OLD AND NEW APPROACHES TO LIBOR MODELING

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Abstract. In this article, we review the construction and properties of some popular approaches to modeling LIBOR rates. We discuss the following frameworks: classical LIBOR market models, forward price models and Markov-functional models. We close with the recently developed affine LIBOR models.

1. Introduction

Interest rate markets are a large and important part of global financial markets. The figures published by the Bank for International Settlements (BIS) show that interest rate derivatives represent more than 60% of the over-the-counter markets over the years, in terms of notional amount; cf. Table 1. Hence, it is important to have models that can adequately describe the dynamics and mechanics of interest rates.

There is a notable difference between interest rate markets and stock or foreign exchange (FX) markets. While in the latter there is a single underlying to be modeled, the stock price or the FX rate, in interest rate markets there is a whole *family* of underlyings to be modeled, indexed by the time of maturity. This poses unique challenges for researchers in mathematical finance and has led to some fascinating developments.

The initial approaches to interest rate modeling considered short rates or instantaneous forward rates as modeling objects, and then deduced from them tradable rates. More recently, *effective rates*, i.e. tradable market rates such as the LIBOR or swap rate, were modeled directly. Models for effective rates consider only a discrete set of maturity dates, the so-called *tenor structure*, which consists of the dates when these rates are fixed. A review of the different approaches to modeling interest rates is beyond the scope of the present article. There are many excellent books available, focusing on the theoretical and practical aspects of interest rate theory. We refer the reader e.g. to Björk (2004), Musiela and Rutkowski (1997), Filipović (2009), or Brigo and Mercurio (2006).

The aim of this article is to review the construction and basic properties of models for LIBOR rates. We consider the following popular approaches: LIBOR market models, forward price models and Markov-functional models, as well as the recently developed class of affine LIBOR models. In section 3 we will present and discuss some basic requirements that models for LIBOR rates should satisfy. These are briefly: *positivity* of LIBOR rates, *arbitrage freeness* and *analytical tractability*.

2000 *Mathematics Subject Classification.* 91B28, 60G44.

Key words and phrases. LIBOR rate, LIBOR market model, forward price model, Markov-functional model, affine LIBOR model.

	Dec 2006	Dec 2007	Jun 2008	Dec 2008
Foreign exchange	40,271	56,238	62,983	49,753
Interest rate	291,581	393,138	458,304	418,678
Equity-linked	7,488	8,469	10,177	6,494
Commodity	7,115	8,455	13,229	4,427
Credit default swaps	28,650	57,894	57,325	41,868
Unallocated	43,026	71,146	81,708	70,742
Total	418,131	595,341	683,726	591,963

Table 1. Amounts outstanding of over-the-counter (OTC) derivatives by risk category and instrument (in billions of US dollars). Source: BIS Quarterly Review, September 2009.

There are two natural starting points for modeling LIBOR rates: the rate itself and the forward price. Although they differ only by an additive and a multiplicative constant, cf. (2.1), the model dynamics are noticeably different, depending on whether the model is based on the LIBOR or the forward price. In addition, the consequences from the point of view of econometrics are also significant.

Modeling LIBOR rates directly, leads to positive rates and arbitrage-free dynamics, but the model is not analytically tractable. On the other hand, models for the forward price are analytically tractable, but LIBOR rates can become negative. The only models that can respect all properties simultaneously are Markov-functional models and affine LIBOR models.

The article is organized as follows: in section 2 we introduce some basic notation for interest rates and in section 3 we describe the basic requirements for LIBOR models. In section 4 we review the construction of LIBOR market models, describe its shortcomings and discuss some approximation methods developed to overcome them. In section 5 we review forward price models and in section 6 we discuss Markov-functional models. Finally, in section 7 we present affine LIBOR models and in section 8 we outline the extensions of LIBOR models to the multi-currency and default risk settings.

2. Interest rate markets – notation

Let us consider a discrete tenor structure $0 = T_0 < T_1 < \cdots < T_N$, with constant tenor length $\delta = T_{k+1} - T_k$. The following notation is introduced for convenience; $\mathcal{K} := \{1, \ldots, N-1\}$ and $\overline{\mathcal{K}} := \{1, \ldots, N\}$. Let us denote by $B(t,T)$ the time-t price of a *zero coupon bond* maturing at time T; by $L(t,T)$ the time-t *forward LIBOR rate* settled at time T and received at time $T+\delta$; and by $F(t,T,U)$ the time-t *forward price* associated to the dates T and U. These fundamental quantities are related by the following basic equation:

$$1 + \delta L(t,T) = \frac{B(t,T)}{B(t,T+\delta)} = F(t,T,T+\delta). \tag{2.1}$$

Throughout this work, $\mathcal{B} = (\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ denotes a complete stochastic basis, where $\mathcal{F} = \mathcal{F}_T$, $\mathbf{F} = (\mathcal{F}_t)_{t\in[0,T]}$, and $T_N \le T < \infty$. We denote by $\mathcal{M}(\mathbb{P})$ the class of martingales on $\mathcal{B}$ with respect to the measure $\mathbb{P}$.

We associate to each date T_k in the tenor structure a *forward martingale measure*, denoted by $\mathbb{P}_{T_k}$, $k \in \overline{\mathcal{K}}$. By the definition of forward measures, cf. Musiela and Rutkowski (1997, Def. 14.1.1), the bond price with maturity T_k is the numeraire for the forward measure $\mathbb{P}_{T_k}$. Thus, we have that forward measures are related to each other via

$$\frac{\mathrm{d}\mathbb{P}_{T_k}}{\mathrm{d}\mathbb{P}_{T_{k+1}}}\Big|_{\mathcal{F}_t} = \frac{F(t,T_k,T_{k+1})}{F(0,T_k,T_{k+1})} = \frac{B(0,T_{k+1})}{B(0,T_k)} \times \frac{B(t,T_k)}{B(t,T_{k+1})}, \tag{2.2}$$

while they are related to the terminal forward measure via

$$\frac{\mathrm{d}\mathbb{P}_{T_k}}{\mathrm{d}\mathbb{P}_{T_N}}\Big|_{\mathcal{F}_t} = \frac{F(t,T_k,T_N)}{F(0,T_k,T_N)} = \frac{B(0,T_N)}{B(0,T_k)} \times \frac{B(t,T_k)}{B(t,T_N)}. \tag{2.3}$$

All forward measures are assumed to be equivalent to the measure $\mathbb{P}$.

