge \dots \wedge T$ then $R \leqq u_1 dd^c u_2 \wedge \dots \wedge T$ (by Proposition 3.2). In order to prove equality, it is enough to prove that for any x_0 there exists an open set $\omega \Subset \Omega, x_0 \in \omega$, a smooth function ϕ in a neighborhood of $\bar{\omega}$, a positive closed form γ such that $dd^c \phi \wedge \gamma$ is strictly positive in a neighborhood of x_0 and $\int_\omega u_1 dd^c u_2 \dots \wedge T \wedge \gamma \wedge dd^c \phi \leqq \underline{\lim}_j \int_\omega u_1^j dd^c u_2^j \wedge \dots \wedge T \wedge \gamma \wedge dd^c \phi$.

If $x_0 \notin \operatorname{supp} T$ this is clear. Assume $x_0 \notin M(u_1) \cap \dots \cap M(u_q)$. Then there exists a ball B such that (u_l^j) is a bounded sequence in a neighborhood of $\bar{B}$ for some $1 \leqq l \leqq q$. Let $B = \{\psi < 0\}, \psi$ plurisubharmonic. Since the result we want is local we can assume that on a neighborhood ω_1 of ∂B, $u_l^j = A\psi$ where A is a large constant. Let h be a function in $C_0^\infty(\omega_1)$ with value 1 in a neighborhood of ∂B. We are going to use repeatedly the identity of Lemma 2.5 with $\phi = |z|^2$ and M empty.

Assume that all the u_k^j s are smooth.

Let T' denote $T \wedge \beta^{p-q}$, we have

$$\int_B u_1 dd^c u_2 \wedge \dots \wedge T' \wedge dd^c \phi$$
$$\leqq \int_B u_1^j dd^c u_2 \wedge \dots \wedge T' \wedge dd^c \phi$$
$$= \int_B \phi(1-h) dd^c u_1^j \wedge dd^c u_2 \wedge \dots \wedge T'$$
$$+ \int_B u_1^j dd^c(\phi h) \wedge dd^c u_2 \wedge \dots \wedge T'$$

$$
\begin{aligned}
&= \int_B u_2 dd^c\phi \wedge dd^c u_1^j \wedge \ldots \wedge T' \\
&\quad - \int_B u_2 dd^c(\phi h) \wedge dd^c u_1^j \wedge \ldots \wedge T' \\
&\quad + \int_B u_1^j dd^c(\phi h) \wedge dd^c u_2 \wedge \ldots \wedge T' \\
&\leqq \int_B u_2^j dd^c\phi \wedge dd^c u_1^j \wedge T' \ldots \\
&\quad - \int_B u_2 dd^c(\phi h) \wedge dd^c u_1^j \wedge \ldots \wedge T' \\
&\quad + \int_B u_1^j dd^c(\phi h) \wedge dd^c u_2 \wedge \ldots \wedge T' \ldots =
\end{aligned}
$$

(where we do not write terms containing $dd^c(\phi h)$)

$$
\begin{aligned}
&= \int_B \phi(1-h) dd^c u_1^j \wedge dd^c u_2^j \wedge dd^c u_3 \wedge \ldots \\
&\quad + \int_B u_2^j dd^c(\phi h) \wedge dd^c u_1^j \wedge \ldots T' + \ldots \\
&= \int_B u_1^j dd^c u_2^j \wedge dd^c u_3 \wedge \ldots \wedge T' \wedge dd^c\phi \\
&\quad - \int_B u_1^j dd^c(\phi h) \wedge dd^c u_2^j \wedge \ldots \wedge T' \\
&\quad + \int_B u_2^j dd^c(\phi h) \wedge dd^c u_1^j \wedge \ldots \wedge T' \\
&\quad - \int_B u_2 dd^c(\phi h) \wedge dd^c u_1^j \wedge \ldots \wedge T' \\
&\quad + \int_B u_1^j dd^c(\phi h) \wedge dd^c u_2 \wedge \ldots \wedge T' \\
&\leqq \int_B u_1^j dd^c u_2^j \wedge \ldots \wedge dd^c u_q^j \wedge T \wedge dd^c\phi +
\end{aligned}
$$

sum of integrals involving $dd^c(\phi h)$.

Recall that if positive measures $\mu_k \mapsto \mu$ weakly on an open Ω set then $\overline{\lim}_{k\mapsto\infty}\mu_k(K) \leqq \mu(K)$ for each compact $K \subset \Omega$ and $\mu(V) \leqq \underline{\lim}\mu_k(V)$ for each open set. Hence, if V is an open set such that $\mu_k(\partial V) = \mu(\partial V) = 0$ for all k, then $\mu(V) = \lim_k \mu_k(V)$. We can assume that all the measures, which are coefficients of the currents we work with, have no mass on ∂B.

