ON 2-DISTRIBUTIONS IN 8-DIMENSIONAL VECTOR BUNDLES OVER 8-COMPLEXES

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Abstract. It is shown that $\mathbb{Z}_2$-index of a 2-distribution in an 8-dimensional spin vector bundle over an 8-complex is independent of the 2-distribution. Necessary and sufficient conditions for the existence of 2-distributions in such vector bundles are given in terms of characteristic classes and a certain secondary cohomology operation. In some cases this operation is computed.

1. Introduction. In [T1] E. Thomas dealt with the question of existence of a 2-distribution with prescribed Euler class in oriented vector bundles of even dimension m over a closed orientable manifold M of the same dimension. If such a 2-distribution exists over the $m-1$ skeleton of M, the obstruction to extending the distribution to all of M lies in

$$H^m(M;\pi_{m-1}(G_{m,2}))\cong \pi_{m-1}(G_{m,2})\cong \mathbb{Z}\oplus\mathbb{Z}_2.$$

E. Thomas computed the $\mathbb{Z}$-index for all even m and the $\mathbb{Z}_2$-index for $m\equiv 2 \mod 4$. He built the Postnikov tower for the fibration $BSO(m-2)\times BSO(2)\to BSO(m)$, found Postnikov invariants and computed the $\mathbb{Z}_2$-obstruction using a generating class and a secondary cohomology operation. For the dimensions $m\equiv 0 \mod 4$ there is no generating class (see [T3]) in general. Nevertheless, in this case the $\mathbb{Z}_2$-index of 2-distributions of tangent bundles was computed by M. Atiyah and J. Dupont in [AD] using K-theory and the Atiyah–Singer index theorem. This index equals $\frac{1}{2}\big(\chi(M)-\sigma(M)\big) \mod 2$, where $\chi(M)$ is the Euler characteristic and $\sigma(M)$ is the signature of M. M. Crabb and B. Steer extended these K-theoretical methods to oriented vector bundles over closed oriented smooth manifolds with only some mild additional assumptions. For similar questions involving non-orientable vector bundles the considerable work has been done by U. Koschorke in [K], M. H. de Paula Leite Mello in [M] and D. Randall in [R].

Our contribution consists in the observation that for arbitrary spin vector bundles in dimension 8 there exist a generating class and a special secondary cohomology operation which make the computation of the $\mathbb{Z}_2$-index possible. This index is

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independent of the 2-distribution and in the case of oriented vector bundles ξ with $w_2(\xi) = 0$ and $w_4(\xi) = w_4(M)$ it turns out to be equal to the index computed in [CS].

In Section 2 we introduce notation, the spin characteristic classes and the secondary cohomology operation Ω. The main result, Theorem 3.1, its consequences and an example are contained in Section 3. They generalize our previous results on the existence of two linearly independent sections in 8-dimensional spin vector bundles contained in [CV1]. Moreover, comparison of Theorem 3.1 and Remark 4.12 in [CS] makes the computation of the given secondary cohomology operation possible on closed smooth spin manifolds. The proof of Theorem 3.1 is given in Section 4.

2. Notation and preliminaries. All vector bundles will be considered over a connected CW-complex X and will be oriented. The mapping $\delta : H^*(X;\mathbb{Z}_2) \to H^*(X;\mathbb{Z})$ is the Bockstein homomorphism associated with the exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 0$. The mapping $\rho_2 : H^*(X;\mathbb{Z}) \to H^*(X;\mathbb{Z}_2)$ is induced from the reduction mod 2.

We will use $w_i(\xi)$ for the i-th Stiefel–Whitney class of the vector bundle ξ, $p_i(\xi)$ for the i-th Pontrjagin class, and $e(\xi)$ for the Euler class. For a complex vector bundle ξ the symbol $c_i(\xi)$ denotes the i-th Chern class. The classifying spaces for special orthogonal groups $SO(n)$, spinor groups $Spin(n)$ and unitary groups $U(n)$ will be denoted by $BSO(n)$, $BSpin(n)$ and $BU(n)$, respectively. The letters w_i, p_i, $e(n)$ and c_i will stand for the characteristic classes of the universal bundles over the classifying spaces $BSO(n)$, $BSpin(n)$ and $BU(n)$, respectively.

