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ON $(0, 1, 2)$ INTERPOLATION IN UNIFORM METRIC

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ABSTRACT. From the well known theorem of G. Faber it follows that for any given matrix of nodes there exists a continuous function for which the Lagrange interpolation polynomial $L_n[f, x]$, generated by the nth row of the matrix, does not tend uniformly to $f(x)$. In this paper we shall provide analogous results for the related operator $H_{n,3}[f, x]$ as defined below.

Let

$$(1) \qquad (-1 \le)x_1 < x_2 < \cdots < x_n(\le 1)$$

be an arbitrary system of nodes of interpolation $(x_k = x_{kn}, k = 1, \ldots, n; n = 1, 2, \ldots)$, and for an arbitrary continuous function $f(x)$ in $[-1, 1]$ (i.e., $f \in C[1, 1]$) and integer $m \ge 1$, consider the $(0, 1, \ldots, m)$ Hermite–Fejér interpolation of order m defined by

$$H_{n,m}^{(j)}(f, x_k) = \delta_{0,j} f(x_k) \ (k = 1, \ldots, n; j = 0, \ldots, m-1).$$

Evidently, $H_{n,m}(f, x)$ is a uniquely determined polynomial of degree at most $mn - 1$.

$H_{n,1}(f, x)$ is the Lagrange interpolation polynomial of $f(x)$; the classical result of G. Faber [4] shows that this cannot be uniformly convergent for all $f \in C[-1, 1]$ for *any* system of nodes (1); while another classical result of P. Erdös and P. Turán [2] asserts that if (1) are the roots of the nth orthogonal polynomial with respect to an arbitrary L^1-integrable weight function $w(x) \ge 0$, then $H_{n,1}(f, x)$ converges in weighted L^2 metric for any $f \in C[-1, 1]$.

For $m = 2$, the situation is different. There exist systems of nodes (1) such that $H_{n,2}(f, x)$ uniformly converges for all $f \in C[-1, 1]$ (e.g., for the roots of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ with $-1 < \alpha, \beta < 0$; see G. Szegö [6], Theorem 14.6). Hence in this case the L^2 convergence follows automatically and is of no interest.

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For $m = 3$ the results are less complete. R. Sakai [5] proved that for the Chebyshev roots $H_{n,3}(f, x)$ cannot converge for all $f \in C[-1, 1]$ (actually he proved this for all odd m's), and later P. Vértesi [7, Theorem 2.7] generalized this result for arbitrary Jacobi nodes (under some additional condition whose validity is checked only for $m \le 5$). Our result here is that *for any system of nodes* (1), $H_{n,3}(f, x)$ *cannot converge uniformly for all* $f \in C[-1, 1]$. This follows from the following more quantitative result on the norm of $H_{n,3}$.

Theorem. *For any system of nodes* (1) *we have*

$$|||H_{n,3}||| \ge c \log n. \tag{2}$$

Proof. An easy calculation shows that

$$H_{n,3}(f, x) = \sum_{k=1}^{n} f(x_k) A_k(x), \tag{3}$$

where

$$A_k(x) = \{1 - 3l_k'(x_k)(x - x_k) + [6l_k'(x_k)^2 - \tfrac{3}{2}l_k''(x_k)](x - x_k)^2\} l_k(x)^3 \quad (k = 1, \dots, n) \tag{4}$$

with the usual notation $l_k(x)$ for the k th fundamental polynomial of Lagrange interpolation. Hence

$$|||H_{n,3}||| = \left\| \sum_{k=1}^{n} |A_k(x)| \right\|$$

where $\| \cdot \|$ is the supremum norm of the corresponding function. Since $1 - 3x + 4.5x^2 > 0$ for any real x, we get from (4)

$$1 - 3l_k'(x_k)(x - x_k) + \left[6l_k'(x_k)^2 - \tfrac{3}{2}l_k''(x_k)\right](x - x_k)^2 \ge \tfrac{3}{2}\left[l_k'(x_k)^2 - l_k''(x_k)\right](x - x_k)^2.$$