On support of $dd^c(\phi h), u_l^j = A\psi$, so we have only $q-1$ nonconstant sequences (u_m^j) so for every integral involving $dd^c(\phi h)$ we have convergence, but as it is shown from the above development the limit of the sum of integrals involving $dd^c(\phi h)$ is zero, (observe that the manipulations are purely algebraic) so

$$
\int u_1 dd^c u_2 \wedge \ldots \wedge T' \wedge dd^c\phi \leqq \underline{\lim}_j \int u_1^j dd^c u_2^j \wedge \ldots \wedge dd^c u_q^j \wedge T' \wedge dd^c\phi .
$$

Now assume $x \in M(u_1) \cap \ldots \cap M(u_q) \cap \operatorname{supp} T := M$. Since this set is in the envelope of $p-q+1$ pseudoconvexity of the complement we can construct finitely many Hartogs figures to absorb successively all of M. Given a Hartogs figure H and $x \in \hat{H} \cap M$, we can construct a smoothly bounded neighborhood $\omega \Subset \hat{H}$ of x and smooth functions $(\phi_i)_{i \leqq s}$ vanishing in a neighborhood of $\partial\omega \cap M$ such that $dd^c\phi_i \wedge \gamma_i \geqq 0$ and $\sum_i^s dd^c\phi_i \wedge \gamma_i > 0$ in a neighborhood of x, where the γ_i's are closed forms of bidegree $(p-q, p-q)$. We compute the integrals in the same way as above, where now T' denotes $T \wedge \gamma_i$. Since the support of ϕh is contained in the set where we have already proved convergence, the sum of integrals involving $dd^c(\phi h)$ is going to converge to zero as $j \mapsto \infty$, and then we add up with respect to $1 \leqq i \leqq s$. Hence,

$$\int u_1 dd^c u_2 \wedge \ldots \wedge \gamma_1 \wedge dd^c\phi \leqq \underline{\lim} \int u_1^j dd^c u_2^j \wedge \ldots \wedge \gamma_1 \wedge dd^c\phi\ .$$

This finishes the convergence proof for general q. The last part follows immediately.

Corollary 3.6 *Let Ω, T, $u_1, \ldots, u_q$ and $u_j \geqq u_k$ be as in theorem. Assume that $\Lambda^{2(p-m+1)}(M(u_{j_1}) \cap \ldots \cap M(u_{j_m}) \cap \operatorname{supp} T) = 0$ for all $j_1, \ldots, j_m \leqq q$. Then $u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T$ has locally bounded mass and $u_1^j dd^c u_2^j \wedge \ldots \wedge dd^c u_q^j \wedge T \mapsto u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T$ in Ω.*

Proof. The assumption on the Hausdorff dimension of $X := M(u_{j_1}) \cap \ldots \cap M(u_{j_m}) \cap \operatorname{supp} T$, implies that X is in the envelope of $p-m+1$ pseudoconvexity of $\Omega \backslash X$, as in the proof of Corollary 3.4 so the result follows from Theorem 3.5.

A weaker form of the corollary was proved recently by Demailly [De]. He requires that $\Lambda^{2p-2m+1}(M(u_{j_1}) \cap \ldots \cap M(u_{j_m}) \cap \operatorname{supp} T) = 0$. The case when $M(u_j)$ is empty, $1 \leqq j \leqq q$ is due to Bedford and Taylor [BT].

Remark 3.7 Theorem 3.5 gives a Thullen type extension for currents of type uT. More precisely, let T be a positive closed current in Ω, of bidimension (p, p). Let u be a plurisubharmonic function in Ω and V an analytic subvariety of dimension p in Ω. If $M(T, u)$ is contained in V and $\Omega \backslash M(T, u)$ intersects every irreducible branch of V then $M(T, u) = \emptyset$. The example of Shiffman–Taylor in [Siu] is a case where V is of codimension 1 in $\mathbb{C}^2$ and $udd^c u$ has locally bounded mass in $\mathbb{C}^2 \backslash V$ and $M(T, u) = V$.

A natural question about the operator $(dd^c)^k$ is the following: Let (u_j) be a sequence of plurisubharmonic functions in Ω. Find the right notion of convergence such that $u_j \mapsto u$ implies $(dd^c u_j)^k \mapsto (dd^c u)^k$ in the sense of currents. Cegrell [Ce] and Lelong [Le] observed that the convergence of (u_j) to u in L^p_{loc}, $p < \infty$, is not enough. We start with a refinement of their example.

Proposition 3.8 *There exists a uniformly bounded sequence (v_p) of plurisubharmonic functions in the unit bidisc $\Delta \subset \mathbb{C}^2$ such that:*
(i) $v_p(z) \mapsto v(z)$ for every $z \in \Delta$ except on a set of Hausdorff dimension 2. More precisely, given any $\varepsilon > 0$ and $\alpha > 0$, there are balls $B(x_i, r_i)$ with $\sum r_i^{2+\varepsilon} < \infty$ such that for $p > p(\varepsilon), v_p > v - \varepsilon$ on $B \backslash \bigcup B(x_i, r_i)$.
(ii) $(dd^c v_p)^2$ does not converge to $(dd^c v)^2$.

Proof. Let $v_p(z,w) := |z^{2p}+w^{2p}|^{1/(2p)}$ and $v(z,w) = \max(|z|,|w|)$. Clearly $0 \leqq v_p \leqq 2^{1/p}v$. Let $\omega := \{|z| \neq |w|\}$. We have uniform convergence on compact subsets of ω. Since through every point q in $\mathbb{C}^2$ there exists a disc Δ_q on which v_p is harmonic, it follows that $(dd^c v_p)^2 = 0$, see [FS1, Lemma 6.9]. On the other hand one computes easily that $(dd^c v)^2 \neq 0$ since $v = 1$ on the boundary of the unit polydisc but we do not have that $1 \leqq v$ as should be the case by maximum principle if $(dd^c v)^2 = 0$, see [BT].

We now study for which points $w = ze^{i\theta}, v_p(z,w)$ does not converge to $v(z,w)$. We have $v_p(z,e^{i\theta}z) = |z||1+e^{2ip\theta}|^{1/(2p)} = 2^{1/2p}|z||\cos p\theta|^{1/(2p)}$.