3. Axioms for LIBOR models

In this section, we present and discuss certain requirements that a model for LIBOR rates should satisfy. These requirements are motivated both by the economic and financial aspects of LIBOR rates, as well as by the practical demands for implementing and using a model in practice. The aim here is to unify the line of thought in Hunt et al. (2000) and in Keller-Ressel et al. (2009).

A model for LIBOR rates should satisfy the following *requirements*:

(**A1**) LIBOR rates should be *non-negative*: $L(t,T) \ge 0$ for all $t \in [0,T]$.
(**A2**) The model should be *arbitrage-free*: $L(\cdot,T) \in \mathcal{M}(\mathbb{P}_{T+\delta})$.
(**A3**) The model should be *analytically tractable*, easy to implement and quick to calibrate to market data.
(**A4**) The model should provide a *good calibration* to market data of liquid derivatives, i.e. caps and swaptions.

Requirements (A1) and (A2) are logical conditions originating from economics and mathematical finance. Below, we briefly elaborate on (A3) and (A4); they stem from practical demands and are more difficult to quantify precisely. In order to clarify their difference, we point out that e.g. the Black–Scholes model obviously satisfies (A3) but not (A4).

Requirement (A3) means that we can price liquid derivatives, e.g. caps and swaptions in “closed form” in the model, so that the model can be calibrated to market data in a fast and easy manner. Ideally, of course we would like to be able to price as many derivatives as possible in closed form. Here, “closed form” is understood in a broad sense meaning e.g. Fourier transform methods; really closed form solutions á la Black–Scholes are typically hard to achieve. Hunt et al. (2000) say that the model is analytically tractable if it is driven by a *low-dimensional* Markov process. In Keller-Ressel et al. (2009), as well as in the present article, we say that a model is analytically tractable if the structure of the driving process is *preserved* under the different forward measures.

Finally, requirement (A4) means that the model is able to describe the observed data accurately, without overfitting them. We will not examine this requirement further in this article. On an intuitive level, since the models we will describe in the sequel are driven by general Markov processes or general semimartingales, we can always find a driving process that provides a good calibration to market data. However, an empirical analysis should be performed in order to identify such a driving process (cf. e.g. Jarrow et al. 2007 and Skovmand 2008, Ch. III).

4. LIBOR market models

LIBOR market models were introduced in the seminal papers of Miltersen et al. (1997), Brace et al. (1997) and Jamshidian (1997). In this framework, LIBOR rates are modeled as the exponential of a Brownian motion under their corresponding forward measure, hence they are log-normally distributed. This is the so-called *log-normal LIBOR market model*. Caplets are then priced by Black's formula (cf. Black 1976), which is in accordance with standard market practice. Later, LIBOR market models were extended to accommodate more general driving processes such as Lévy processes, stochastic volatility processes and general semimartingales, in order to describe more accurately the market data; cf. Jamshidian (1999), Glasserman and Kou (2003), Eberlein and Özkan (2005), Andersen and Brotherton-Ratcliffe (2005), Belomestny and Schoenmakers (2010) and Belomestny et al. (2009), to mention just a fraction of the existing literature.

Consider an initial tenor structure of non-negative LIBOR rates $L(0,T_k)$, $k \in \overline{K}$, and let $\lambda(\cdot,T_k) : [0,T_k] \to \mathbb{R}$ denote the volatility of the forward LIBOR rate $L(\cdot,T_k)$, $k \in K$; the volatilities are assumed deterministic, for simplicity. Let H denote a semimartingale on $(\Omega,\mathcal{F},\mathbf{F},\mathbb{P}_{T_N})$ with triplet of semimartingale characteristics $\mathbb{T}(H|\mathbb{P}_{T_N}) = (B,C,\nu)$ and $H_0 = 0$ a.s.; H satisfies certain integrability assumptions which are suppressed for brevity (e.g. finite exponential moments, absolutely continuous characteristics). The process H is driving the dynamics of LIBOR rates and is chosen to have a *tractable* structure under $\mathbb{P}_{T_N}$ (e.g. H is a Lévy or an affine process).

In LIBOR market models, forward LIBOR rates are modeled as follows:

$$L(t,T_k) = L(0,T_k)\exp\left(\int_0^t \beta(s,T_k)\mathrm{d}s + \int_0^t \lambda(s,T_k)\mathrm{d}H_s\right), \tag{4.1}$$

where $\beta(\cdot,T_k)$ is the drift term that makes $L(\cdot,T_k) \in \mathcal{M}(\mathbb{P}_{T_{k+1}})$, for all $k \in K$. Therefore, the model clearly satisfies requirements (A1) and (A2).

Now, using Theorem 2.18 in Kallsen and Shiryaev (2002), we have that

$$\begin{aligned}\beta(s,T_k) = &-\lambda(s,T_k)b_s^{T_{k+1}} - \frac{1}{2}\lambda^2(s,T_k)c_s \\ &- \int_{\mathbb{R}} \left(\mathrm{e}^{\lambda(s,T_k)x} - 1 - \lambda(s,T_k)x\right)F_s^{T_{k+1}}(\mathrm{d}x),\end{aligned} \tag{4.2}$$

such that indeed $L(\cdot,T_k) \in \mathcal{M}(\mathbb{P}_{T_{k+1}})$. Here $(b_s^{T_{k+1}}, c_s, F_s^{T_{k+1}})$ denote the differential characteristics of H under $\mathbb{P}_{T_{k+1}}$. Therefore, in order to completely understand the dynamics of the model we have to calculate the characteristics $(b_s^{T_{k+1}}, c_s, F_s^{T_{k+1}})$.

