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ω^ω-Base and infinite-dimensional compact sets in locally convex spaces

Taras Banakh[1,2] · **Jerzy Kąkol**[3,4] · **Johannes Philipp Schürz**[5]

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Abstract
A locally convex space (lcs) E is said to have an ω^ω-base if E has a neighborhood base $\{U_\alpha : \alpha \in \omega^\omega\}$ at zero such that $U_\beta \subseteq U_\alpha$ for all $\alpha \leq \beta$. The class of lcs with an ω^ω-base is large, among others contains all (LM)-spaces (hence (LF)-spaces), strong duals of distinguished Fréchet lcs (hence spaces of distributions $D'(\Omega)$). A remarkable result of Cascales-Orihuela states that every compact set in an lcs with an ω^ω-base is metrizable. Our main result shows that every uncountable-dimensional lcs with an ω^ω-base contains an infinite-dimensional metrizable compact subset. On the other hand, the countable-dimensional vector space φ endowed with the finest locally convex topology has an ω^ω-base but contains no infinite-dimensional compact subsets. It turns out that φ is a unique infinite-dimensional locally convex space which is a $k_{\mathbb{R}}$-space containing no infinite-dimensional compact subsets. Applications to spaces $C_p(X)$ are provided.

Keywords Locally convex space · ω^ω-base · Free space · Networks

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✉ Johannes Philipp Schürz
johannes.schuerz@tuwien.ac.at

Taras Banakh
t.o.banakh@gmail.com

Jerzy Kąkol
kakol@amu.edu.pl

1 Ivan Franko National University of Lviv, Lviv, Ukraine

2 Jan Kochanowski University in Kielce, Kielce, Poland

3 Faculty of Mathematics and Informatics, A. Mickiewicz University, 61-614 Poznan, Poland

4 Institute of Mathematics Czech Academy of Sciences, Prague, Czech Republic

5 Faculty of Mathematics and Geoinformation, TU Wien, 1040 Vienna, Austria

1 Introduction

A topological space X is said to have a neighborhood ω^ω-base at a point $x \in X$ if there exists a neighborhood base $(U_\alpha(x))_{\alpha\in\omega^\omega}$ at x such that $U_\beta(x) \subseteq U_\alpha(x)$ for all $\alpha \leq \beta$ in ω^ω. We say that X has an ω^ω-base if it has a neighborhood ω^ω-base at each point of X. Evidently, a topological group (particularly topological vector space (tvs)) has an ω^ω-base if it has a neighborhood ω^ω-base at the identity. The classical metrization theorem of Birkhoff and Kakutani states that a topological group G is metrizable if and only if G is first-countable. Then, as easily seen, if $(U_n)_{n\in\omega}$ is a neighborgood base at the identity of G, then the family $\{U_\alpha : \alpha \in \omega^\omega\}$ formed by sets $U_\alpha = U_{\alpha(0)}$ forms an ω^ω-base (at the identity) for G. Locally convex spaces (lcs) with an ω^ω-base are known in Functional Analysis since 2003 when Cascales, Kąkol, and Saxon [7] characterized quasi-barreled lcs with an ω^ω-base. In several papers (see [16] and the references therein) spaces with an ω^ω-base were studied under the name *lcs with a* $\mathfrak{G}$*-base*, but here we prefer (as in [4]) to use the more self-suggesting terminology of ω^ω-bases.

In [8] Cascales and Orihuela proved that compact subsets of any lcs with an ω^ω-base are metrizable. This refers, among others, to each (LM)-space, i.e. a countable inductive limit of metrizble lcs, since (LM)-spaces have an ω^ω-base. Also the following *metrization theorem* holds together a number of topological conditions.

Theorem 1.1 [16, Corollary 15.5] *For a barrelled lcs E with an ω^ω-base, the following conditions are equivalent.*

(1) *E is metrizable;*
(2) *E is Fréchet-Urysohn;*
(3) *E is Baire-like;*
(4) *E does not contain a copy of φ, i.e. an $\aleph_0$-dimensional vector space endowed with the finest locally convex topology.*

Hence every Baire lcs with an ω^ω-base is metrizable. The space φ appearing in Theorem 1.1 has the following properties:

(1) φ is the strong dual of the Fréchet-Schwartz space $\mathbb{R}^\omega$.
(2) All compact subsets in φ are finite-dimensional.
(3) φ is a complete bornological space,

see [16,21,23].

Being motivated by above's results, especially by a remarkable theorem of Cascales-Oruhuela mentioned above, one can ask for a possible large class of lcs E for which every infinite-dimensional subspace of E contains an infinite-dimensional compact (metrizable) subset. Surely, each metrizable lcs trivially fulfills this request. We prove however the following general

Theorem 1.2 *Every uncountably-dimensional lcs E with ω^ω-base contains an infinite-dimensional metrizable compact subset.*

Theorem 1.2 will be proved in Sect. 4. An alternative proof will be presented in Sect. 5 as a consequence of Theorem 5.2.

The uncountable dimensionality of the space E in Theorem 1.2 cannot be replaced by the infinite-dimensionality of E: the space φ is infinite-dimensional, has an ω^ω-base and contains no infinite-dimensional compact subsets. However, φ is a unique locally convex $k_{\mathbb{R}}$-space with this property. Recall [20] that a topological space X is a $k_{\mathbb{R}}$*-space* if a function $f : X \to \mathbb{R}$ is continuous whenever for every compact subset $K \subseteq X$ the restriction $f \restriction K$ is continuous. We prove the following

Theorem 1.3 *An lcs E is topologically isomorphic to the space φ if and only if E is a $k_{\mathbb{R}}$-space containing no infinite-dimensional compact subsets.*

Theorem 1.3 implies that an lcs is topologically isomorphic to φ if and only if it is homeomorphic to φ. This topological uniqueness property of the space φ was first proved by the first author in [2].

The following characterization of the space φ can be derived from Theorems 1.2 and 2.1. It shows that φ is a unique bornological space for which the uncountable dimensionality in Theorem 1.2 cannot be weakened to infinite dimensionality.

Theorem 1.4 *An lcs E is topologically isomorphic to the space φ if and only if E is bornological, has an ω^ω-base and contains no infinite-dimensional compact subset.*

Theorem 1.2 provides a large class of concrete (non-metrizable) lcs containing infinite-dimensional compact sets.