Let $E = \{e^{i\theta}; \liminf_p |\cos p\theta|^{1/p} < 1\}$. Let $\varepsilon_k = 1-1/k$. Then $E \subset \bigcup_k \bigcap_N \bigcup_{p>N} \{|\cos p\theta| < \varepsilon_k^p\} =: \bigcup E_k$. The set $\{|\cos p\theta| < \varepsilon_k^p\}$ is contained in a union of $2p$ intervals of length at most $3\varepsilon_k^p/p$ for $p \geqq p(k)$. Given $\alpha > 0$ we have $\Lambda^\alpha(E) \leqq \sum_k \Lambda^\alpha(E_k)$, and $\Lambda^\alpha(E_k) \leqq c\sum_{p>N} p(\varepsilon_k^p)^\alpha = o(1)$ as $N \mapsto \infty$ so for every $k, \Lambda^\alpha(E_k) = 0$. Hence $\Lambda^\alpha(E) = 0$. It follows easily that $v_p(z,w) \mapsto v(z,w)$ except on a set of Hausdorff dimension 2. Indeed, given any $\varepsilon > 0$ and $\alpha > 0$, there are balls $B(x_i,r_i)$ with $\sum r_i^{2+\alpha} < \varepsilon$ and such that for $p > p(\varepsilon), v_p \geqq v - \varepsilon$ on $B \backslash \bigcup_i (B(x_i,r_i))$.

The previous example is quite sharp as the following result * shows.

Theorem 3.9 *Let T_j be a sequence of positive closed currents of bidimension (p,p) in an open set Ω in $\mathbb{C}^n$. Let A be a subset in Ω with $\Lambda^{2p}(A) = 0$. Let (u_j) be a bounded sequence of plurisubharmonic functions and u a plurisubharmonic function in Ω, whose restriction to $\Omega \backslash A$ is continuous. Assume*

(i) *$T_j \mapsto T$ as currents.*
(ii) *$u_j \mapsto u$ in L^1_{loc}.*
(iii) *For every compact $K \subset \Omega \backslash A$ and $\varepsilon > 0$ there exists $j(K)$ such that for $j \geqq j(K) u_j \geqq u - \varepsilon$.*

Then $u_j T_j \mapsto uT$ and $dd^c u_j \wedge T_j \mapsto dd^c u \wedge T$.

Proof. Observe that we do not assume that A is closed.

Let S be a limit point of the bounded sequence of currents $u_j T_j$. By Proposition 3.2 we know that $S \leqq uT$. Let γ be a positive form with compact support in $\omega \Subset \omega_1 \Subset \Omega$. Since (u_j) is bounded, we can assume that $0 \leqq u_j \leqq M$. We have to show that $\int uT \wedge \gamma \leqq \liminf \int u_j T_j \wedge \gamma$.

Since the mass of (T_j) is bounded by a constant C on ω_1, for every ball $B(x,r) \subset \omega_1$, we have [Le1]

$$\sigma_{T_j} B(x,r) \leqq Cr^{2p} . \tag{1}$$

Fix $\varepsilon > 0$ and let $B(x_j,r_j)$ be a sequence of balls such that $A \subset \bigcup B(x_i,r_i)$ and $\sum_i r_i^{2p} < \varepsilon$. It follows from (1) that there exists an open set $U \supset A \cap \omega$ such that $\sigma_{T_j}(U) < \varepsilon$ independently of j. Let $K = \operatorname{supp} \gamma \cap (\Omega \backslash U)$. Since $u_j T_j \geqq 0$,

**Note.* We thank Russakovski for pointing out that a weaker version of Theorem 3.9 has been proved by Ronkin [Ro].

we have for $j \geqq j(K)$

$$\int u_j T_j \wedge \gamma = \int_{\omega \setminus U} (u_j - u) T_j \wedge \gamma + \int_{\omega \setminus U} u T_j \wedge \gamma + \int_U u_j T_j \wedge \gamma$$
$$\geqq -C\varepsilon + \int_{\omega \setminus U} u T_j \wedge \gamma .$$

Let $\tilde{u}$ be a continuous extension of u from $\operatorname{supp}\gamma \setminus U$ to $\operatorname{supp}\gamma, 0 \leqq \tilde{u} \leqq M$. Since for all $j, \sigma_{T_j}(U) < \varepsilon$, the last integral is close to $\int_\omega \tilde{u} T_j \wedge \gamma$ which converges to $\int uT \wedge \gamma$. We have then proved that

$$\int uT \wedge \gamma \leqq \lim_j \int u_j T_j \wedge \gamma$$

which is what we wanted to show.

4 Bezout theorem for currents in $\mathbb{P}^k$

Let ω be the standard Kahler form on $\mathbb{P}^k$, normalized such that $\int_{\mathbb{P}^k} \omega^k = 1$. If $\pi : \mathbb{C}^{k+1} \mapsto \mathbb{P}^k$ is the canonical projection, then $\pi^* \omega = dd^c \log \|z\|$ where $z = (z_0, z_1, \ldots, z_k)$ and $\| \ \|$ is the Euclidean norm in $\mathbb{C}^{k+1}$. Given a positive closed current T of bidimension (p,p) we define $\|T\| := \int T \wedge \omega^p$. When T corresponds to integration on an analytic manifold V, then $\|T\|$ is just the $2p$-volume of V. We will need the following standard result.