These characteristics follow readily from Girsanov's theorem for semimartingales (cf. Jacod and Shiryaev 2003, III.3.24) once we have the density between the measure changes at hand. It is convenient to express this density as a stochastic exponential. Keeping (2.3) in mind, and denoting (4.1) as follows

$$\mathrm{d}L(t,T_k) = L(t-,T_k)\mathrm{d}\widetilde{H}^k_t, \tag{4.3}$$

i.e. $\widetilde{H}^k$ is the exponential transform of the exponent in (4.1), we get from (2.1) that

$$\begin{aligned} \mathrm{d}F(t,T_k,T_{k+1}) &= \delta\,\mathrm{d}L(t,T_k) \\ &= \delta L(t-,T_k)\mathrm{d}\widetilde{H}^k_t \\ &= F(t-,T_k,T_{k+1})\frac{\delta L(t-,T_k)}{1+\delta L(t-,T_k)}\mathrm{d}\widetilde{H}^k_t \\ \Rightarrow F(t,T_k,T_{k+1}) &= F(0,T_k,T_{k+1})\,\mathcal{E}\left(\int_0^{\cdot}\frac{\delta L(s-,T_k)}{1+\delta L(s-,T_k)}\mathrm{d}\widetilde{H}^k_s\right)_t. \end{aligned} \tag{4.4}$$

Therefore, the density between the measure changes takes the form

$$\frac{\mathrm{d}\mathbb{P}_{T_{k+1}}}{\mathrm{d}\mathbb{P}_{T_N}}\Big|_{\mathcal{F}_t} = \frac{B(0,T_N)}{B(0,T_{k+1})}\times\prod_{l=k+1}^{N-1}\mathcal{E}\left(\int_0^{\cdot}\frac{\delta L(s-,T_l)}{1+\delta L(s-,T_l)}\mathrm{d}\widetilde{H}^l_s\right)_t. \tag{4.5}$$

This calculation reveals the problem of LIBOR market models: the density process between the measure changes – and thus the characteristics of H under the forward measures – does not depend *only* on the dynamics of $\widetilde{H}^k$, or equivalently on the dynamics of $\int\lambda(s,T_k)\mathrm{d}H_s$, as is the case in e.g. HJM models. It also crucially depends on *all subsequent* LIBOR rates, as the product and the terms $\frac{\delta L(\cdot,T_l)}{1+\delta L(\cdot,T_l)}$ in (4.5) clearly indicate. This means, in particular, that the structure of the model is *not preserved* under the different forward measures; e.g. if H is a Lévy or an affine process under the terminal measure $\mathbb{P}_{T_N}$, then H is neither a Lévy nor an affine process under any other forward measure – not even a time-inhomogeneous version of those. Therefore, LIBOR market models do not satisfy requirement (A3).

The semimartingale H, that drives the dynamics of LIBOR rates, has the following canonical decomposition under the terminal martingale measure $\mathbb{P}_{T_N}$

$$H_t = B_t + \int_0^t\sqrt{c_s}\mathrm{d}W_s + \int_0^t\int_{\mathbb{R}}x(\mu^H-\nu)(\mathrm{d}s,\mathrm{d}x), \tag{4.6}$$

(cf. Jacod and Shiryaev 2003, II.2.38 and Karatzas and Shreve 1991, Theorem 3.4.2) where W denotes the $\mathbb{P}_{T_N}$-Brownian motion and μ^H denotes the random measure of the jumps of H. The $\mathbb{P}_{T_N}$-compensator of μ^H is ν and $C=\int_0^{\cdot}c_s\mathrm{d}s$. Straightforward calculations using the density in (4.5) (cf. e.g. Kluge 2005 or Papapantoleon and Siopacha 2009) yield that the

$\mathbb{P}_{T_{k+1}}$-Brownian motion $W^{T_{k+1}}$ is related to the $\mathbb{P}_{T_N}$-Brownian motion via

$$W_t^{T_{k+1}} = W_t - \int_0^t \left(\sum_{l=k+1}^{N-1} \frac{\delta L(t-, T_l)}{1+\delta L(t-, T_l)} \lambda(t, T_l) \right) \sqrt{c_s} \mathrm{d}s, \tag{4.7}$$

while the $\mathbb{P}_{T_{k+1}}$-compensator of μ^H, $\nu^{T_{k+1}}$, is related to the $\mathbb{P}_{T_N}$-compensator of μ^H via

$$\nu^{T_{k+1}}(\mathrm{d}s, \mathrm{d}x) = \left(\prod_{l=k+1}^{N-1} \gamma(s, x, T_l) \right) \nu(\mathrm{d}s, \mathrm{d}x), \tag{4.8}$$

where

$$\gamma(s, x, T_l,) = \frac{\delta L(s-, T_l)}{1+\delta L(s-, T_l)} \left(\mathrm{e}^{\lambda(s, T_l)x} - 1 \right) + 1. \tag{4.9}$$

In addition, the drift term of the LIBOR rate $L(\cdot, T_k)$ relative to the $\mathbb{P}_{T_N}$ differential characteristics of H, i.e. (b, c, F), is

$$\begin{aligned} \widehat{\beta}(s, T_k) = &-\frac{1}{2}\lambda^2(s, T_k)c_s - c_s\lambda(s, T_k) \sum_{l=k+1}^{N-1} \frac{\delta L(s-, T_l)}{1+\delta L(s-, T_l)} \lambda(s, T_l) \\ &- \int_{\mathbb{R}} \left(\left(\mathrm{e}^{\lambda(s, T_k)x} - 1 \right) \prod_{l=k+1}^{N-1} \gamma(s, x, T_l) - \lambda(s, T_k)x \right) F_s(\mathrm{d}x). \end{aligned} \tag{4.10}$$

The consequences of the intractability of LIBOR market models are the following. When the driving process is a *continuous* semimartingale, then

- caplets can be priced in closed form;
- swaptions and other multi-LIBOR products cannot be priced in closed form;
- Monte-Carlo simulations are particularly time consuming, since one is dealing with coupled high dimensional stochastic processes.

When the driving process is a *general* semimartingale, then

- even caplets cannot be priced in closed form, let alone swaptions or other multi-LIBOR derivatives;
- the Monte-Carlo simulations are equally time consuming.

Several approximation methods have been developed in the literature in order to overcome these problems. We briefly review three of the proposed methods below; for other methods and empirical comparison we refer the interested reader to the review paper by Joshi and Stacey (2008).