Corollary 1.5 *Every uncountable-dimensional subspace of an (LM)-space contains an infinite-dimensional compact set.*

Let X be a Tychonoff space. By $C_p(X)$ and $C_k(X)$ we denote the space of continuous real-valued functions on X endowed with the pointwise and the compact-open topology, respectively. The problem of characterization of Tychonoff spaces X whose function spaces $C_p(X)$ and $C_k(X)$ admit an ω^ω-base is already solved. Indeed, by [16, Corollary 15.2] $C_p(X)$ has an ω^ω-base if and only if X is countable. The space $C_k(X)$ has an ω^ω-base if and only if X admits a fundamental compact resolution [11], for necessary definitions see below. Since every Čech-complete Lindelöf space X is a continuous image of a Polish space under a perfect map (and the latter space admits a fundamental compact resolution), the space $C_p(X)$ has an ω^ω-base. So, we have another concrete application of Theorem 1.2.

Example 1.6 Let X be an infinite Čech-complete Lindelöf space. Then every uncountable-dimensional subspace of $C_k(X)$ contains an infinite-dimensional metrizable compact set.

In Sect. 2 we show that all (bornological) lcs containing no infinite-dimensional compact subsets are bornologically (and topologically) isomorphic to a free lcs over discrete topological spaces. Consequently, in Sects. 3 and 4 we study the free lcs $L(\kappa)$ over infinite cardinals κ, including $L(\omega) = \varphi$. We introduce two concepts: the (κ, λ)-tall bornology and the $(\kappa, \lambda)_p$-equiconvergence, which will be used to obtain bornological and topological characterizations of $L(\kappa)$. Both concepts apply to prove Theorem 1.2. To this end, we shall prove that each topological (vector) space with an ω^ω-base is $(\omega_1, \omega)_p$-equiconvergent (and has (ω_1, ω)-tall bornology). Another property implying the $(\omega_1, \omega)_p$-equiconvergence is the existence of a countable $\mathsf{cs}^\bullet$-network (see Theorem 4.2), which follows from the existence of an ω^ω-base according to Proposition 3.3. Linear counterparts of $\mathsf{cs}^\bullet$-networks are radial networks introduced in Sect. 5, whose main result is Theorem 5.2 implying Theorem 1.2. Some applications of Theorem 1.2 to function spaces $C_p(X)$ are provided in Sect. 6.

2 Locally convex spaces containing no infinite-dimensional compact subsets

In this section we study lcs containing no infinite-dimensional compact subsets. We shall show that all such (bornological) spaces are bornologically (and topologically) isomorphic to free lcs over discrete topological spaces.

Recall that for a topological space X its *free locally convex space* is an lcs $L(X)$ endowed with a continuous function $\delta : X \to L(X)$ such that for any continuous function $f : X \to E$ to an lcs E there exists a unique linear continuous map $T : L(X) \to E$ such that $T \circ \delta = f$. The set X forms a Hamel basis for $L(X)$ and δ is a topological embedding, see [22]; we also refer to [5] and [4] for several results and references concerning this concept; [5, Theorem 5.4] characterizes those X for which $L(X)$ has an ω^ω-base.

Let E be a tvs. A subset $B \subseteq E$ is called *bounded* if for every neighborhood $U \subseteq E$ of zero there exists $n \in \mathbb{N}$ such that $B \subseteq nU$. The family of all bounded sets of E is called the *bornology* of E. A linear operator $f : E \to F$ between two tvs is called *bounded* if for any bounded set $B \subseteq E$ its image $f(B)$ is bounded in F.

Two tvs E and F are

- *topologically isomorphic* if there exists a linear bijective function $f : E \to F$ such that f and f^{-1} are continuous;
- *bornologically isomorphic* if there exists a linear bijective function $f : E \to F$ such that f and f^{-1} are bounded.

An lcs E is called *bornological* if each bounded linear operator from E to an lcs F is continuous. A linear space E is called κ*-dimensional* if E has a Hamel basis of cardinality κ. In this case we write $\kappa = \dim(E)$.

An lcs E is *free* if it carries the finest locally convex topology. In this case E is topologically isomorphic to the free lcs $L(\kappa)$ over the cardinal $\kappa = \dim(E)$ endowed with the discrete topology.

The study around the free lcs $L(\omega) = \varphi$ has attracted specialists for a long time. For example, Nyikos observed [21] that each sequentially closed subset of $L(\omega)$ is closed although the sequential closure of a subset of φ need not be closed. Consequently, $L(\omega)$ is a concrete "small" space without the Fréchet-Urysohn property. Applying the Baire category theorem one shows that $L(\omega)$ is not a Baire-like space (in sense of Saxon [23]) and a barrelled lcs E is Baire-like if E does not contain a copy of $L(\omega)$, see [23]. Although $L(\omega)$ is not Fréchet-Urysohn, it provides some extra properties since all vector subspaces in $L(\omega)$ are closed. In [17] we introduced the property for an lcs E (under the name C_3^-) stating that the sequential closure of every linear subspace of E is sequentially closed, and we proved [17, Corollary 6.4] that the only infinite-dimensional Montel (DF)-space with property C_3^- is $L(\omega)$ (yielding a remarkable result of Bonet and Defant that the only infinite-dimensional Silva space with property C_3^- is $L(\omega)$). This implies that barrelled (DF)-spaces and (LF)-spaces satisfying property C_3^- are exactly of the form M, $L(\omega)$, or $M \times L(\omega)$ where M is metrizable, [17, Theorems 6.11, 6.13].

The following simple theorem characterizes lcs containing no infinite-dimensional compact subsets.