Proposition 4.1 *Let R be a closed positive $(1,1)$ current on $\mathbb{P}^k$.*
(i) There exists a plurisubharmonic function v on $\mathbb{C}^{k+1}$ such that $\pi^* R = dd^c v, v(\lambda z) = c \log |\lambda| + v(z)$ for $\lambda \in \mathbb{C}, z \in \mathbb{C}^k$.
(ii) If v is a plurisubharmonic function on $\mathbb{C}^{k+1}$, satisfying the previous homogeneity condition, then there is a decreasing sequence (v_ε) of plurisubharmonic functions on $\mathbb{C}^{k+1}$ satisfying the homogeneity condition, such that $v_\varepsilon \in C^\infty(\mathbb{C}^{k+1} \setminus \{0\})$ and $v_\varepsilon \searrow v$.

Proof. A proof of (i) can be found in Theorem 4.1 of [FS1], see also [LG].
For simplicity assume $c = 1$. If α is an approximation of the identity in $\mathbb{C}^{k+1}$ depending only on $\|z\|$, we define

$$v_\varepsilon(z) := \frac{1}{\|z\|^{2(k+1)}} \int v(w) \alpha_\varepsilon \left(\frac{z-w}{\|z\|} \right) d\lambda(w) = \int v(z - \|z\| w) \alpha_\varepsilon(w) d\lambda(w)$$

where λ denotes Lebesgue measure. The proof is then the same as in Theorem 7.6 of Lelong and Gruman [LG].

Proposition 4.2 *Let R be a positive closed current of bidegree $(1,1)$ on $\mathbb{P}^k$. Let v be a plurisubharmonic function on $\mathbb{C}^{k+1}$ satisfying:*
(i) $\pi^* R = dd^c v$
(ii) $v(\lambda z) = c \log |\lambda| + v(z), \lambda \in \mathbb{C}$.
Then $\|R\| = c$.

Proof. We can assume $c = 1$. Let (v_ε) be the sequence of plurisubharmonic functions constructed in Proposition 4.1. Consider the positive closed current on $\mathbb{P}^k, (R_\varepsilon)$, defined by the relation $\pi^* R_\varepsilon = dd^c v_\varepsilon$. Let $u_\varepsilon[z_0 : \ldots : z_k] = v_\varepsilon(z_0, \ldots, z_k) - \log \|z\|$. The function u_ε is smooth and well defined on $\mathbb{P}^k$. Moreover $R_\varepsilon - \omega = dd^c u_\varepsilon$. Since $v_\varepsilon \mapsto v$ in L^1_{loc}, then $R_\varepsilon \mapsto R$ in the sense of currents. Using Stoke's theorem we get:

$$\begin{aligned} \|R\| &= \int R \wedge \omega^{k-1} = \lim_{\varepsilon \mapsto 0} \int R_\varepsilon \wedge \omega^{k-1} \\ &= \lim_{\varepsilon \mapsto 0} \int \omega^k + dd^c u_\varepsilon \wedge \omega^{k-1} = \int \omega^k = 1 \, . \end{aligned}$$

Remark 4.3 Let P be a homogeneous polynomial of degree d in $\mathbb{C}^{k+1}$. Let $V = \{z \in \mathbb{P}^k, P(z) = 0\}$ and $[V]$ the current of integration on V. Since $\pi^*[V] = dd^c \log |P|$, we get $\text{vol}(V) = \|V\| = \int [V] \wedge \omega^{k-1} = d$.

Let T be a positive closed current of bidimension (p, p) in $\mathbb{P}^k$. Let $R_1, \ldots, R_q$ be closed positive currents of bidegree $(1,1)$. Locally in $\mathbb{P}^k$ each R_j can be written as $dd^c u_j$ where u_j is a plurisubharmonic function. We will say that $(T, R_1, \ldots, R_q)$ admit a wedge product if for every $x \in \mathbb{P}^k$, there exists an open set ω containing x, and plurisubharmonic functions $(u_1, \ldots, u_q)$ on ω such that for every $j, 1 \leqq j \leqq q, R_j = dd^c u_j$, and moreover for all $i_1, \ldots, i_m \in [1, q]$, the set

$$X := \operatorname{Supp} T \cap M(u_{i_1}) \cap \ldots \cap M(u_{i_m}) \cap \omega$$

is in the envelope of $p - m + 1$ pseudoconvexity of $\omega \backslash X$.

Under this assumption and using Theorem 3.5 we can define the wedge product $T \wedge R_1 \wedge \ldots \wedge R_q$.

Theorem 4.4 *Let T be a positive closed current of bidimension (p, p) on $\mathbb{P}^k$. Let $R_1, \ldots, R_q$ be positive closed currents of bidegree $(1,1)$ on $\mathbb{P}^k$. Assume that $(T, R_1, \ldots, R_q)$ admit a wedge product. Then*

$$\|T \wedge R_1 \ldots \wedge R_q\| = \|T\| \|R_1\| \ldots \|R_q\| \, .$$

In particular $T \wedge R_1 \wedge \ldots \wedge R_q$ is nonzero and $\operatorname{supp} T \cap \operatorname{supp}(R_1) \cap \ldots \cap \operatorname{supp}(R_q) \neq \emptyset$.

Proof. Without loss of generality we can assume that $\|T\| = \|R_1\| = \ldots = \|R_q\| = 1$. Assume $\pi^* R_q = dd^c v, v(\lambda z) = \log |\lambda| + v(z)$. Then as in the proof of Proposition 4.2, $R_q = \lim_{\varepsilon \mapsto 0} R_q^\varepsilon$ where $\pi^* R_q^\varepsilon = dd^c v_\varepsilon$. The assumption that $(T, R_1, \ldots, R_q)$ admit a wedge product and Theorem 3.5 imply that $T \wedge R_1 \wedge \ldots \wedge R_q = \lim_{\varepsilon \mapsto 0} T \wedge R_1 \wedge \ldots \wedge R_{q-1} \wedge R_q^\varepsilon$. (Indeed convergence of currents has to be checked locally, and $v_\varepsilon \searrow v$ in a given chart in $\mathbb{P}^k$.)