4.1. **"Frozen drift" approximation.** The first and easiest solution to the problem is the so-called "frozen drift" approximation, where one replaces the random terms in (4.10) or (4.5) by their deterministic initial values, i.e.

$$\frac{\delta L(s-, T_l)}{1+\delta L(s-, T_l)} \approx \frac{\delta L(0, T_l)}{1+\delta L(0, T_l)}. \tag{4.11}$$

This approximation was first proposed by Brace et al. (1997), and has been used by many authors ever since. Under this approximation, the measure change becomes a structure preserving one – observe that the density in (4.5)

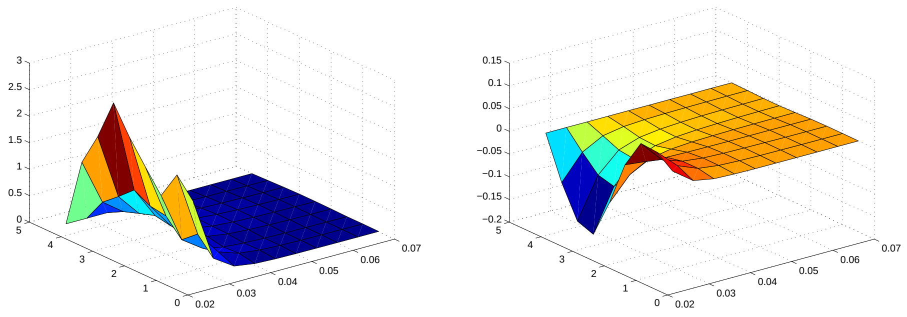

FIGURE 1. Difference in implied volatilities between the actual LIBOR and the frozen drift prices (left), and between the actual LIBOR and the Taylor approximation prices (right). Source: Papapantoleon and Siopacha (2009).

depends now only on the driving process and the volatility structure – and the resulting *approximate* LIBOR market model is analytically tractable; e.g. caplets and swaptions can now be priced in closed form even in models driven by semimartingales with jumps.

However, this method yields poor empirical results. Comparing the prices of either liquid options, or long-dated options, using the frozen drift approximation to the prices obtained by the simulation of the actual dynamics for the LIBOR rates, we can observe notable differences both in terms of prices and in terms of implied volatilities. See Figure 1 for an example. We refer to Kurbanmuradov et al. (2002), Siopacha and Teichmann (2010), and Papapantoleon and Siopacha (2009) for further numerical illustrations.

4.2. **Log-normal approximations.** The following approximation schemes for the log-normal LIBOR market model were developed by Kurbanmuradov et al. (2002). Consider the log-normal LIBOR market model driven by a one-dimensional Brownian motion, for simplicity. The dynamics of LIBOR rates (expressed under the terminal measure) take the form

$$\mathrm{d}L(t,T_k) = L(t,T_k)\big(\lambda_t(T_k)\mathrm{d}W_t + \beta_t(T_k)\mathrm{d}t\big), \tag{4.12}$$

where the drift term equals

$$\beta_t(T_k) = -\lambda_t(T_k)\sum_{l=k+1}^{N-1}\frac{\delta L(t-,T_l)}{1+\delta L(t-,T_l)}\lambda_t(T_l), \tag{4.13}$$

cf. (4.10); w.l.o.g. we can set $c \equiv 1$. A very crude log-normal approximation is to "neglect" the non-Gaussian terms in the SDE, i.e. to set $\beta_t(T_k) \equiv 0$. Of course, this approximation does not yield satisfactory results – in principle, results are even worse than the frozen drift approximation.

One can develop more refined log-normal approximations as follows: let $f(x) = \frac{\delta x}{1+\delta x}$, and define the process Z, which equals the terms that need to be approximated; i.e.

$$Z(t,T_k) = \frac{\delta L(t,T_k)}{1+\delta L(t,T_k)} = f\big(L(t,T_k)\big). \tag{4.14}$$

Applying Itô's formula, we derive the SDE that $Z(\cdot, T_k)$ satisfies

$$\begin{aligned} \mathrm{d}Z(t,T_k) &= f'(L(t,T_k))L(t,T_k)\lambda_t(T_k)\mathrm{d}W_t \\ &\qquad + \big\{f'(L(t,T_k))L(t,T_k)\beta_t(T_k) + \frac{1}{2}f''(L(t,T_k))L^2(t,T_k)\lambda_t^2(T_k)\big\}\mathrm{d}t \\ &\qquad =: A_k(t,Z)\mathrm{d}t + B_k(t,Z)\mathrm{d}W_t, \end{aligned} \tag{4.15}$$

with the initial condition $Z(0,T_k) = f(L(0,T_k))$. Note that the coefficients A_k and B_k can be calculated explicitly, by solving (4.14) for L and substituting into (4.15). Moreover, Z in A_k and B_k denotes the dependence on the whole vector $Z = [Z(\cdot,T_1),\dots,Z(\cdot,T_N)]$. The first and second *Picard iterations* for the solution of this SDE are

$$Z^0(t,T_k) = Z(0,T_k) = \frac{\delta L(0,T_k)}{1+\delta L(0,T_k)}, \tag{4.16}$$

and

$$Z^1(t,T_k) = Z(0,T_k) + \int_0^t A_k(s,Z^0)\mathrm{d}s + \int_0^t B_k(s,Z^0)\mathrm{d}W_s. \tag{4.17}$$

Note that Z^0 is constant, while $Z^1(t,T_k)$ has a Gaussian distribution since the coefficients $A_k(\cdot,Z^0)$ and $B_k(\cdot,Z^0)$ are deterministic.

Now, replacing the random terms $Z(\cdot,T_k)$ in $\beta(T_k)$ with the Picard iterates $Z^0(\cdot,T_k)$ and $Z^1(\cdot,T_k)$ leads to two different *log-normal approximations* to the dynamics of LIBOR rates. Obviously, the approximation by Z^0 is nothing else than the frozen drift approximation. The approximation by Z^1 is again log-normal, cf. (4.13) and (4.14), and yields very good empirical results. This latter approximation has the advantage that the law of the approximate LIBOR rate is known at any time t, hence the time-consuming Monte Carlo simulations can be avoided. For the empirical and numerical analysis of these approximations we refer to Kurbanmuradov et al. (2002) and Schoenmakers (2005, Ch. 6).

4.3. **Strong Taylor approximation.** Another approximation method has been recently developed by Siopacha and Teichmann (2010) and Papapantoleon and Siopacha (2009). The main idea behind this method is to replace the random terms in the drift (4.10) by their first-order strong Taylor approximations. The Taylor approximation is developed by a perturbation of the SDE for the LIBOR rates and a subsequent Taylor expansion.