We say that $x \in H^*(X;\mathbb{Z})$ is an element of order i ($i = 2, 3, 4, \dots$) if and only if $x \neq 0$ and i is the least positive integer such that $ix = 0$ (if it exists).

The Eilenberg–MacLane space with n-th homotopy group G will be denoted $K(G,n)$, and ι_n will stand for the fundamental class in $H^n(K(G,n);G)$. Writing the fundamental class, it will be always clear which group G we have in mind.

Now we summarize the results on cohomologies of $BSpin(6)$ and $BSpin(8)$. For details see [Q] and [CV1].

Lemma 2.1. *The cohomology rings of $BSpin(6)$ are*

$$H^*(BSpin(6);\mathbb{Z}_2) \cong \mathbb{Z}_2[w_4, w_6, \varepsilon],$$
$$H^*(BSpin(6);\mathbb{Z}) \cong \mathbb{Z}[q_1, q_2, e(6)],$$

where q_1, q_2 and ε are uniquely determined by the relations

$$p_1 = 2q_1 \quad , \quad p_2 = q_1^2 + 4q_2 \quad , \quad \varepsilon = \rho_2 q_2.$$

Moreover

$$\rho_2 q_1 = w_4 \quad , \quad \rho_2 e(6) = w_6.$$

Lemma 2.2. *The mod 2 cohomology ring of $BSpin(8)$ is*

$$H^*(BSpin(8);\mathbb{Z}_2) \cong \mathbb{Z}_2[w_4, w_6, w_7, w_8, \varepsilon].$$

The only nonzero integer cohomology groups through dimension 8 are

$$
\begin{array}{ll}
H^0(BSpin(8);\mathbb{Z}) \cong \mathbb{Z} & \\
H^4(BSpin(8);\mathbb{Z}) \cong \mathbb{Z} & \textit{with generator } q_1 \\
H^7(BSpin(8);\mathbb{Z}) \cong \mathbb{Z}_2 & \textit{with generator } \delta w_6 \\
H^8(BSpin(8);\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} & \textit{with generators } q_1^2, q_2, e(8),
\end{array}
$$

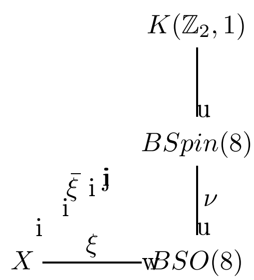
where q_1*,* q_2 *and* ε *are defined by the relations*

$$p_1 = 2q_1 \quad , \quad p_2 = q_1^2 + 2e(8) + 4q_2 \quad , \quad \rho_2 q_2 = \varepsilon.$$

Moreover

$$\rho_2 q_1 = w_4 \quad , \quad \rho_2 e(8) = w_8.$$

Denote ν the standard fibration $BSpin(n) \to BSO(n)$. Let ξ be an 8-dimensional oriented vector bundle over a CW-complex X with $w_2(\xi) = 0$. Then there is a mapping $\bar{\xi} : X \to BSpin(8)$ such that the following diagram is commutative.



We define

$$q_1(\xi) = \bar{\xi}^* q_1.$$

The definition is correct since for two liftings $\bar{\xi}_1$, $\bar{\xi}_2$ of ξ we have $\bar{\xi}_1^* q_1 = \bar{\xi}_2^* q_1$, see [CV1, Section 3].

Further, we define

$$Q_2(\xi) = \{\bar{\xi}^* q_2; \nu \circ \bar{\xi} = \xi\}.$$

The indeterminacy of this class is equal to

$$\mathrm{Indet}(Q_2, \xi, X) = \{\delta(w_6(\xi)x) + q_1(\xi)\delta x^3 + \delta x^7; x \in H^1(X;\mathbb{Z}_2)\},$$

see [CV1]. As an easy consequence we get

Lemma 2.3. *Let one of the following conditions be satisfied*

(i) $H^8(X;\mathbb{Z})$ *has no element of order* 2*,*

(ii) X *is simply connected.*

Then

$$\mathit{Indet}(Q_2, \xi, X) = 0.$$

If the indeterminacy of $Q_2(\xi)$ is zero, we shall write $q_2(\xi)$ instead of $Q_2(\xi)$, to emphasize this fact.