We can also assume that $p + q = k$. Hence using Stokes theorem we get if $u_\varepsilon[z] = v_\varepsilon(z) - \log |z|$

$$\begin{aligned} \int T \wedge R_1 \wedge \ldots \wedge R_q &= \lim_{\varepsilon \mapsto 0} \int T \wedge R_1 \wedge \ldots \wedge R_{q-1} \wedge R_q^\varepsilon \\ &= \lim_{\varepsilon \mapsto 0} \int T \wedge R_1 \wedge \ldots \wedge R_{q-1} \wedge (\omega + dd^c u_\varepsilon) \\ &= \int T \wedge R_1 \wedge \ldots \wedge R_{q-1} \wedge \omega \, . \end{aligned}$$

Repeating the procedure we then show that

$$\int T \wedge R_1 \wedge \ldots \wedge R_q = \int T \wedge \omega^q = 1 .$$

Remark 4.5 Let $P_1, \ldots, P_q$ be homogeneous polynomials of degree $d_1, \ldots, d_q$ in $\mathbb{C}^{k+1}$. Let $V_j = \{z \in \mathbb{P}^k; P_j(z) = 0\}$. Assume $\operatorname{codim}(V_{j_1} \cap \ldots \cap V_{j_m}) = m$. Then the currents $([V_1], \ldots, [V_q])$ admit a wedge product. Indeed if u_j is such that on a given chart on $\mathbb{P}^k, dd^c u_j = [V_j]$, then $M(u_{i_1}) \cap \ldots \cap M(u_{i_m})$ has dimension $k - m$, so is in the $k - m + 1$ envelope of the complement. So Theorem 4.4 and Remark 4.3 gives in this case Bezout's theorem. See Demailly [De] who shows that $[V_1] \wedge \ldots \wedge [V_q]$ is the current of integration of the intersection with multiplicity.

Theorem 4.6 *Let T be a nonzero, positive closed current of bidegree (p, p) in P^k. Let $X = \operatorname{supp} T$. For any algebraic variety $V = \{h_1 = \ldots = h_l = 0\}, X \cap V \neq \emptyset$ provided $p + l \leqq k$ and provided V is a geometric complete intersection.*

Proof. Let ω be the Kahler form on $\mathbb{P}^k$. Replacing T by $T \wedge \omega^{k-(p+l)}$ we can assume that $p + l = k$. We assume that $X \cap V = \emptyset$.

Define

$$u[z_0 : \ldots : z_k] = \log \frac{\|z\|^{2s}}{|h_1|^2 + \ldots + |h_l|^2}$$

where we have assumed that the polynomials h_j are homogenous of degree s.

The function u is smooth in $\mathbb{P}^k \backslash V$. Let S be the current in $\mathbb{P}^k$ defined by the relation:

$$\pi^* S = (dd^c \log(|h_1|^2 + \ldots |h_l|^2))^{l-1} .$$

We show at first that in $\mathbb{P}^k \backslash V$, we have

$$dd^c u \wedge S = 2s\omega \wedge S .$$

In local coordinates

$$dd^c u \wedge S = dd^c \log \|z\|^{2s} \wedge S - (dd^c \log \|h\|^2)^l$$

and the last term is zero outside V. Indeed, if for example $h_1 \neq 0, \log \|h\|^2 = \log |h_1|^2 + \log \left(1 + \frac{|h_2|^2}{|h_1|^2} + \ldots + \frac{|h_l|^2}{|h_1|^2}\right)$. If we consider the variety $\frac{h_m}{h_1} = c^{te}$, we see that $dd^c \log \|h\|^2$ has at least $k - l + 1$ zero eigenvalues, the result follows.

Let θ be a nonnegative test function with support in $\mathbb{P}^k \backslash V$ and with value 1 in a neighborhood of X. Since S is smooth and closed in a neighborhood of $\operatorname{supp} T$, the current $T \wedge S$ is well defined and

$$\langle T \wedge S, \theta dd^c u \rangle = \langle T \wedge S, dd^c(u\theta) \rangle = \langle dd^c T \wedge S, u\theta \rangle = 0$$

and hence $T \wedge S \wedge \omega = 0$. We can vary slightly the h_j's to obtain varieties V_t close to V and still not intersecting X. We obtain for the corresponding forms

S_i, that $T \wedge S_i \wedge \omega = 0$. It is clear that if we take enough perturbations of V, this implies that $T = 0$, a contradiction.

Theorem 4.7 *Let T be a positive, closed $(1,1)$ current on $\mathbb{P}^k$. Assume $\pi^*T = dd^c v$ with v plurisubharmonic and continuous on $\mathbb{C}^{k+1}\backslash\{0\}$. Let $X_l = \operatorname{supp} T^l$. Then for $2l \leqq k, X_l$ is connected.*

Proof. We first observe that a positive closed current S of bidegree (l,l) is a limit of smooth positive closed currents. Indeed it is enough to take an approximation of identity in $\operatorname{Aut}(\mathbb{P}^k)$ and average the current S with respect to the approximation of identity. Let $\rho_\varepsilon(g)$ be an approximation of identity on the group $U(k)$ which acts transitively on $\mathbb{P}^k$. Let μ be the Haar measure for $U(k)$. For a positive (l,l) current S define S_ε by

$$S_\varepsilon = \int_{U(k)} \rho_\varepsilon(g)(g_*S)d\mu(g) .$$

If $\rho_\varepsilon \geqq 0$, S_ε is positive, closed and smooth [deRh]. The previous formula has to be understood as follows: If ϕ is a test form,

$$\langle S_\varepsilon, \phi\rangle = \int \rho_\varepsilon(g)\langle S, g^*\phi\rangle d\mu(g) .$$

It is also known, Dold [Do] or de Rham [deRh], that there exists a constant c such that $S - c\omega^l = dd^c H$ where H has integrable coefficients. Observe that since $S \geqq 0$ then $c \geqq 0$.