Theorem 2.1 *For an lcs E the following conditions are equivalent:*

(1) *Each compact subset of E has finite topological dimension.*
(2) *Each bounded linearly independent set in E is finite.*
(3) *E is bornologically isomorphic to a free lcs.*
If E is bornological, then the conditions (1)–(3) *are equivalent to*
(4) *E is free.*

Proof (1) $\Rightarrow$ (2) Suppose that each compact subset of E has finite topological dimension. Assuming that E contains an infinite bounded linearly indendent set, we can find a bounded linearly independent set $\{x_n\}_{n\in\omega}$ consisting of pairwise distinct points x_n. Then the sequence $(2^{-n}x_n)_{n\in\omega}$ converges to zero and

$$K = \bigcup_{n\in\omega} \Big\{ \sum_{k=n}^{2n} t_k x_k : (t_k)_{k=n}^{2n} \in \prod_{k=n}^{2n} [0, 2^{-k}] \Big\}$$

is an infinite-dimensional compact set in E, which contradicts our assumption.

(2) $\Rightarrow$ (3) Let τ be the finest locally convex topology on E. Then the identity map $(E, \tau) \to E$ is continuous and hence bounded. If each bounded linearly independent set in E is finite, then each bounded set $B \subseteq E$ is contained in a finite-dimensional subspace of E and hence is bounded in the topology τ. This means that the identity map $E \to (E, \tau)$ is bounded and hence E is bornologically isomorphic to the free lcs (E, τ).

(3) $\Rightarrow$ (1) If E is bornologically isomorphic to a free lcs F then each bounded linearly independent set in E is finite, since the free lcs F has this property.

The implication (4) $\Rightarrow$ (3) is trivial. If E is bornological then the implication (3) $\Rightarrow$ (4) follows from the continuity of bounded linear operators on bornological spaces. □

The free lcs over discrete topological spaces are not unique lcs possessing no infinite-dimensional compact sets. A subset B of a topological space X is called *functionally bounded* if for any continuous real-valued function $f : X \to \mathbb{R}$ the set $f(B)$ is bounded.

Proposition 2.2 *For a Tychonoff space X the following conditions are equivalent:*

(1) *each compact subset of the free lcs $L(X)$ has finite topological dimension;*
(2) *each bounded linearly independent set in $L(X)$ is finite;*
(3) *each functionally bounded subset of X is finite.*

Proof The equivalence (1) $\Leftrightarrow$ (2) follows from the corresponding equivalence in Theorem 2.1. The implication (3) $\Rightarrow$ (1) follows from [6, Lemma 10.11.3], and (2) $\Rightarrow$ (3) follows from the observation that each functionally bounded set in an lcs is bounded. □

3 Bornological and topological characterizations of the spaces $L(\kappa)$

In this section, given an infinite cardinal κ we characterize the free lcs $L(\kappa)$ using some specific properties of the bornology and the topology of the space $L(\kappa)$.

Let κ, λ be two cardinals. An lcs E is defined to have (κ, λ)-*tall bornology* if every subset $A \subseteq E$ of cardinality $|A| = \kappa$ contains a bounded subset $B \subseteq A$ of cardinality $|A| = \lambda$.

Theorem 3.1 *Let κ be an infinite cardinal. For an lcs E the following conditions are equivalent:*

(1) *E is bornologically isomorphic to the free lcs $L(\kappa)$;*
(2) *each bounded linearly independent set in E is finite and the bornology of E is (κ^+, ω)-tall but not (κ, ω)-tall.*
 If E is bornological, then the conditions (1)–(2) *are equivalent to*
(3) *E is topologically isomorphic to $L(\kappa)$.*

Proof (1) $\Rightarrow$ (2): Assume that E is bornologically isomorphic to $L(\kappa)$. Then E has algebraic dimension κ and each bounded linearly independent set in E is finite (since this is true in $L(\kappa)$).

To see that the bornology of E is (κ^+, ω)-tall, take any set $K \subseteq E$ of cardinality $|K| = \kappa^+$. Since E has algebraic dimension κ, there exists a cover $(B_\alpha)_{\alpha\in\kappa}$ of E by κ many compact sets. By the Pigeonhole Principle, there exists $\alpha \in \kappa$ such that $|K \cap B_\alpha| = \kappa^+$. This means that the bornology of E is (κ^+, κ^+)-tall and hence (κ^+, ω)-tall.

To see that the bornology of the space E is not (κ, ω)-tall, observe that the Hamel basis κ of $L(\kappa)$ has the property that no infinite subset of κ is bounded in $L(\kappa)$. Since E is bornologically isomorphic to $L(\kappa)$, the image of κ in E is a subset of cardinality κ containing no bounded infinite subsets and witnessing that E is not (κ, ω)-tall.

(2) $\Rightarrow$ (1): Assume that each bounded linearly independent set in E is finite and the bornology of E is (κ^+, ω)-tall but not (κ, ω)-tall. Let B be a Hamel basis of E. We claim that $|B| = \kappa$. Assuming that $|B| > \kappa$, we conclude that E is not (κ^+, ω)-tall, which is a contradiction. Assuming that $|B| < \kappa$, we conclude that E is the union of $< \kappa$ many bounded sets and hence is (κ, κ)-tall by the Pigeonhole Principle. But this contradicts our assumption. Therefore $|B| = \kappa$. Let $h : \kappa \to B$ be any bijection and $\bar{h} : L(\kappa) \to E$ be the unique extension of h to a linear continuous operator. Since B is a Hamel basis for E, the operator $\bar{h}$ is bijective. Since each bounded set in E is contained in a finite-dimensional linear subspace, the operator $\bar{h}^{-1} : E \to L(\kappa)$ is bounded and hence $\bar{h} : L(\kappa) \to E$ is a bornological isomorphism.

If the space E is bornological, then the equivalence (1) $\Leftrightarrow$ (3) follows from the bornological property of E and $L(\kappa)$. □

The (κ, ω)-tallness of the bornology of an lcs E has topological counterparts introduced in the following definition.