Let us denote the log-LIBOR rates by $G(\cdot,T_k) := \log L(\cdot,T_k)$. Then, they satisfy the linear SDE (under the terminal measure)

$$\mathrm{d}G(t,T_k) = \widehat{\beta}(t,T_k)\mathrm{d}t + \lambda(t,T_k)\mathrm{d}H_t, \tag{4.18}$$

with initial condition $G(0,T_k) = \log L(0,T_k)$; cf. (4.1) and (4.10). We perturb this SDE by a real parameter ϵ, i.e.

$$\mathrm{d}G^\epsilon(t,T_k) = \epsilon\big(\widehat{\beta}^\epsilon(t,T_k)\mathrm{d}t + \lambda(t,T_k)\mathrm{d}H_t\big), \tag{4.19}$$

where $G^\epsilon(0,T_k) = G(0,T_k)$ for all ϵ. The superscript ϵ in the drift term $\widehat{\beta}^\epsilon(\cdot,T_k)$ is a reminder that this term is also perturbed by ϵ, since it contains all subsequent LIBOR rates; see (4.10) again. Now, the *first order strong Taylor approximation* of G^ϵ, denoted by $\mathbf{T}G^\epsilon$, is

$$\mathbf{T}G^\epsilon(t,T_k) = G(0,T_k) + \epsilon\frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0}G^\epsilon(t,T_k). \tag{4.20}$$

We denote the "first variation" process $\frac{\partial}{\partial\epsilon}\big|_{\epsilon=0}G^\epsilon(\cdot,T_k)$ by $Y(\cdot,T_k)$, and then we can deduce, after some calculations, that it has the decomposition

$$Y(t,T_k) = \int_0^t \widehat{\beta}^0(s,T_k)\mathrm{d}s + \int_0^t \lambda(s,T_k)\mathrm{d}H_s, \tag{4.21}$$

where $\widehat{\beta}^0(s,T_k) := \widehat{\beta}^\epsilon(s,T_k)|_{\epsilon=0}$. Hence, this is a *deterministic* drift term, obtained by replacing the random terms in (4.10) by their deterministic initial values. In particular, we can easily deduce from (4.21) that, for example, if H is a Lévy process then $Y(\cdot,T_k)$ is a *time-inhomogeneous* Lévy process.

Concluding, we have developed the following approximation scheme:

$$\log L(t,T_k) \approx \log L(0,T_k) + Y(t,T_k), \tag{4.22}$$

where $Y(\cdot,T_k)$ has the decomposition (4.21); compare with (4.1).

The advantage of this method is threefold: (a) it is universal, and can be applied to LIBOR models with stochastic volatility and/or jumps, (b) it is faster and easier to simulate than the actual SDE for the LIBOR rates, and (c) the empirical performance is very satisfactory; cf. Figure 1 and the aforementioned articles for further numerical illustrations. The drawback is that it is based on Monte Carlo simulations, hence computational times can become long.

5. Forward price models

Forward price models were developed by Eberlein and Özkan (2005), and further investigated by Kluge (2005); see also Eberlein and Kluge (2007). We consider a setting similar to LIBOR market models, i.e. an initial tenor structure of non-negative LIBOR rates, $\lambda(\cdot,T_k)$ denotes the volatility of the forward LIBOR rate $L(\cdot,T_k)$, and H denotes a semimartingale on $(\Omega,\mathcal{F},\mathbf{F},\mathbb{P}_{T_N})$ with triplet of characteristics (B,C,ν); again some assumptions on H are suppressed. The process H is driving the dynamics of LIBOR rates and has a *tractable* structure under $\mathbb{P}_{T_N}$ (e.g. H is a Lévy or an affine process).

Instead of modeling the forward LIBOR rate directly, one now models the forward price in a similar fashion; that is

$$1+\delta L(t,T_k) = (1+\delta L(0,T_k))\exp\left(\int_0^t \beta(s,T_k)\mathrm{d}s + \int_0^t \lambda(s,T_k)\mathrm{d}H_s\right), \tag{5.1}$$

where again the drift term is such that $L(\cdot,T_k) \in \mathcal{M}(\mathbb{P}_{T_{k+1}})$, for all $k \in K$; i.e. $\beta(\cdot,T_k)$ has similar form to (4.2). Therefore the model obviously satisfies requirement (A2).

Now, from (2.3) and (5.1), we get that the density between the forward measures is

$$\frac{\mathrm{d}\mathbb{P}_{T_{k+1}}}{\mathrm{d}\mathbb{P}_{T_N}}\Big|_{\mathcal{F}_t} = \frac{B(0,T_N)}{B(0,T_{k+1})} \times \prod_{l=k+1}^{N-1} \big(1+\delta L(t,T_l)\big) \tag{5.2}$$

$$= \frac{B(0,T_N)}{B(0,T_{k+1})} \times \exp\left(\int_0^t \sum_{l=k+1}^{N-1} \beta(s,T_l)\mathrm{d}s + \int_0^t \sum_{l=k+1}^{N-1} \lambda(s,T_l)\mathrm{d}H_s \right).$$

Observe that this density only depends on the driving process H and the volatility structures, hence we can deduce that the measure changes between forward measures are *Esscher transformations*; cf. Kallsen and Shiryaev (2002) for the Esscher transform. Analogously to eqs. (4.6)–(4.8), we have now that the $\mathbb{P}_{T_{k+1}}$-Brownian motion is related to the $\mathbb{P}_{T_N}$-Brownian motion via

$$W_t^{T_{k+1}} = W_t - \int_0^t \left(\sum_{l=k+1}^{N-1} \lambda(t,T_l) \right) \sqrt{c_s}\mathrm{d}s, \tag{5.3}$$

while the $\mathbb{P}_{T_{k+1}}$-compensator of μ^H, is related to the $\mathbb{P}_{T_N}$-compensator of μ^H via

$$\nu^{T_{k+1}}(\mathrm{d}s,\mathrm{d}x) = \exp\left(x \sum_{l=k+1}^{N-1} \lambda(s,T_l) \right) \nu(\mathrm{d}s,\mathrm{d}x). \tag{5.4}$$

Thus the structure of the driving process is preserved. For example, if H is a Lévy or an affine process under $\mathbb{P}_{T_N}$, then it becomes a *time-inhomogeneous* Lévy or affine process respectively under any forward measure $\mathbb{P}_{T_{k+1}}$. This implies that requirement (A3) is satisfied, so that caplets and swaptions can be priced in closed form. In this class of models, we have the additional benefit that we can even price some exotic path-dependent options easily using Fourier transform methods; see Kluge and Papapantoleon (2009) for an example.

The main shortcoming of forward price models is that *negative* LIBOR rates can occur, similarly to HJM models, since

$$1+\delta L(t,T_k) > 0 \;\not\Rightarrow\; L(t,T_k) > 0 \qquad \text{for all } t \in [0,T_k].$$

Therefore, this model can violate requirement (A1).