Since T has locally a continuous potential u, i.e. $dd^c u = T$, we can define T^l for any $l \leqq k$. Assume there exists two disjoint open sets U_1, U_2 each one intersecting $X_l = \operatorname{supp} T^l$ and such that $X_l \cap \partial U_1 = X_l \cap \partial U_2 = \emptyset$. Let $S_1 = T^l|U_1$ and $S_2 = T^l|U_2$. Clearly S_1 is a positive closed current. Let (S_1^n) be a sequence of smooth positive closed currents $S_1^n \mapsto S_1$. Let B be a ball where $T = dd^c u$. Since all the currents are positive, hence with measure coefficients, we have $uS_1^n \mapsto uS_1$, on B hence $dd^c u \wedge S_1^n \mapsto dd^c u \wedge S_1$. Inductively we get $(dd^c u)^l \wedge S_1^n \mapsto (dd^c u)^l \wedge S_1$, hence $T^l \wedge S_1^n \mapsto T^l \wedge S_1$. Since T^l has no mass on ∂U_2 we also have $S_2 \wedge S_1^n \mapsto S_2 \wedge S_1$.

Let $s = k - 2l$ and let $S_2 = c_2\omega^l + dd^c H_2$. We have

$$\begin{aligned}\int S_2 \wedge S_1 \wedge \omega^s &= \lim_n \int S_2 \wedge S_1^n \wedge \omega^s \\ &= \lim_n \int c_2 S_1^n \wedge \omega^{l+s} + \int dd^c H_2 \wedge S_1^n \wedge \omega^s .\end{aligned}$$

Since S_1^n are smooth, the last integral is zero. Hence

$$\int S_2 \wedge S_1 \wedge \omega^s = c_2 \int \omega^{l+s} \wedge S_1 .$$

But $S_2 \neq 0$ implies $c_2 \neq 0$ and $S_1 \neq 0$ implies that $S_1 \wedge \omega^{l+s} \neq 0$. Consequently the previous relation shows that $S_1 \wedge S_2 \neq 0$, contradicting the assumption that $\operatorname{supp} S_1 \cap \operatorname{supp} S_2 = \emptyset$.

5 Examples from holomorphic dynamics in $\mathbb{P}^k$

For background on this section we refer to [FS1], [FS2] and [HP].

Let f be a holomorphic surjective map on $\mathbb{P}^k$. Let $F = [F_0 : \ldots : F_k]$ be a lifting of f to $\mathbb{C}^{k+1}$. Let d be the common degree of the polynomials F_j. The fact that f is holomorphic implies that $F^{-1}(0) = (0)$ and hence, there exists a constant $c > 0$ such that $1/c\|z\|^d \leqq \|F(z)\| \leqq c\|z\|^d$. Hence out of 0 we have

$$|1/d^{n+1} \log \|F^{n+1}\| - 1/d^n \log \|F^n\|| \leqq c_1/d^n$$

and the function

$$G(z) := \lim_{n\mapsto\infty} 1/d^n \log \|F^n(z)\|$$

is continuous on $\mathbb{C}^{k+1}\backslash\{0\}$ since the convergence is uniform. Clearly we have:

(i) $G(\lambda z) = \log|\lambda| + G(z), \lambda \in \mathbb{C}$
(ii) G is plurisubharmonic in $\mathbb{C}^{k+1}$
(iii) $G((F(z)) = dG(z)$.

Definition 5.1 *Let $f : \mathbb{P}^k \mapsto \mathbb{P}^k$ be in the space H_d of holomorphic self maps on $\mathbb{P}^k$ given by polynomials of degree d. For $0 \leqq l \leqq k-1$, a point $p \in \mathbb{P}^k$ belongs to the Fatou set F_l if there exists a neighborhood $U(p)$ such that for every $q \in U(p)$ there exists a complex variety X_q of codimension l such that $\{f^n|X_q\}$ is equicontinuous. Observe that F_0 is the largest open set where f^n is equicontinuous. We call F_0 the Fatou set. We have $F_0 \subset F_1 \subset \ldots \subset F_{k-1}$.*

Let $J_l := \mathbb{P}^k\backslash F_l$, we call J_l the Julia set of order l.

Let T be the $(1,1)$ positive closed current defined on $\mathbb{P}^k$ by the relation $\pi^*T = dd^cG$. Since G is continuous on $\mathbb{C}^{k+1}\backslash\{0\}, T$ has a continuous potential on any chart in $\mathbb{P}^k$, for example if $z_0 \neq 0, T = dd^cG(1, z_1, \ldots, z_k)$. Hence we can define the closed positive currents of bidegree (l,l) by the relation $\pi^*(T^l) = (dd^cG)^l$ for any $1 \leqq l \leqq k$. We define $J'_{l-1} = \operatorname{supp} T^l, 1 \leqq l \leqq k$, and $F'_{l-1} = \mathbb{P}^k\backslash J'_{l-1}$.