Definition 3.2 Let κ, λ be cardinals. We say that a topological space X is

- $(\kappa, \lambda)_p$*-equiconvergent at a point* $x \in X$ if for any indexed family $\{x_\alpha\}_{\alpha\in\kappa} \subseteq \{s \in X^\omega : \lim_{n\to\infty} s(n) = x\}$, there exists a subset $\Lambda \subseteq \kappa$ of cardinality $|\Lambda| = \lambda$ such that for every neighborhood $O_x \subseteq X$ of x there exists $n \in \omega$ such that the set $\{\alpha \in \Lambda : x_\alpha(n) \notin O_x\}$ is finite;
- $(\kappa, \lambda)_k$*-equiconvergent at a point* $x \in X$ if for any indexed family $\{x_\alpha\}_{\alpha\in\kappa} \subseteq \{s \in X^\omega : \lim_{n\to\infty} s(n) = x\}$, there exists a subset $\Lambda \subseteq \kappa$ of cardinality $|\Lambda| = \lambda$ such that for every neighborhood $O_x \subseteq X$ of x there exists $n \in \omega$ such that for every $m \geq n$ and $\alpha \in \Lambda$ we have $x_\alpha(m) \in O_x$;
- $(\kappa, \lambda)_p$*-equiconvergent* if X is $(\kappa, \lambda)_p$-equiconvergent at every point $x \in X$;
- $(\kappa, \lambda)_k$*-equiconvergent* if X is $(\kappa, \lambda)_k$-equiconvergent at every point $x \in X$.

It is easy to see that every $(\kappa, \lambda)_k$-equiconvergent space is $(\kappa, \lambda)_p$-equiconvergent. The following observation will be used below.

Proposition 3.3 *If an lcs E is $(\kappa, \lambda)_p$-equiconvergent, then its bornology is (κ, λ)-tall.*

Proof Given a subset $K \subseteq E$ of cardinality $|K| = \kappa$, for every $\alpha \in K$ consider the convergent sequence $x_\alpha \in X^\omega$ defined by $x_\alpha(n) = 2^{-n}\alpha$. Assuming that the lcs E is $(\kappa, \lambda)_p$-equiconvergent, we can find a subset $L \subseteq K$ of cardinality $|L| = \lambda$ such that for every neighborhood of zero $U \subseteq E$ there exists $n \in \omega$ such that the set $\{\alpha \in L : 2^{-n}\alpha \notin U\}$ is finite. We claim that the set L is bounded. Indeed, for every neighborhood $U \subseteq E$ of zero, we find a neighborhood $V \subseteq E$ of zero such that $[0, 1] \cdot V \subseteq U$. By our assumption, there exists $n \in \omega$ such that the set $F = \{\alpha \in K : 2^{-n}\alpha \notin V\}$ is finite. Find $m \geq n$ such that $2^{-m}\alpha \in U$ for every $\alpha \in F$. Then $2^{-m}L \subseteq 2^{-m}(L \setminus F) \cup 2^{-m}F \subseteq ([0, 1] \cdot V) \cup U = U$, and hence the set L is bounded. □

Nevertheless, it seems that the following question remains open.

Problem 3.4 Assume that the bornology of an lcs E is (ω_1, ω)-tall. Is it true that E is $(\omega_1, \omega)_p$-equiconvergent?

Below we prove the following topological counterpart to Theorem 3.1.

Theorem 3.5 *Let κ be an infinite cardinal. For an lcs E the following conditions are equivalent:*

(1) *E is bornologically isomorphic to $L(\kappa)$;*
(2) *each compact subset of E has finite topological dimension, E is $(\kappa^+, \omega)_k$-equiconvergent but not $(\kappa, \omega)_p$-equiconvergent.*
(3) *each compact subset of E has finite topological dimension, E is $(\kappa^+, \omega)_p$-equiconvergent but not $(\kappa, \omega)_k$-equiconvergent.*
If E is bornological, then the conditions (1)–(3) *are equivalent to*
(4) *E is topologically isomorphic to $L(\kappa)$.*

Proof (1) $\Rightarrow$ (2): Assume that E is bornologically isomorphic to $L(\kappa)$. By Theorems 3.1 each bounded linearly independent set in E is finite, and by Theorem 2.1, each compact subset of E is finite-dimensional. The linear space E has algebraic dimension κ, being isomorphic to the linear space $L(\kappa)$. Let B be a Hamel basis for the space E.

To show that E is $(\kappa^+, \omega)_k$-equiconvergent, fix an indexed family $\{x_\alpha\}_{\alpha\in\kappa^+} \subseteq \{s \in E^\omega : \lim_{n\to\infty} s(n) = 0\}$. Since bounded linearly independent sets in E are finite, for every $\alpha \in \kappa^+$ there exists a finite set $F_\alpha \subseteq B$ such that the bounded set $x_\alpha[\omega]$ is contained in the linear hull of F_α. Since $|B| = \kappa < \kappa^+$, by the Pigeonhole Principle, for some finite set $F \subseteq B$ the set $A = \{\alpha \in \kappa^+ : F_\alpha = F\}$ is uncountable. Let $[F]$ be the linear hull of the finite set F in the linear space E.

Consider the ordinal $\omega + 1 = \omega \cup \{\omega\}$ endowed with the compact metrizable topology generated by the linear order. For every $\alpha \in A$ let $\bar{x}_\alpha : \omega + 1 \to [F]$ be the continuous function such that $\bar{x}_\alpha{\restriction}\omega = x_\alpha$ and $\bar{x}_\alpha(\omega) = 0$. Let $C_k(\omega + 1, [F])$ be the space of continuous functions from $\omega + 1$ to $[F]$, endowed with the compact-open topology. Since A is uncountable and the space $C_k(\omega + 1, [F]) \supseteq \{\bar{x}_\alpha\}_{\alpha\in A}$ is Polish, there exists a sequence $\{\alpha_n\}_{n\in\omega} \subseteq A$ of pairwise distinct ordinals such that the sequence $(\bar{x}_{\alpha_n})_{n\in\omega}$ converges to $\bar{x}_{\alpha_0}$ in the function space $C_k(\omega + 1, [F])$. Then the set $\Lambda = \{\alpha_n\}_{n\in\omega} \subseteq \kappa^+$ witnesses that E is $(\kappa^+, \omega)_k$-equiconvergent to zero and by the topological homogeneity, E is (κ^+, ω)-equiconvergent. By Theorem 3.1, the bornology of the space E is not (κ, ω)-tall. By Proposition 3.3, the space E is not $(\kappa, \omega)_p$-equiconvergent.