Lemma 2.4 (Computation of $q_1(\xi)$). *If $H^4(X;\mathbb{Z})$ has no element of order 4, then the class $q_1(\xi)$ is uniquely determined by the relations*

$$2q_1(\xi) = p_1(\xi)$$
$$\rho_2 q_1(\xi) = w_4(\xi).$$

Proof. See [CV1, Lemma 3.2].

Lemma 2.5 (Computation of $q_2(\xi)$). *If $H^8(X;\mathbb{Z})$ has no element of order 2, then the class $q_2(\xi)$ is uniquely determined by the relation*

$$16q_2(\xi) = 4p_2(\xi) - p_1^2(\xi) - 8e(\xi).$$

Proof. See [CV1, Lemma 3.3].

On integral classes u of dimension 4 we have

$$Sq^2\rho_2(\delta Sq^2\rho_2 u) = Sq^2Sq^1Sq^2\rho_2 u = Sq^2Sq^3\rho_2 u = Sq^1Sq^4\rho_2 u + Sq^4Sq^1\rho_2 u$$
$$= Sq^1\rho_2 u^2 = 0.$$

Let Ω denote a secondary operation associated with the relation

$$(Sq^2\rho_2)\circ(\delta Sq^2\rho_2) = 0. \tag{2.6}$$

Its indeterminacy on the CW-complex X is

$$\mathrm{Indet}(\Omega, X) = Sq^2\rho_2 H^6(X;\mathbb{Z}).$$

The operation is not uniquely specified by the above relation, for $\Omega' = \Omega + Sq^4$ is the second operation also associated with (2.6). We normalize the operation in the same way as in [T2]. Let $\mathbb{H}P^2$ denote the quaternionic projective plane. We can regard $\mathbb{H}P^2$ as 8-skeleton of the classifying space for the special unitary group $SU(2)$. Let $x \in H^4(\mathbb{H}P^2;\mathbb{Z})$ denote the restriction of the universal Chern class c_2 to $\mathbb{H}P^2$. Then $H^*(\mathbb{H}P^2;\mathbb{Z}) \cong \mathbb{Z}[x]/x^3$. We will let Ω denote the unique operation associated with (2.6) such that

$$\rho_2 x^2 \in \Omega(x). \tag{2.7}$$

According to [T2] this operation satisfies the following

Lemma 2.8. *(i) Let $u, v \in H^4(X;\mathbb{Z})$ be elements from the domain of Ω. Then*

$$\Omega(u+v) = \Omega(u) + \Omega(v) + \{u\cdot v\},$$

where $\{u \cdot v\}$ denotes the image of $\rho_2(u\cdot v)$ in $H^8(X;\mathbb{Z}_2)/Sq^2\rho_2H^6(X;\mathbb{Z})$.
(ii) Let w be any element in $H^4(X;\mathbb{Z})$. Then $2w$ belongs to the domain of Ω, and $\Omega(2w) = \{w^2\}$.

In some special cases the secondary operation can be computed directly.

Lemma 2.9. *Let α be a complex vector bundle over CW-complex. Then*

$$\rho_2\big(c_4(\alpha) + c_2^2(\alpha) + c_2(\alpha)c_1^2(\alpha)\big) \in \Omega(c_2(\alpha)).$$

Proof. See [T2, (2.7)].

Lemma 2.10. *In $H^8(BSpin(6); \mathbb{Z}_2)$*

$$\Omega(q_1) = \rho_2 q_2.$$

Proof. See [CV1, Section 6].

Let β_3 be the canonical 3-dimensional complex vector bundle over $BU(3)$ and let β_1 be 1-dimensional complex vector bundle determined uniquely by its first Chern class $c_1(\beta_3)$. Consider $\beta = \beta_3 \oplus \beta_1$ over $BU(3)$. This is a 4-dimensional complex vector bundle with the following Chern and Pontrjagin classes.

$$\begin{aligned}
c_1(\beta) &= 2c_1 \\
c_2(\beta) &= c_2(\beta_3) + c_1(\beta_3)c_1(\beta_1) = c_2 + c_1^2 \\
c_3(\beta) &= c_3(\beta_3) + c_2(\beta_3)c_1(\beta_1) = c_3 + c_2c_1 \\
c_4(\beta) &= c_3(\beta_3)c_1(\beta_1) = c_3c_1 \\
p_1(\beta) &= 2c_1^2 - 2c_2 \\
p_2(\beta) &= 2c_3c_1 - 4c_1(c_3 + c_2c_1) + (c_2 + c_1^2)^2.
\end{aligned}$$