6. Markov-functional models

Markov-functional models were introduced in the seminal paper of Hunt, Kennedy, and Pelsser (2000). In contrast to the other approaches described in this review, the aim of Markov-functional models is not to model some fundamental quantity, e.g. LIBOR or swap rates, directly. Instead, Markov-functional models are constructed by inferring the model dynamics, as well as their functional forms, through matching the model prices to the market prices of certain liquid derivatives. That is, they are *implied interest rate models*, and should be thought of in a fashion similar to local volatility models and implied trees in equity markets.

The main idea behind Markov-functional models is that bond prices and the numeraire are, at any point in time, a function of a *low-dimensional* Markov process under some martingale measure. The functional form for the bond prices is selected such that the model accurately calibrates to the relevant market prices, while the freedom to choose the Markov process makes the model realistic and tractable. Moreover, the functional form for the numeraire can be used to reproduce the marginal laws of swap rates or other relevant instruments for the calibration.

More specifically, let $(M, \mathbf{M})$ denote a numeraire pair, and consider a (time-inhomogeneous) Markov process $X = (X_t)_{0 \le t \le T}$ under the measure $\mathbf{M}$. In the framework of Markov-functional models, one assumes that bond prices have the functional form

$$B(t, S) = \mathrm{B}(t, S; X_t), \quad 0 \le t \le \partial_S \le S, \tag{6.1}$$

where ∂_S denotes some "boundary curve". In applications, the boundary curve typically has the form

$$\partial_S = \begin{cases} S, & S \le T_*, \\ T_*, & S > T_*, \end{cases} \tag{6.2}$$

where T_* is a common time of maturity. One further assumes that the numeraire M is also a function of the driving Markov process X, i.e.

$$M_t = \mathrm{M}(t; X_t), \quad 0 \le t \le T. \tag{6.3}$$

Therefore, in order to specify a Markov-functional model, the following quantities are required:

(P1) the law of X under the measure $\mathbf{M}$;
(P2) the functional form $\mathrm{B}(\partial_S, S; \cdot)$ for $S \in [0, T]$;
(P3) the functional form $\mathrm{M}(t; \cdot)$ for $0 \le t \le T$.

In applications, the Markov process is specified first and is typically a diffusion process with time-dependent volatility. Then, the functional forms for the bond prices and the numeraire are implied by calibrating the model to market prices of liquid options. The choice of the calibrating instruments depends on the exotic derivative that should be priced or hedged with the model. If the exotic depends on LIBOR rates, e.g. the flexible cap, then the model is calibrated to digital caplets, which leads to the *Markov-functional LIBOR model*. If the exotic depends on swap rates, e.g. the Bermudan swaption, then the model is calibrated to digital swaptions, which leads to the *Markov-functional swap rate model*. Let us point out that the functional forms are typically not known in closed form, and should be computed numerically. These models typically satisfy requirements (A1), (A2) and (A3). For further details and concrete applications we refer the reader to the books by Hunt and Kennedy (2004) and Fries (2007), and the references therein.

Remark 6.1. Let us point out that forward price models and affine LIBOR models, that will be introduced in section 7, belong to the class of Markov-functional models, while LIBOR market models do not. In LIBOR market models the LIBOR rates are functions of a *high-dimensional* Markov process.

6.1. **Markov-functional LIBOR model.** In order to gain a better understanding of the construction of Markov-functional models, we will briefly describe a Markov-functional model calibrated to LIBOR rates. This model is called the *Markov-functional LIBOR model.*

The set of relevant market rates are LIBOR rates $L(\cdot, T_k), k \in K$. We will consider the numeraire pair $(M, \mathbf{M}) = (B(\cdot, T_N), \mathbb{P}_{T_N})$.

In order to be consistent with Black's formula for caplets, we assume that $L(\cdot, T_{N-1})$ is a log-normal martingale under $\mathbb{P}_{T_N}$, i.e.

$$\mathrm{d}L(t, T_{N-1}) = \sigma(t, T_{N-1}) L(t, T_{N-1}) \mathrm{d}W_t, \tag{6.4}$$

where W denotes a standard Brownian motion under $\mathbb{P}_{T_N}$ and $\sigma(\cdot, T_{N-1})$ is a deterministic, time-dependent volatility function. Hence, we have that

$$L(t, T_{N-1}) = L(0, T_{N-1}) \exp\Big(- \frac{1}{2}\Sigma_t + X_t \Big), \tag{6.5}$$

where $\Sigma = \int_0^{\cdot} \sigma^2(s, T_{N-1})\mathrm{d}s$, and X is a deterministic time-change of the Brownian motion, that satisfies

$$\mathrm{d}X_t = \sigma(t, T_{N-1})\mathrm{d}W_t. \tag{6.6}$$

We will use X as the driving process of the model, which specifies (P1).

Regarding (P2), the boundary curve is exactly of the form (6.2) with $T_* = T_{N-1}$, hence we need to specify $\mathrm{B}(T_i, T_i; X_{T_i})$ for $i \in K$, which is trivially the unit map. We also need to specify $\mathrm{B}(T_{N-1}, T_N; X_{T_{N-1}})$; using (2.1) and (6.5) we get that

$$\mathrm{B}(T_{N-1}, T_N; X_{T_{N-1}}) = \frac{1}{1 + \delta L(0, T_{N-1}) \exp\big(- \frac{1}{2}\Sigma_{T_{N-1}} + X_{T_{N-1}} \big)}. \tag{6.7}$$

Then, we can recover bond prices in the interior of the region bounded by ∂_S using the martingale property:

$$\mathrm{B}(t, S; X_t) = \mathrm{B}(t, T_N; X_t) \mathbb{E}_{T_N}\left[\frac{\mathrm{B}(\partial_S, S; X_{\partial_S})}{\mathrm{B}(\partial_S, T_N; X_{\partial_S})} \Big| \mathcal{F}_t \right]. \tag{6.8}$$

Now, it remains to specify the functional form $\mathrm{B}(T_i, T_N; X_{T_i})$, $i \in K$, for the numeraire, cf. (P3). In the framework of the Markov-functional LIBOR model, this is done by deriving the numeraire from LIBOR rates and inferring the functional forms of the LIBOR rates via calibration to market prices. Equation (2.1) combined with (6.2) and the fact that $B(T_i, T_{i+1})$ is a function of X_{T_i}, cf. (6.8), yield that $L(T_i, T_i)$ is also a function of X_{T_i}. The functional form is

$$\begin{aligned} 1 + \delta \mathrm{L}(T_i, T_i; X_{T_i}) &= \frac{1}{\mathrm{B}(T_i, T_{i+1}; X_{T_i})} \\ &= \frac{1}{\mathrm{B}(T_i, T_N; X_{T_i}) \mathbb{E}_{T_N}\left[\frac{1}{\mathrm{B}(T_{i+1}, T_N; X_{T_{i+1}})} \Big| \mathcal{F}_{T_i} \right]}. \end{aligned} \tag{6.9}$$