The implication (2) $\Rightarrow$ (3) is trivial. To prove that (3) $\Rightarrow$ (1), assume that each compact subset of E has finite topological dimension and E is $(\kappa^+, \omega)_p$-equiconvergent but not $(\kappa, \omega)_k$-equiconvergent. Let B be a Hamel basis in E. By Theorem 2.1, the space E is bornologically isomorphic to $L(|B|)$. Applying the (already proved) implication (1) $\Rightarrow$ (2), we conclude that E is $(|B|^+, \omega)_k$-equiconvergent, which implies that $|B| \geq \kappa$ (as E is not $(\kappa, \omega)_k$-equiconvergent). Assuming that $|B| > \kappa$, we can see that the family $\{x_b\}_{b\in B} \subseteq E^\omega$ of the sequences $x_b(n) = 2^{-n}b$ witnesses that E is not $(|B|, \omega)_p$-equiconvergent and hence not $(\kappa^+, \omega)_p$-equiconvergent, which contradicts our assumption. So, $|B| = \kappa$ and E is bornologically isomorphic to $L(\kappa)$. If the space E is bornological, then the equivalence (1) $\Leftrightarrow$ (4) follows from the bornological property of E and $L(\kappa)$. □

Observe that the purely topological properties (2), (3) in Theorem 3.5 characterize the free lcs $L(\kappa)$ up to bornological equivalence. We do not know whether the topological structure of the space $L(\kappa)$ determines this lcs uniquely up to a topological isomorphism.

Problem 3.6 Assume that an lcs E is homeomorphic to the free lcs $L(\kappa)$ for some cardinal κ. Is E topologically isomorphic to $L(\kappa)$?

By [2] the answer to this problem is affirmative for $\kappa = \omega$. This affirmative answer can also be derived from the following topological characterizations of the space $L(\omega) = \varphi$. This characterization has been announced in the introduction as Theorem 1.3.

Theorem 3.7 *An lcs E is topologically isomorphic to the free lcs $L(\omega)$ if and only if E is an infinite-dimensional $k_{\mathbb{R}}$-space containing no infinite-dimensional compact subset.*

Proof The "only if" part follows from known topological properties of the space $L(\omega) = \varphi$ mentioned in the introduction. To prove the "if" part, assume that an lcs E is a $k_{\mathbb{R}}$-space and each compact subset of E is finite-dimensional. Choose a Hamel basis B in E and consider the linear continuous operator $T : L(B) \to E$ such that $T(b) = b$ for each $b \in B$. Since B is a Hamel basis, the operator T is injective. We claim that the operator $T^{-1} : E \to L(B)$ is bounded. By Theorem 2.1 the linear hull of each compact subset $K \subseteq E$ is finite-dimensional, which implies that the restriction $T^{-1}{\restriction}K$ is continuous. Since E is a $k_{\mathbb{R}}$-space, T^{-1} is continuous and hence T is a topological isomorphism. Then the free lcs $L(B)$ is a $k_{\mathbb{R}}$-space. Applying [15], we conclude that B is countable and hence E is topologically isomorphic to $L(\omega)$. □

A Tychonoff space X is called *Ascoli* if the canonical map $\delta : X \to C_k(C_k(X))$ assigning to each point $x \in X$ the Dirac functional $\delta_x : C_k(X) \to \mathbb{R}$, $\delta_x : f \mapsto f(x)$, is continuous. By [3], the class of Ascoli spaces includes all Tychonoff $k_{\mathbb{R}}$-spaces. By [15] a Tychonoff space X is countable and discrete if and only if its free lcs $L(X)$ is Ascoli.

Problem 3.8 Assume that an infinite-dimensional lcs E is Ascoli and contains no infinite-dimensional compact subsets. Is E topologically isomorphic to the space $L(\omega)$?

4 Equiconvergence of topological spaces and proof of Theorem 1.2

In this section we establish two results related to equiconvergence in topological spaces.

Theorem 4.1 *If a topological space X admits an ω^ω-base at a point $x \in X$, then X is $(\omega_1, \omega)_k$-equiconvergent at the point x.*

Proof Let $(U_f)_{f\in\omega^\omega}$ be an ω^ω-base at x. To show that X is $(\omega_1, \omega)_k$-equiconvergent at x, fix an indexed family

$$\{x_\alpha\}_{\alpha\in\omega_1} \subseteq \{s \in X^\omega : \lim_{n\to\infty} s(n) = x\}$$

of sequences that converge to x. For every $\alpha \in \omega_1$ consider the function $\mu_\alpha : \omega^\omega \to \omega$ assigning to each $f \in \omega^\omega$ the smallest number $n \in \omega$ such that $\{x_\alpha(m)\}_{m\geq n} \subseteq U_f$. It is easy to see that the function $\mu_\alpha : \omega^\omega \to \omega$ is monotone.

For every $n \in \omega$ and finite function $t \in \omega^n$, let $\omega^\omega_t = \{f \in \omega^\omega : f{\restriction}n = t\}$. By [4, Lemma 2.3.5], for every $f \in \omega^\omega$ there exists $n \in \omega$ such that $\mu_\alpha[\omega^\omega_{f\restriction n}]$ is finite. Let T_α be the set of all finite functions $t \in \omega^{<\omega} = \bigcup_{n\in\omega} \omega^n$ such that $\mu_\alpha[\omega^\omega_t]$ is finite but for any $\tau \in \omega^{<\omega}$ with $\tau \subset t$ the set $\mu_\alpha[\omega^\omega_\tau]$ is infinite. It follows from [4, Lemma 2.3.5] that for every $f \in \omega^\omega$ there exists a unique $t_f \in T_\alpha$ such that $t_f \subset f$.

Let $\delta_\alpha(f) = \max \mu_\alpha[\omega^\omega_{tf}] \geq \mu_\alpha(f)$. It is clear that the function $\delta_\alpha : \omega^\omega \to \omega$ is continuous and hence δ_α is an element of the space $C_p(\omega^\omega, \omega)$ of continuous functions from ω^ω to ω. Here we endow ω^ω with the product topology. The function space $C_p(\omega^\omega, \omega)$ is endowed with the topology of poitwise convergence. By Michael's Proposition 10.4 in [19], the space $C_p(\omega^\omega, \omega)$ has a countable network.

Consider the function $\delta : \omega_1 \to C_p(\omega^\omega, \omega)$, $\delta : \alpha \mapsto \delta_\alpha$, and observe that $\delta_\alpha(f) \geq \mu_\alpha(f)$ for any $\alpha \in \omega_1$ and $f \in \omega^\omega$.