As a real vector bundle β has dimension 8 and $w_2(\beta) = 0$. Its spin characteristic classes are

$$\begin{aligned}
q_1(\beta) &= c_1^2 - c_2 \\
q_2(\beta) &= -c_3c_1.
\end{aligned} \tag{2.11}$$

Since $\delta Sq^2\rho_2 q_1(\beta) = \delta\rho_2(c_3 + c_2c_1) = 0$, we can apply the secondary operation Ω to $q_1(\beta)$. According to Lemma 2.8 and Lemma 2.9, we get

$$\begin{aligned}
\Omega(q_1(\beta)) =& \Omega(c_1^2 - c_2) = \Omega(c_1^2 + c_2 + (-2c_2)) = \\
& \Omega(c_2(\beta)) + \Omega(-2c_2) = \Omega(c_2(\beta)) + \{c_2^2\} = \\
& \rho_2\big(c_4(\beta) + c_2^2(\beta) + c_2(\beta)c_1^2(\beta)\big) + \{c_2^2\} = \\
& \rho_2(c_3c_1 + c_2^2 + c_1^4) + \{c_2^2\} = \{c_3c_1 + c_1^4\} = \\
& \{Sq^2\rho_2c_3 + Sq^2\rho_2c_1^3\} = \\
& \text{Indet }(\Omega, BU(3)).
\end{aligned}$$

Thus we have proved

Lemma 2.12. *For the 8-dimensional vector bundle β defined above*

$$\Omega(q_1(\beta)) = Sq^2\rho_2H^6(BU(3); \mathbb{Z}).$$

Let M be a smooth 8-dimensional spin manifold, i. e. $w_1(M) = w_2(M) = 0$. We denote $q_1(M)$ and $q_2(M)$ the spin characteristic classes of the tangent bundle. In [CV1] the following lemma was derived.

Lemma 2.13. *Let M be a closed connected smooth spin manifold of dimension 8 and let $H^4(M;\mathbb{Z})$ have no element of order 4. Then*

$$\Omega(q_1(M)) = 0.$$

3. Existence of 2-distributions. Let ξ and η be 8 and 2-dimensional vector bundles, respectively. We will say that there is a 2-distribution η in ξ if there is a 6-dimensional vector bundle ζ such that

$$\xi \cong \eta \oplus \zeta.$$

Under an oriented Poincaré duality complex of formal dimension 8 we understand a CW-complex X satisfying Poincaré duality with respect to some fundamental class $\mu \in H_8(X;\mathbb{Z})$. Our main result is the following

Theorem 3.1. *Let ξ be an 8-dimensional oriented vector bundle over a connected oriented Poincaré duality complex X of formal dimension 8 with $w_2(\xi) = 0$. Then in ξ there exists a 2-distribution whose Euler class is u if and only if there is $v \in H^6(M;\mathbb{Z})$ such that*

(i) $\rho_2 v = w_6(\xi) + w_4(\xi)\rho_2 u + \rho_2 u^3$ *and* $uv = e(\xi)$,
(ii) $\rho_2 q_2(\xi) \in \Omega(q_1(\xi))$,

where $q_1(\xi)$ and $q_2(\xi)$ are the spin characteristic classes and Ω is the secondary cohomology operation defined in Section 2.

Remark. The assumptions on the CW-complex X ensure only that the indeterminacy of the second spin characteristic class of ξ is zero. In fact, we will prove the statement of Theorem 3.1 for connected CW-complexes if the condition (ii) is replaced by

(ii') $\rho_2 Q_2(\xi) \cap \Omega(q_1(\xi)) \neq \varnothing$.

Further, notice that (i) implies $\delta w_6(\xi) = 0$ because $w_4(\xi) = \rho_2 q_1(\xi)$ and $\delta\rho_2 = 0$.

Taking $u = 0$ we get necessary and sufficient conditions for the existence of two linearly independent sections in the vector bundle ξ. (See [CV1], Theorem 5.1.)