Theorem 5.2 *Let $f : \mathbb{P}^k \mapsto \mathbb{P}^k$ be a surjective holomorphic map of degree $d \geqq 2$ on $\mathbb{P}^k$. Let T be the positive closed $(1,1)$ current associated to f. Then*

(i) *J'_{k-1} is nonempty and for every $0 \leqq l < k, f(J'_l) = f^{-1}(J'_l) = J'_l$. Moreover $J'_l \subset J_l$ for $0 \leqq l < k$.*
(ii) *Support $T = J'_0 = J_0$ and F_0 is a domain of holomorphy. For $0 \leqq l < k, F'_l$ is $(k-l-1)$ pseudoconvex.*
(iii) *J'_{l-1} is connected if $2l \leqq k$.*
(iv) *If X is an algebraic variety of dimension r, which is a complete intersection, then $T^r \wedge [X]$ is a positive measure of total mass $\operatorname{vol}(X)$. Moreover X intersects every component of* $\operatorname{supp} T^r$.

Proof. It follows from Proposition 4.2 that $\|T\| = 1$ since $G(\lambda z) = \log|\lambda| + G(z)$. Theorem 4.4 implies that for every $1 \leqq l \leqq k \|T^l\| = 1$, in particular T^k is a probability measure, hence $J'_{k-1} = \operatorname{supp} T^k$ is nonempty.

Let $G_n := 1/d^n \log \|F^n\|$ and observe that G_n are smooth and converge uniformly to G on $\mathbb{C}^{k+1}\setminus\{0\}$. If $(dd^c G_n)^l \mapsto 0$, then $(dd^c G_n \circ F)^l = (dd^c(G_{n+1}d))^l \mapsto 0$, hence F_l' and J_l' are totally invariant.

We prove now that $J_l' \subset J_l, 0 \leqq l < k$. We have to show that if $p \in F_l$ then T^{l+1} vanishes in a neighborhood of p. Let U be a neighborhood of $p, q \in U$, and let X_q be an analytic variety of codimension l through q such that $f^n|X_q$ is equicontinuous. Let $f^{n_l}|X_q \mapsto h$. This means that we can find non zero holomorphic functions λ_{n_l} on X_q such that $\frac{F^{n_l}}{\lambda_{n_l}}$ converge to a holomorphic function H. We can write

$$G_{n_l} = 1/d^{n_l} \log \|F^{n_l}/\lambda_{n_l}\| + 1/d^{n_l} \log |\lambda_{n_l}| \,.$$

The first sequence converges uniformly to zero on $\pi^{-1}(X_q)$ and the sequence $1/d^n \log |\lambda_{n_l}|$ is harmonic on any analytic disc in $\pi^{-1}(X_q)$. So G is harmonic on any analytic disc in $\pi^{-1}(X_q)$. Hence if G_1 is a local section of G in U, G_1 is harmonic on any analytic disc contained in X_q. Assume that U is a polydisc $\Delta^{k-l-1} \times \Delta^{l+1}$. For every $a \in \Delta^{k-l-1}$ and every q in $a \times \Delta^{l+1}, X_q^a := a \times \Delta^{l+1} \cap X_q$ is at least of dimension one, and G_1 is harmonic on any analytic disc contained in X_q^a. Therefore Lemma 6.9 in [FS2] implies that $(dd^c G_1(a,w))^{l+1} = 0$ in $a \times \Delta^{l+1}$. Since a was arbitrary, standard results in slicing theory applied to the current T^{l+1} [HS] or [Fe], imply that T^{l+1} is zero on U.

(ii) We show that supp $T = J_0$. We have that supp $T \subset J_0$, in this case no slicing is needed since the first part of the proof shows that G is pluriharmonic on $\pi^{-1}(F_0)$. Let U be an open set disjoint from supp T. We want to prove that $U \subset F_0$. We can assume U is a ball. We know that G is pluriharmonic on $\pi^{-1}(U)$, hence $G = \log |h|$ where h is holomorphic on $\pi^{-1}(U)$. Since

$$|G - G_n| \leqq c/d^n$$

we get

$$e^{-c} \leqq \frac{\|F^n\|}{|h^{d^n}|} \leqq e^c \,.$$

So the family (f^n) is normal on U.

The fact that $F_0 = \mathbb{P}^k\setminus J_0$ is pseudoconvex follows from Corollary 2.6. Since the Levi problem has a positive solution for domains in $\mathbb{P}^k$, it follows that F_0 is a domain of holomorphy. Corollary 2.6 also implies that F_{l-1}', the complement of supp T^l is $(k-l)$-pseudoconvex.

(iii) Theorem 4.7 shows that J_{l-1}' is connected if $l \leqq 2k$.

(iv) Since T has locally a continuous potential, T^r and $[X]$ admit a wedge product. The current $[X]$ is given locally as $dd^c \log |P_1| \wedge \ldots \wedge dd^c \log |P_{k-r}|$ where (P_l) are homogeneous polynomials. So we can apply Theorem 4.4 and we get that $\|T^r \wedge [X]\| = \|[X]\|$. Let Y be a component of suppT^r. Assume $Y \cap X = \emptyset$. Let U be a neighborhood of Y such that $U \cap X = \emptyset$. Let $S = T^r|U, S$ is a positive closed current of bidegree (r,r) to which we can apply Theorem 4.6, hence supp S intersects X, a contradiction.