Since the space $C_p(\omega^\omega, \omega)$ has countable network, there exists a sequence $\{\alpha_n\}_{n\in\omega} \subseteq \omega_1$ of pairwise distinct ordinals such that the sequence $(\delta_{\alpha_n})_{n\in\omega}$ converges to δ_{α_0} in the function space $C_p(\omega^\omega, \omega)$. We claim that the sequence $(x_{\alpha_n})_{n\in\omega}$ witnesses that X is $(\omega_1, \omega)_k$-equiconvergent at x. Given any open neighborhood $O_x \subseteq X$ of x, find $f \in \omega^\omega$ such that $U_f \subseteq O_x$. Since the sequence $(x_{\alpha_0}(n))_{n\in\omega}$ converges to x, there exists $m \in \omega$ such that $\{x_{\alpha_0}(n)\}_{n\geq m} \subseteq U_f$. Since the sequence $(\delta_{\alpha_n})_{n\in\omega}$ converges to δ_{α_0} in $C_p(\omega^\omega, \omega)$ we can replace m by a larger number and additionally assume that $\delta_{\alpha_n}(f) = \delta_{\alpha_0}(f)$ for all $n \geq m$. Choose a number $l \geq \delta_{\alpha_0}(f)$ such that for every $n < m$ and $k \geq l$ we have $x_{\alpha_n}(k) \in O_x$. On the other hand, for every $n \geq m$ and $k \geq l$ we have $k \geq l \geq \delta_{\alpha_0}(f) = \delta_{\alpha_n}(f) \geq \mu_{\alpha_n}(f)$ and hence $x_{\alpha_n}(k) \in U_f \subseteq O_x$. □

Another property implying the $(\omega_1, \omega)_p$-equiconvergence is the existence of a countable $\mathsf{cs}^\bullet$-network. First we introduce the necessary definitions.

Let x be a point of a topological space X. We say that a sequence $\{x_n\}_{n\in\omega} \subseteq X$ *accumulates at* x if for each neighborhood $U \subseteq X$ of x the set $\{n \in \omega : x_n \in U\}$ is infinite.

A family $\mathcal{N}$ of subsets of X is defined to be

- an s^*-*network at* x if for any neighborhood $O_x \subseteq X$ of x and any sequence $\{x_n\}_{n\in\omega} \subseteq X$ that accumulates at x there exists $N \in \mathcal{N}$ such that $N \subseteq O_x$ and the set $\{n \in \omega : x_n \in N\}$ is infinite;
- a cs^*-*network at* $x \in X$ if for any neighborhood $O_x \subseteq X$ of x and any sequence $\{x_n\}_{n\in\omega} \subseteq X$ that converges to x there exists $N \in \mathcal{N}$ such that $N \subseteq O_x$ and the set $\{n \in \omega : x_n \in N\}$ is infinite;
- a $\mathsf{cs}^\bullet$-*network at* x if for any neighborhood $O_x \subseteq X$ of x and any sequence $\{x_n\}_{n\in\omega} \subseteq X$ that converges to x there exists $N \in \mathcal{N}$ such that $N \subseteq O_x$ and N contains some point x_n.
- a *network at* x if for any neighborhood $O_x \subseteq X$ the union $\bigcup\{N \in \mathcal{N} : N \subseteq O_x\}$ is a neighborhood of x;

It is clear that for any family $\mathcal{N}$ of subsets of a topological space X and any $x \in X$ we have the following implications.

$$\begin{array}{ccc} (\mathcal{N} \text{ is an } \mathsf{s}^*\text{-network at } x) & & (\mathcal{N} \text{ is a network at } x) \\ \Downarrow & & \Downarrow \\ (\mathcal{N} \text{ is a } \mathsf{cs}^*\text{-network at } x) & \Longrightarrow & (\mathcal{N} \text{ is a } \mathsf{cs}^\bullet\text{-network at } x) \end{array}$$

Theorem 4.2 *If a topological space* X *has a countable* $\mathsf{cs}^{\cdot}$*-network at a point* $x \in X$, *then* X *is* $(\omega_1, \omega)_p$*-equiconvergent at* x.

Proof Let $\mathcal{N}$ be a countable $\mathsf{cs}^\bullet$-network at x and

$$\{x_\alpha\}_{\alpha\in\omega_1} \subseteq \{s \in X^\omega : \lim_{n\to\infty} s(n) = x\}.$$

Endow the ordinal $\omega + 1 = \omega \cup \{\omega\}$ with the discrete topology. For every $\alpha \in \omega_1$ consider the function $\delta_\alpha : \mathcal{N} \to \omega + 1$ assigning to each $N \in \mathcal{N}$ the smallest number $n \in \omega$ such that $x_\alpha(n) \in N$ if such number n exists, and ω if $x_n \notin N$ for all $n \in \omega$. Since $(\omega + 1)^{\mathcal{N}}$ is a metrizable separable space, the uncountable set

$$\{\delta_\alpha\}_{\alpha\in\omega_1} \subseteq (\omega + 1)^{\mathcal{N}}$$

contains a non-trivial convergent sequence. Consequently, we can find a sequence $(\alpha_n)_{n\in\omega}$ of pairwise distinct countable ordinals such that the sequence $(\delta_{\alpha_n})_{n\in\omega}$ converges to δ_{α_0} in the Polish space $(\omega + 1)^{\mathcal{N}}$. We claim that the sequence $(x_{\alpha_n})_{n\in\omega}$ witnesses that the space X is $(\omega_1, \omega)_p$-equiconvergent. Fix any neighborhood $U \subseteq X$ of zero.

Since $\mathcal{N}$ is an $\mathsf{cs}^\bullet$-network, there exists $N \in \mathcal{N}$ and $n \in \omega$ such that $x_n \in N \subseteq U$. Hence

$$d := \delta_{\alpha_0}(N) \le n.$$

Since the sequence $(\delta_{\alpha_n})_{n\in\omega}$ converges to δ_{α_0}, there exists $l \in \omega$ such that

$$\delta_{\alpha_k}(N) = \delta_{\alpha_0}(N) = d$$

for all $k \ge l$. Then for every $k \ge l$ we have $x_{\alpha_k}(d) \in N \subseteq U$. □

The following proposition (connecting ω^ω-bases with networks) is a corollary of Theorem 6.4.1 in [4].