Rearranging, we get the following functional form for the numeraire

$$\mathrm{B}(T_i, T_N; X_{T_i}) = \frac{1}{(1 + \delta \mathrm{L}(T_i, T_i; X_{T_i})) \mathbb{E}_{T_N}\left[\frac{1}{\mathrm{B}(T_{i+1}, T_N; X_{T_{i+1}})} \Big| \mathcal{F}_{T_i} \right]}. \tag{6.10}$$

This formula provides a backward induction scheme to calculate $\mathrm{B}(T_i, T_N; \cdot)$ from $\mathrm{B}(T_{i+1}, T_N; \cdot)$ for any value of the Markov process; the induction starts from $B(T_N, T_N) = 1$.

The calibrating instruments are digital caplets with payoff $1_{\{L(T_i,T_i)>\mathcal{K}\}}$, $i \in K$, and their market values are provided by Black's formula; we denote them by $\mathbb{V}_0(T_i, \mathcal{K})$. Assuming that the map $\xi \mapsto \mathrm{L}(T_i, T_i; \xi)$ is strictly increasing, there exists a unique strike $\mathcal{K}(T_i, x^*)$ such that the set equality

$$\big\{X_{T_i} > x^*\big\} = \big\{\mathrm{L}(T_i, T_i; X_{T_i}) > \mathcal{K}(T_i, x^*)\big\} \tag{6.11}$$

holds almost surely. Define the model prices

$$\mathbb{U}_0(T_i, x^*) = B(0, T_N)\mathrm{I\!E}_{T_N}\left[\frac{\mathrm{B}(T_i, T_{i+1}; X_{T_i})}{\mathrm{B}(T_i, T_N; X_{T_i})} 1_{\{X_{T_i}>x^*\}}\right], \tag{6.12}$$

which have to be calculated numerically. Therefore, we can equate market and model prices

$$\mathbb{V}_0(T_i, \mathcal{K}(T_i, x^*)) = \mathbb{U}_0(T_i, x^*), \tag{6.13}$$

where the strike $\mathcal{K}(T_i, x^*)$ is determined by Black's formula using some numerical algorithm.

Hence, we have specified, numerically at least, the functional form for the LIBOR rates, which yields also the functional form for the numeraire via (6.10). This completes the specification of the Markov-functional LIBOR model. This model satisfies requirement (A3), in the sense of Hunt et al. (2000), since all bond prices are functions of a one-dimensional diffusion.

7. Affine LIBOR models

Affine LIBOR models were recently developed by Keller-Ressel, Papapantoleon, and Teichmann (2009) with the aim of combining the advantages of LIBOR market models and forward price models, while circumventing their drawbacks. We provide here a more general outline of this framework, which is based on two key ingredients: martingales *greater than 1*, which are *increasing* in some parameter.

The construction of martingales greater than 1 is done as follows: let Y_T^u be an $\mathcal{F}_T$-measurable, integrable random variable, taking values in $[1, \infty)$, and set

$$M_t^u = \mathrm{I\!E}[Y_T^u | \mathcal{F}_t], \qquad 0 \le t \le T. \tag{7.1}$$

Then, using the tower property of conditional expectations, it easily follows that $M^u = (M_t^u)_{0 \le t \le T}$ is a martingale. Moreover, it obviously holds that $M_t^u \ge 1$ for all $t \in [0, T]$.

In addition, assume that the map $u \mapsto Y_T^u$ is *increasing*; then, we immediately get that the map

$$u \mapsto M_t^u \tag{7.2}$$

is also increasing, for all $t \in [0, T]$.

Now, using the family of martingales M^u we can model quotients of bond prices as follows. Consider a *decreasing* sequence $(u_k)_{k \in \overline{K}}$ and set

$$\frac{B(t, T_k)}{B(t, T_N)} = M_t^{u_k}, \qquad t \in [0, T_k],\ k \in \overline{K}, \tag{7.3}$$

requiring that the initial values of the martingales fit today's observed market prices, i.e. $\frac{B(0,T_k)}{B(0,T_N)} = M_0^{u_k}$. Since M^u is increasing in u, we have that

$$M_t^{u_k} \geq M_t^{u_l} \quad \text{for} \quad k \leq l \Leftrightarrow u_k \geq u_l. \tag{7.4}$$

Hence, we can deduce that bond prices are decreasing as functions of time of maturity, i.e. $B(t,T_k) \geq B(t,T_l)$ for $k \leq l$.

Turning our attention to LIBOR rates, we get that

$$1 + \delta L(t,T_k) = \frac{B(t,T_k)}{B(t,T_{k+1})} = \frac{M_t^{u_k}}{M_t^{u_{k+1}}} \geq 1, \tag{7.5}$$

for all $t \in [0,T_k]$ and all $k \in K$; this is a trivial consequence of (7.4). Moreover, the martingale property of the LIBOR rate under its corresponding forward measure follows easily from the structure of the measure changes (2.3), and the structure of the martingales. Indeed, we have that

$$\begin{aligned} 1 + \delta L(\cdot,T_k) &= \frac{M^{u_k}}{M^{u_{k+1}}} \in \mathcal{M}(\mathbb{P}_{T_{k+1}}) \\ \text{since} \quad \frac{M^{u_k}}{M^{u_{k+1}}} &\cdot \frac{\mathrm{d}\mathbb{P}_{T_{k+1}}}{\mathrm{d}\mathbb{P}_{T_N}} = \frac{M^{u_k}}{M^{u_{k+1}}} \cdot \frac{M^{u_{k+1}}}{M_0^{u_{k+1}}} \in \mathcal{M}(\mathbb{P}_{T_N}). \end{aligned} \tag{7.6}$$

Therefore, we have just described a broad framework for modeling LIBOR rates, in which requirements (A1) and (A2) are satisfied. The next step is to show that requirement (A3) is also satisfied. We will not pursue this in full generality, instead we will consider a specific form for the variable Y_T^u, and thus for the martingales M^u. In addition, the model is driven by an affine process, and is henceforth called the *affine LIBOR model*.