We next show that f cannot be hyperbolic on J_l' for $l < k-1$. Let $f \in H_d, f : \mathbb{P}^k \mapsto \mathbb{P}^k$. We define first hyperbolicity, see Ruelle [Ru]. Let $K \subset \mathbb{P}^k$ be a compact set. We assume that K is *surjectively forward invariant*, that is $f(K) = K$. The space $\hat{K} = K^{\mathbb{N}}$ of orbits $\{x_n\}_{n=-\infty}^{0}, f(x_n) = x_{n+1}$ is compact in the product topology. By the tangentbundle T_k of $\hat{K}$ we mean the space of (x, ξ) where $x = \{x_n\} \in \hat{K}$ and $\xi \in T_{\mathbb{P}^2}(x_0)$ is a tangent vector. We give this tangentbundle the obvious topology. Then f lifts to a homeomorphism $\hat{f} : \hat{K} \mapsto \hat{K}$ and f' lifts to a map f' on T_k in the obvious way.

Definition 5.3 *Let $K \subset \mathbb{P}^k$ be a compact surjectively forward invariant set. Then f is* **hyperbolic** *on K if there exists a continuous splitting $E^u \oplus E^s$ of the tangent space bundle of $\hat{K}$ such that $\hat{f}'$ preserves the splitting and for some constants $C, c > 0, \lambda > 1, \mu < 1$ depending on the choice of a Hermitian metric on $\mathbb{P}^2$,*

$$|D\hat{f}^n(\xi)| \geqq c\lambda^n |\xi|, \xi \in E^u$$

$$|D\hat{f}^n(\xi)| \leqq C\mu^n |\xi|, \xi \in E^s, \quad n = 1, 2, \ldots$$

Theorem 5.4 *Let $f : \mathbb{P}^k \mapsto \mathbb{P}^k, f \in H_d$. If $\operatorname{supp} \mu = J_{k-1}'$ is hyperbolic, then the unstable dimension of J_{k-1}' is equal to k.*

Proof. Suppose the stable dimension is equal to $l \geqq 1$. Then through every point $x \in J_{k-1}'$ there is an analytic disc Δ_x on which the family (f^n) is equicontinuous. Hence if G_1 is a local potential for μ, i.e. $(dd^c G_1)^k = \mu$, then G_1 is harmonic on Δ_x. The theorem will then be a consequence of the following lemma.

Lemma 5.5 *Let u be a continuous plurisubharmonic function in a ball $\bar{B} \subset \mathbb{C}^k$. Let $(dd^c u)^k = \nu$. Assume that through every point x of $X := \operatorname{supp} \nu$ there is an analytic disc Δ_x such that $u|\Delta_x$ is harmonic, then $\nu = 0$.*

Proof. We are going to prove that u is maximal and hence $\nu = 0$. Let v be a continuous function on $\bar{B}$ which is plurisubharmonic on B, with $v \leqq u$ on ∂B. Let $M := \sup_B v - u$, assume $M > 0$. Let $K = \{z; z \in B, (v-u)(z) = M\}$. Let p be a peak point for a function $h \in C(K)$ which is a uniform limit of K of holomorphic polynomials, i.e., $h(p) = 1$ and $|h| < 1$ on $K \backslash \{p\}$. If $p \in X$, then since $v \leqq u + M$ and $u|\Delta_p$ is harmonic then $v - u = M$ on Δ_p, hence $\Delta_p \subset K$, contradicting that p is a peak point for h. So $p \in B \backslash X$. Let $B(p,r) \Subset B$ be a ball such that $B(p,r) \cap X = \emptyset$. Since $(dd^c u)^k = 0$ on $B(p,r)$, it follows from [GS] that given any $z \in B(p,r)$, there exists a probability measure σ_z supported on $\partial B(p,r)$ such that $u(z) = \int u d\sigma_z$, and moreover for any continuous function on $\bar{B}(p,r)$, plurisubharmonic on $B(p,r)$

$$\phi(z) \leqq \int \phi d\sigma_z .$$

Since $M = v(p) - u(p) \leqq \int (u-v) d\sigma_p \leqq M$, we have that support σ_p is contained in K, and p cannot be a peak point for h, since $h(p) = \int h d\sigma_p$, hence K is empty and $\nu = 0$.

Theorem 5.6 *Let $f: \mathbb{P}^k \mapsto \mathbb{P}^k$ be a holomorphic map of degree $d \leqq 2$. Then f cannot be hyperbolic on $\mathbb{P}^k$ nor on J_l' for $l < k - 1$.*

Proof. Assume f is hyperbolic on $\mathbb{P}^k$. Since the critical set is nonempty the fibre dimension of E^u is $\leqq k-1$. If $\dim E^s = k$ then all periodic orbits are attractive. Pick one, p, with immediate basin of attraction Ω. Since f is surjective, $\partial\Omega$ is a non empty, compact, forward invariant subset of $\mathbb{P}^k$. Hyperbolicity implies that orbits of points $q \in \Omega$ close to $\partial\Omega$ converge to $\partial\Omega$ contradicting that they are in the attractive basin of p. Hence $1 \leqq \dim E^s \leqq k-1$. Let $\dim E^s = l$. Then we have a foliation of $\mathbb{P}^k$ by stable manifolds of dimension l, and on each manifold the family (f^n) is equicontinuous, so $\mathbb{P}^k \subset F_{k-l}$. Since $F_{k-l} \subset F_{k-1}$, we get $J_{k-1} = \emptyset$ which is a contradiction.

Let $l < k-1$ and assume J_l' is hyperbolic. Theorem 5.2(iv) shows that every component of J_l' intersects C, the critical set. Hence the fibre dimension of $E^u \leqq k-1$, so $\dim E^s \geqq l$. This implies that through every point p in $J_l' \supset \operatorname{supp}\mu$ there exists an analytic disc Δ_p on which G_1 is harmonic, G_1 is a local solution of $dd^c G_1 = T$.

So Lemma 5.6 applies and $\mu = 0$, a contradiction.

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