Proposition 4.3 *If $(U_\alpha)_{\alpha\in\omega^\omega}$ is an ω^ω-base at a point x of a topological space X, then $(\bigcap_{\beta\in\uparrow\alpha} U_\beta)_{\alpha\in\omega^{<\omega}}$ is a countable s^*-network at x. Here $\uparrow\alpha = \{\beta \in \omega^\omega : \alpha \subset \beta\}$ for any $\alpha \in \omega^{<\omega} = \bigcup_{n\in\omega} \omega^n$.*

As a consequence of the results presented above about the $(\kappa, \lambda)_p$-equiconvergence and the (κ, λ)-tall bornology for an lcs E, we propose the following proof of Theorem 1.2.

Proof of Theorem 1.2 If an lcs E has an ω^ω-base, then by Theorem 4.1, the space E is $(\omega_1, \omega)_k$-equiconvergent and hence $(\omega_1, \omega)_p$-equiconvergent. The $(\omega_1, \omega)_p$-equiconvergence of E also follows from Proposition 4.3 and Theorem 4.2. Next, by Proposition 3.3, the space E has (ω_1, ω)-tall bornology, which means that each uncountable set in E contains an infinite bounded set. If E has an uncountable Hamel basis H, then H contains an infinite bounded linearly independent set, and by Theorem 2.1 the space E contains an infinite-dimensional compact set. □

5 Radial networks and another proof of Theorem 1.2

A family $\mathcal{N}$ of subsets of a linear topological space E is called a *radial network* if for every neighborhood of zero $U \subseteq E$ and every every $x \in E$ there exist a set $N \in \mathcal{N}$ and a nonzero real number ε such that $\varepsilon \cdot x \in N \subseteq U$.

The following theorem is a "linear" modification of Theorem 4.2.

Theorem 5.1 *If an lcs E has a countable radial network, then each uncountable subset in E contains an infinite bounded subset.*

Proof Let $\mathcal{N}$ be a countable radial network in E, and let A be an uncountable set in E. Endow the ordinal $\omega + 1 = \omega \cup \{\omega\}$ with the discrete topology.

For every $\alpha \in A$ consider the function $\delta_\alpha : \mathcal{N} \to \omega + 1$ assigning to each $N \in \mathcal{N}$ the ordinal

$$\delta_\alpha(N) = \min\{n \in \omega + 1 : 2^{-n} \cdot \alpha \in [-1, 1] \cdot N\}.$$

Here we assume that $2^{-\omega} = 0$.

Since $(\omega + 1)^{\mathcal{N}}$ is a metrizable separable space, the uncountable set $\{\delta_\alpha\}_{\alpha \in A} \subseteq (\omega + 1)^{\mathcal{N}}$ contains a non-trivial convergent sequence. Consequently, we can find a sequence $\{\alpha_n\}_{n\in\omega} \subseteq A$ of pairwise distinct points of A such that the sequence $(\delta_{\alpha_n})_{n\in\omega}$ converges to δ_{α_0} in the Polish space $(\omega + 1)^{\mathcal{N}}$.

We claim that the set $\{\alpha_n\}_{n\in\omega}$ is bounded in X. Fix any neighborhood $U \subseteq X$ of zero.

Since $\mathcal{N}$ is a radial network, there exist a set $N \in \mathcal{N}$ and a nonzero real number ε such that $\varepsilon \cdot \alpha_0 \in N \subseteq U$. Then $d := \delta_{\alpha_0}(N) \in \omega$. Since the sequence $(\delta_{\alpha_n})_{n\in\omega}$ converges to δ_{α_0}, there exists $l \in \omega$ such that $\delta_{\alpha_k}(N) = \delta_{\alpha_0}(N)$ for all $k \geq l$. Then for every $k \geq l$ we have

$$2^{-d} \cdot \alpha_k \in [-1, 1] \cdot N \subseteq [-1, 1] \cdot U$$

and hence $\{\alpha_k\}_{k\geq l} \subseteq [-2^d, 2^d] \cdot U$, which implies that the family $(\alpha_n)_{n\in\omega}$ is bounded in X. □

The implication (1) ⇒ (7) in the following theorem provides an alternative proof of Theorem 1.2, announced in the introduction.

Theorem 5.2 *For an lcs E consider the following properties:*

(1) *E has an ω^ω-base;*
(2) *E has a countable s^*-network at zero;*
(3) *E has a countable cs^*-network at zero;*
(4) *E has a countable $\mathsf{cs}^\bullet$-network at zero;*
(5) *E has a countable radial network at zero;*
(6) *each uncountable set in E contains an infinite bounded subset;*
(7) *E contains an infinite-dimensional compact set.*

Then (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5) ⇒ (6). *If E has uncountable Hamel basis, then* (6) ⇒ (7).

Proof The implication (1) $\Rightarrow$ (2) follows from Proposition 4.3. The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are trivial and (4) $\Rightarrow$ (5) follows from the observation that every $\mathsf{cs}^\bullet$-network at zero in the space E is a radial network for E. The implication (5) $\Rightarrow$ (6) is proved by Theorem 5.1.

If E has an uncountable Hamel basis H, then by (6), there exists an infinite bounded set $B \subseteq H$. By Theorem 2.1, the space E contains an infinite-dimensional compact set. □

Problem 5.3 Is there an lcs E that has a countable radial network but does not have a countable $\mathsf{cs}^\bullet$-network at zero?

6 Applications to spaces $C_p(X)$

A family $\{B_\alpha : \alpha \in \omega^\omega\}$ of bounded (compact) sets covering an lcs E is called a *bounded (compact) resolution* if $B_\alpha \subseteq B_\beta$ for each $\alpha \le \beta$. If additionally every bounded (compact) subset of E is contained in some B_α, we call the family $\{B_\alpha : \alpha \in \omega^\omega\}$ a *fundamental bounded (compact) resolution* of E.