7.1. **Affine processes.** Let $X = (X_t)_{0 \leq t \leq T}$ be a stochastically continuous, time-homogeneous Markov process with state space $D = \mathbb{R}^d_{\geqslant 0}$, starting from $x \in D$. The process X is called *affine* if the moment generating function satisfies

$$\mathbb{E}_x\big[\mathrm{e}^{\langle u, X_t\rangle}\big] = \exp\big(\phi_t(u) + \langle \psi_t(u), x\rangle\big), \tag{7.7}$$

for some functions $\phi : [0,T] \times \mathcal{I}_T \to \mathbb{R}$ and $\psi : [0,T] \times \mathcal{I}_T \to \mathbb{R}^d$, and all $(t,u,x) \in [0,T] \times \mathcal{I}_T \times D$, where

$$\mathcal{I}_T := \Big\{ u \in \mathbb{R}^d : \mathbb{E}_x\big[\mathrm{e}^{\langle u, X_T\rangle}\big] < \infty, \text{ for all } x \in D \Big\}. \tag{7.8}$$

We will assume in the sequel that $0 \in \mathcal{I}_T^\circ$. The functions ϕ and ψ satisfy the semi-flow property

$$\begin{aligned} \phi_{t+s}(u) &= \phi_t(u) + \phi_s(\psi_t(u)) \\ \psi_{t+s}(u) &= \psi_s(\psi_t(u)), \end{aligned} \tag{7.9}$$

with initial condition

$$\phi_0(u) = 0 \quad \text{and} \quad \psi_0(u) = u, \tag{7.10}$$

for all $(t+s,u) \in [0,T] \times \mathcal{I}_T$. Equivalently, ϕ and ψ satisfy generalized Riccati differential equations. For comprehensive expositions of affine processes we refer the reader to Duffie et al. (2003) and Keller-Ressel (2008).

7.2. **Affine LIBOR model.** In the affine LIBOR model, the random variable Y_T^u has the following form:

$$Y_T^u = \mathrm{e}^{\langle u, X_T \rangle}, \tag{7.11}$$

where $u \in \mathbb{R}^d_{\geqslant 0}$ and X_T is a random variable from an $\mathbb{R}^d_{\geqslant 0}$-valued affine process X. Hence, $Y_T^u \geq 1$, while the map $u \mapsto Y_T^u$ is obviously increasing; note that inequalities involving vectors are understood componentwise.

Using the Markov property of affine processes, we can deduce that the martingales M^u have the form

$$\begin{aligned} M_t^u &= \mathrm{I\!E}\big[\mathrm{e}^{\langle u, X_T \rangle} | \mathcal{F}_t\big] \\ &= \exp\big(\phi_{T-t}(u) + \langle \psi_{T-t}(u), X_t \rangle\big). \end{aligned} \tag{7.12}$$

Therefore, LIBOR rates have the following evolution:

$$1 + \delta L(t, T_k) = \frac{M_t^{u_k}}{M_t^{u_{k+1}}} = \exp\big(A_{k,t} + \langle B_{k,t}, X_t \rangle\big), \tag{7.13}$$

where

$$\begin{aligned} A_{k,t} &:= \phi_{T_N - t}(u_k) - \phi_{T_N - t}(u_{k+1}) \\ B_{k,t} &:= \psi_{T_N - t}(u_k) - \psi_{T_N - t}(u_{k+1}). \end{aligned} \tag{7.14}$$

Let us point that, under reasonable assumptions on the driving affine process, we can prove that the affine LIBOR model can fit *any* term structure of initial LIBOR rates; cf. Proposition 6.1 in Keller-Ressel et al. (2009).

Now, regarding requirement (A3), let us turn our attention to the structure of the driving process under the different forward measures. Using the connections between forward measures (2.3), the Markov property of affine processes, and the flow equations (7.9), we can show that

$$\begin{aligned} \mathrm{I\!E}_{T_k}\Big[\mathrm{e}^{\langle v, X_r \rangle} \big| \mathcal{F}_s\Big] &= \mathrm{I\!E}_{T_N}\Big[\frac{M_r^{u_k}}{M_s^{u_k}} \mathrm{e}^{\langle v, X_r \rangle} \big| \mathcal{F}_s\Big] \\ &= \exp\Big(\phi_{r-s}(\psi_{T_N - r}(u_k) + v) - \phi_{r-s}(\psi_{T_N - r}(u_k)) \\ &\qquad + \langle \psi_{r-s}(\psi_{T_N - r}(u_k) + v) - \psi_{r-s}(\psi_{T_N - r}(u_k)), X_s \rangle\Big); \end{aligned} \tag{7.15}$$

cf. Keller-Ressel et al. (2009, eq. (6.15)). This means that X becomes a *time-inhomogeneous affine* process under *any* forward measure. Note that the measure changes are again Esscher transformations, similarly to forward price models. Consequently the affine LIBOR model satisfies requirements (A1), (A2) *and* (A3).

The pricing of caplets and swaptions in the affine LIBOR model using Fourier transform methods is described in Keller-Ressel et al. (2009). In addition, closed-form valuation formulas – in terms of the χ^2-distribution function – are derived when the driving affine process is the Cox–Ingersoll–Ross (CIR) process.

8. Extensions

The different approaches for modeling LIBOR rates have been extended in two different directions: (i) to model simultaneously LIBOR rates for different currencies and the corresponding foreign exchange rates, and (ii) to jointly model default-free and defaultable LIBOR rates.

8.1. **Multiple currencies.** The log-normal LIBOR market model has been extended to a multi-currency setting by Schlögl (2002) and by Mikkelsen (2002). The Lévy LIBOR model and the Lévy forward price model have been extended to model multiple currencies and foreign exchange rates by Eberlein and Koval (2006). A multi-factor approach to multiple currency LIBOR models has been presented in Benner et al. (2009). Markov-functional models have been extended to the multi-currency setting by Fries and Rott (2004) and Fries and Eckstädt (2010).

8.2. **Default risk.** The log-normal LIBOR market model has been first extended to model default risk by Lotz and Schlögl (2000), who used a deterministic hazard rate to model the time of default. Eberlein et al. (2006), borrowing also ideas from Schönbucher (2000), constructed a model for default-free and defaultable rates where they use time-inhomogeneous Lévy processes as the driving motion and the "Cox construction" to model the time of default (cf. e.g. Bielecki and Rutkowski (2002) for the Cox construction). This has been extended to a model where defaultable bonds can have rating migrations by Grbac (2010).

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