Corollary 3.2. *Let ξ be an 8-dimensional oriented vector bundle over a connected oriented Poincaré duality complex X of formal dimension 8 with $w_2(\xi) = 0$ and $w_8(\xi) \neq 0$. Then in ξ there exists a 2-distribution whose Euler class is u if and only if there is $v \in H^6(M;\mathbb{Z})$ such that*

$$\rho_2 v = w_6(\xi) + w_4(\xi)\rho_2 u + \rho_2 u^3 \text{ and } uv = e(\xi).$$

Proof. In the proof of Theorem 3.1 it will be shown that under the condition (i) of Theorem 3.1, $w_8(\xi) \in \text{Indet}\ (\Omega, X)$. Hence, if $w_8(\xi) \neq 0$, $\text{Indet}\ (\Omega, X) = H^8(X;\mathbb{Z}_2)$ and (ii) of Theorem is satisfied.

Corollary 3.3. *Let M be a closed connected smooth spin manifold of dimension 8 and let ξ be an 8-dimensional oriented vector bundle over M with $w_2(\xi) = 0$ and $w_4(\xi) = w_4(M)$. Suppose $H^4(M;\mathbb{Z})$ has no element of order 4. Then in ξ there exists a 2-distribution whose Euler class is u if and only if there is $v \in H^6(M;\mathbb{Z})$ such that*

(I) $\rho_2 v = w_6(M) + w_4(M)\rho_2 u + \rho_2 u^3$ *and* $uv = e(\xi)$
(II) $\{4p_2(\xi) - 8e(\xi) - 2p_1(\xi)p_1(M) + p_1^2(M)\}[M] \equiv 0 \quad \textit{mod } 32.$

Proof. First, $w_4(\xi) = w_4(M)$ implies $w_6(\xi) = w_6(M)$. So it is sufficient to show that under the conditions of Corollary 3.3 the formula (II) is equivalent to (ii) of Theorem 3.1.

Since $\rho_2 q_1(\xi) = w_4(\xi) = w_4(M) = \rho_2 q_1(M)$ there is $y \in H^4(M;\mathbb{Z})$ such that $2y = q_1(\xi) - q_1(M)$, and consequently

$$4y = p_1(\xi) - p_1(M).$$

Due to Lemmas 2.8 and 2.13 we get

$$\Omega(q_1(\xi)) = \Omega(q_1(M) + 2y) = \Omega(q_1(M)) + \Omega(2y) = \rho_2 y^2.$$

Then (ii) of Theorem 3.1 is equivalent to

$$\rho_2 q_2(\xi) = \rho_2 y^2.$$

Since $H^8(M;\mathbb{Z}) \cong \mathbb{Z}$, using the reduction mod 32, this is the same as

$$\begin{aligned} 0 &= \rho_{32}(16q_2(\xi) + (p_1(\xi) - p_1(M))^2) = \\ &= \rho_{32}(4p_2(\xi) - p_1^2(\xi) - 8e(\xi) + p_1^2(\xi) - 2p_1(\xi)p_1(M) + p_1^2(M)), \end{aligned}$$

which is the formula (II) in Corollary 3.3.

Remark. Corollary 3.3 is also a consequence of more general Remark 4.12 proved by Crabb and Steer in [CS] using K-theory and the Atiyah–Singer index theorem. They have shown that for orientable m-dimensional vector bundle ξ over a closed connected oriented smooth m-manifold M with $m \equiv 0 \pmod 4$, $m \geq 8$ and $w_2(\xi) = w_2(M)$ and arbitrary oriented 2-dimensional vector bundle η over M the index of an injection $\lambda : \eta/M \setminus S \to \xi/M \setminus S$ with finite singularities S is

$$E(\lambda) \oplus \frac{1}{2}(e(\xi)[M] + \sigma(\xi)) \bmod 2 \in \mathbb{Z} \oplus \mathbb{Z}_2 \tag{3.4}$$