Example 6.1 Let E be a metrizable lcs with a decreasing countable base $(U_n)_{n\in\omega}$ of absolutely convex neighbourhoods of zero. For $\alpha = (n_k)_{k\in\omega} \in \omega^\omega$ put $B_\alpha = \bigcap_{k\in\omega} n_k U_k$ and observe that $\{B_\alpha : \alpha \in \omega^\omega\}$ is a fundamental bounded resolution in E.

A Tychonoff space X is called *pseudocompact* if each continuous real-valued function on X is bounded.

The first part of the following (motivating) result has been proved in [18]; since this is not published yet, we add a short proof.

Proposition 6.2 *For a Tychonoff space X the following assertions are equivalent:*

(1) *The space $C_k(X)$ is covered by a sequence of bounded sets.*
(2) *The space $C_p(X)$ is covered by a sequence of bounded sets.*
(3) *X is pseudocompact.*
Moreover, the following assertions are equivalent:
(4) *$C_p(X)$ is covered by a sequence of bounded sets but is not covered by a sequence of functionally bounded sets.*
(5) *X is pseudocompact and contains a countable subset which is not closed in X or not C^*-embedded in X.*

Proof (1) $\Rightarrow$ (2) is clear. (2) $\Rightarrow$ (3): Assume $C_p(X)$ is covered by a sequence of bounded sets but X is not psudocompact. Then $C_p(X)$ contains a complemented copy of $\mathbb{R}^\omega$, see [1]. But $\mathbb{R}^\omega$ cannot be covered by a sequence of bounded sets, otherwise would be σ-compact. (3) $\Rightarrow$ (1): If X is pseudocompact, then for every $n \in \mathbb{N}$ the set $B_n = \{f \in C(X) : \sup_{x\in X} |f(x)| \le n\}$ is bounded in $C_k(X)$ and $\bigcup_{n\in\mathbb{N}} B_n = C_k(X)$.

The equivalence (4) $\Leftrightarrow$ (5) follows from [24, Problem 399]: $C_p(X)$ is covered by a sequence of functionally bounded subsets o $C_p(X)$ if and only if X is pseudocompact and every countable subset of X is closed and C^*-embedded in X. □

Example 6.3 $C_p([0,\omega_1))$ is covered by a sequence of bounded sets but is not covered by a sequence of functionally bounded sets.

By [10], $C_p(X)$ has a bounded resolution if and only if there exists a K-analytic space L such that $C_p(X) \subseteq L \subseteq \mathbb{R}^X$. The problem when $C_p(X)$ has a fundamental bounded resolution is easier. As a simple application of Theorem 1.2 we prove the following

Proposition 6.4 *For a Tychonoff space X consider the following assertions:*

(1) *$C_p(X)$ admits a fundamental bounded resolution $\{B_\alpha : \alpha \in \omega^\omega\}$.*
(2) *X is countable.*
(3) *$\mathbb{R}^X = \bigcup_{\alpha\in\omega^\omega} \overline{B_\alpha}^{\mathbb{R}^X}$ for a fundamental bounded resolution $\{B_\alpha : \alpha \in \omega^\omega\}$ in $C_p(X)$.*
(4) *The strong (topological) dual $L_\beta(X)$ of $C_p(X)$ is a cosmic space, i.e. a continuous image of a metrizable separable space.*
(5) *$C_p(X)$ is a large subspace of $\mathbb{R}^X$, i.e. for every mapping $f \in \mathbb{R}^X$ there is a bounded set $B \subseteq C_p(X)$ such that $f \in \overline{B}^{\mathbb{R}^X}$.*

Then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (5) *but* (5) $\Rightarrow$ (2) *fails even for compact spaces X.*

The implication (1) $\Rightarrow$ (2) was recently proved by Ferrando, Gabriyelyan and Kąkol [9] (with the help of cs0*-networks). We will derive this implication from Theorem 1.2.

Proof (1) $\Rightarrow$ (2): If $C_p(X)$ has a fundamental bounded resolution $\{B_\alpha : \alpha \in \omega^\omega\}$, then the sets $U_\alpha = \{\xi \in L_\beta(X) : \sup_{f\in B_\alpha} |\xi(f)| \le 1\}$ form an ω^ω-base in $L_\beta(X)$. By [14], every bounded set in $L_\beta(X)$ is finite-dimensional. Applying Theorem 1.2, we conclude that the Hamel basis X of the lcs $L_\beta(X)$ is countable. (2) $\Rightarrow$ (1) is clear. (2) $\Rightarrow$ (3)$\wedge$(5): Since $C_p(X)$ is dense in the metrizable space $\mathbb{R}^X$, the claims hold. (2) $\Rightarrow$ (4): If X is countable, then $L_\beta(X)$ has a fundamental sequence of compact sets covering $L_\beta(X)$ and [19, Proposition 7.7] implies that $L_\beta(X)$ is an $\aleph_0$-space, hence cosmic. (4) $\Rightarrow$ (2): If $L_\beta(X)$ is cosmic, then it is separable, and [12, Corollary 2.5] shows that X is countable. (5) $\not\Rightarrow$ (2): $C_p(X)$ over every Eberlein scattered, compact X satisfies (5), see [13]. □

Item (5) in Proposition 6.4 is strictly connected with the following result.

Theorem 6.5 [12,13] *For a Tychonoff space X, the following conditions are equivalent*:

(i) *$C_p(X)$ is distinguished, i.e. the strong dual $L_\beta(X)$ of the space $C_p(X)$ is bornological.*
(ii) *The strong dual $L_\beta(X)$ of the space $C_p(X)$ is a Montel space.*
(iii) *$C_p(X)$ is a large subspace of $\mathbb{R}^X$.*
(iv) *The strong dual $L_\beta(X)$ of the space $C_p(X)$ carries the finest locally convex topology.*

The following is a linear counterpart to item (4) in Proposition 6.4.

Remark 6.6 A Tychonoff space X is finite if and only if $L_\beta(X)$ is a continuous linear image of a metrizable lcs.

Indeed, if X is finite, nothing is left to prove. Conversely, assume that $L_\beta(X)$ is a continuous linear image of a metrizable lcs E (by a one-to-one map). But $L_\beta(X)$ has only finite-dimensional bounded sets and E fails this property. Hence X is finite.

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