Here

$$l_k'(x_k)^2 - l_k''(x_k) = \left(\sum_{i \ne k} \frac{1}{x_k - x_i}\right)^2 - \sum_{\substack{i, j \ne k \\ i \ne j}} \frac{1}{(x_k - x_i)(x_k - x_j)} = \sum_{i \ne k} \frac{1}{(x_k - x_i)^2},$$

i.e.,

$$|A_k(x)| \ge \frac{3}{2} \frac{(x - x_k)^2}{(x_k - x_{k-1})^2} |l_k(x)|^3 \quad (2 \le k \le n). \tag{5}$$

In the rest of the proof of Theorem 1 we use a modification of the idea of proof of Theorem 1 in P. Erdös and P. Turán [3]. We distinguish two cases.

Case 1. There is a k_0, $1 \le k_0 \le n$ such that

$$|l_{k_0}(\xi_0)| = \|l_{k_0}\| \ge n^2 \quad (\xi_0 \in [-1, 1]).$$

Then by Markov's theorem

$$(6)\qquad |l_{k_0}(x)| \ge \tfrac{1}{2}\|l_{k_0}\| \ge \tfrac{1}{2}n^2 \quad \left(|x-\xi_0| \le \frac{1}{2n^2}\right).$$

Thus choosing a $\xi_1 \in [-1, 1]$ such that $|\xi_1 - x_{k_0}| \ge \frac{1}{2n^2} \ge |\xi_1 - \xi_0|$ we obtain from (5) and (6)

$$|A_{k_0}(\xi_1)| \ge \frac{3}{2}\cdot\frac{1}{4}\cdot\left(\frac{1}{2n^2}\right)^2\left(\frac{1}{2}n^2\right)^3 = \frac{3}{2^8}n^2,$$

which is stronger than (2).

Case 2. $\|l_k\| < n^2$ $(k = 1, 2, \dots, n)$. In this case, according to a result of P. Erdös [1], we have (with $x_k = \cos\theta_k$)

$$\left|\sum_{\theta_k\in I} 1 - \frac{|I|}{\pi}n\right| \le \log^2 n \quad (I \subseteq [0, \pi]).$$

Thus

$$\sum_{\theta_k\in I} 1 \ge \frac{1}{15}|I|n \text{ if } |I| \ge 4\frac{\log^2 n}{n}, \quad I \subseteq [0, \pi] \text{ and } n \ge n_0.$$

Therefore using the harmonic-geometric-arithmetic mean inequalities we get

(7)

$$\begin{aligned}\sum_{\theta_k\in I}\frac{1}{(x_k - x_{k-1})^2} &\ge \frac{\sum_{\theta_k\in I} 1}{\left(\prod_{\theta_k\in I}(\theta_{k-1}-\theta_k)\right)^{2/\sum_{\theta_k\in I}1}} \ge \frac{\left(\sum_{\theta_k\in I} 1\right)^3}{\left[\sum_{\theta_k\in I}(\theta_{k-1}-\theta_k)\right]^2} \\ &\ge \frac{\left(\frac{1}{15}|I|n\right)^3}{|I|^2} > 10^{-4}|I|n^3 \text{ if } |I| \ge 4\frac{\log^2 n}{n}, \quad I \subseteq [0, \pi],\ n \ge n_0.\end{aligned}$$

Now let

$$(8)\quad \omega_n(x) = \prod_{k=1}^{n}(x - x_k),\ I_n = \left[-\frac{1}{\log n}, \frac{1}{\log n}\right],\ I_n' = \left[-\frac{1}{2\log n}, \frac{1}{2\log n}\right],$$

$$(9)\quad M_n = \max_{|x|\le 1}|\omega_n(x)| = |\omega_n(\xi)|,\ \overline{M}_n = \max_{x\in I_n}|\omega_n(x)| = |\omega_n(\xi_0)| \quad (x = \cos\theta).$$

According to Lemma 1 in [3], we have

$$(10)\qquad \max_{\theta\in I_n'}|\omega_n'(x)| = O(\eta_n n M_n), \text{ where } \eta_n = \max\left(\frac{1}{\log^2 n}, \frac{\overline{M}_n}{M_n}\right).$$

Subcase 2a. $\overline{M}_n \le M_n/\log^2 n$. Then by (10)

$$\max_{\theta\in I_n'}|\omega_n'(x)| = O\left(\frac{nM_n}{\log^2 n}\right).$$

Choosing $I = I_n'$ in (7), we obtain from (5), (8), (9), and (7)

$$\sum_{\theta_k \in I_n'} |A_k(\xi)| \geq \frac{3}{4} \sum_{\theta_k \in I_n'} \left| \frac{\omega_n(\xi)}{\omega_n'(x_k)} \right|^3 \frac{1}{(x_k - x_{k-1})^2}$$
$$\geq c_1 \frac{\log^6 n}{n^3} \sum_{\theta_k \in I_n'} \frac{1}{(x_k - x_{k-1})^2} \geq c_2 \log^5 n.$$

Subcase 2b. $\overline{M}_n > M_n/\log^2 n$. Then similarly as above,

$$\max_{\theta \in I_n'} |\omega_n'(x)| = O(n\overline{M}_n),$$

and

$$\tag{11} \begin{aligned} \sum_{\theta_k \in I_n'} |A_k(\xi_0)| &\geq \frac{3}{2} \sum_{\theta_k \in I_n'} \left| \frac{\omega_n(\xi_0)}{\omega_n'(x_k)} \right|^3 \frac{1}{|\xi_0 - x_k|(x_k - x_{k-1})^2} \\ &\geq \frac{c_3}{n^3} \sum_{\theta_k \in I_n'} \frac{1}{|\xi_0 - x_k|(x_k - x_{k-1})^2}. \end{aligned}$$

Let $\xi_0 = \cos\theta_0$ and $0 < \theta_0 \leq \frac{\pi}{2}$. Define

$$I_{n,\lambda} = \left[\theta_0 + 4\lambda \frac{\log^2 n}{n}, \theta_0 + 4(\lambda+1)\frac{\log^2 n}{n}\right] \subseteq I_n'$$
$$\left(\lambda = 0, 1, \dots, \left[\frac{n}{10\log^3 n}\right] = m\right).$$

Then by (7) (with $I = I_{n,\lambda}$)

$$\sum_{\theta_k \in I_n'} \frac{1}{|\xi_0 - x_k|(x_k - x_{k-1})^2} \geq \sum_{\lambda=1}^{m} \sum_{\theta_k \in I_{n,\lambda}} \frac{1}{|\theta_0 - \theta_k|(x_k - x_{k-1})^2}$$
$$\geq c_4 \frac{n}{\log^2 n} \sum_{\lambda=1}^{m} \frac{1}{\lambda} \sum_{\theta_k \in I_{n,\lambda}} \frac{1}{(x_k - x_{k-1})^2}$$
$$\geq c_5 \frac{n}{\log^2 n} \cdot \log n \cdot \log^2 n \cdot n^2$$
$$= c_5 n^3 \log n,$$

i.e., by (11)

$$\sum_{\theta_k \in I_n'} |A_k(\xi_0)| \geq c_3 c_5 \log n.$$

The theorem is completely proved.

Remark. We conjecture that the statement of our theorem remains true for any odd m.

REFERENCES

1. P. Erdös, *On the uniform distribution of the roots of certain polynomials*, Ann. of Math. **43** (1942), 59–64.
2. P. Erdös and P. Turán, *On interpolation*. I, Ann. of Math. **38** (1937), 142–155.
3. ——, *An extremal problem in the theory of interpolation*, Acta Math. Acad. Sci. Hung. **12** (1961), 221–233.
4. G. Faber, *Über die interpolatorische Darstellung stetiger Funktionen*, Jahresber. der Deutschen Math. Ver. **23** (1914), 190–210.
5. R. Sakai, *Hermite-Fejér interpolation prescribing higher order derivatives*, J. Approx. Theory (to appear).
6. G. Szegö, *Orthogonal Polynomials*, vol. 23, Amer. Math. Soc. Colloq. Publ., Providence, RI, 1974.
7. P. Vértesi, *Hermite-Fejér interpolations of higher order*. I, Acta Math. Hungarian Academy, (1–2) **59** (1989), 135–152.

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