where $E(\lambda) = \{e(\xi) - e(\lambda) \cdot e(\eta)\}[M]$, $e(\lambda)$ being the Euler class of the partial complement of η, $\sigma(\xi) = \{2^{\frac{m}{2}} \widehat{A}(M) \cdot \widehat{B}(\xi)\}[M]$, $\widehat{A}$ being $\widehat{A}$-genus given by $\prod_{j=1}^{\frac{m}{2}} \frac{1}{2} y_j (\sinh \frac{1}{2} y_j)^{-1}$, $\widehat{B}$ is given by $\prod_{j=1}^{\frac{m}{2}} \cosh \frac{1}{2} y_j$ and the Pontrjagin classes are the elementary symmetric polynomials in the squares y_j^2. In the case $m = 8$ the condition for vanishing of the $\mathbb{Z}_2$-index reads as

$$\{7p_1^2(M) - 4p_2(M) + 60p_2(\xi) + 15p_1^2(\xi) - 30p_1(\xi)p_1(M) + 8 \cdot 45e(\xi)\}[M] \equiv 0 \bmod 32.$$

Since for M a spin manifold and ξ a trivial vector bundle the $\mathbb{Z}_2$-index vanish, we get

$$\{7p_1^2(M) - 4p_2(M)\}[M] \equiv 0 \bmod 32.$$

That is why under the conditions of Corollary 3.3, using the notation from its proof we get

$$\begin{aligned}
8\cdot 45\{e(\xi)[M]+\sigma(\xi)\} \equiv &\{60p_2(\xi)+15p_1^2(\xi)-30p_1(\xi)p_1(M)+8\cdot 45e(\xi)\}[M]\\
\equiv &\{15p_1^2(\xi)+120e(\xi)+240q_2(\xi)+15p_1^2(\xi)\\
&-30p_1(\xi)p_1(M)+8\cdot 45e(\xi)\}[M]\\
\equiv &\{30p_1^2(\xi)-30p_1(\xi)p_1(M)+240q_2(\xi)\}[M]\\
\equiv &\{2p_1(\xi)p_1(M)-2p_1^2(\xi)-16q_2(\xi)\}[M]\\
\equiv &\{-2(2q_1(\xi))\cdot 4y-16q_2(\xi)\}[M]\\
\equiv &\{16q_1(\xi)y-16q_2(\xi)\}[M] \bmod 32.
\end{aligned}$$

This is equivalent to

$$\rho_2 q_2(\xi)=\rho_2(q_1(\xi)y)=w_4(M)\rho_2 y=Sq^4\rho_2 y=\rho_2 y^2,$$

which is just the condition equivalent to condition (II) from Corollary 3.3. (See the above proof.)

Moreover, we can use the comparison of Remark 4.12 in [CS] with our Theorem 3.1 to compute the secondary cohomology operation Ω on closed connected smooth spin manifolds.

Theorem 3.5. *Let M be a closed connected smooth spin manifold of dimension 8. Then*

$$\Omega(z)=\rho_2\frac{1}{2}\{zq_1(M)-z^2)\}$$

for every $z\in H^4(M;\mathbb{Z})$ such that $\delta Sq^2\rho_2 z=0$.

Proof. According to [CV2], Theorem 2, for every $z\in H^4(M;\mathbb{Z})$ there is an 8-dimensional oriented vector bundle ξ with $w_2(\xi)=0$, $q_1(\xi)=z$ and $e(\xi)=0$ and $p_2(\xi)=y$ if and only if $\rho_4 y=\rho_4 z^2$ and $P_3^1\rho_3 2z=\rho_3(2y-4z^2)$ where P_3^1 is the Steenrod cohomology operation mod 3. Since $H^8(M;\mathbb{Z})\cong\mathbb{Z}$, it is easy to see that for every z, there is $y\in H^8(M;\mathbb{Z})$ such that the both conditions are satisfied. Moreover, for such a vector bundle $\delta w_6(\xi)=\delta Sq^2\rho_2 z=0$.

Due to Crabb and Steer the vector bundle ξ has two linearly independent sections (trivial subbundle η) if and only if

$$\frac{1}{2}\sigma(\xi)\equiv 0 \bmod 2.$$

Theorem 3.1 states that ξ has two linearly independent sections if and only if

$$\Omega(q_1(\xi))-\rho_2 q_2(\xi)=0.$$

(Here $\mathrm{Indet}(\Omega,M)=Sq^2\rho_2 H^6(M;\mathbb{Z})=w_2(M)\rho_2 H^6(M;\mathbb{Z})=0$.) That is why

$$\frac{1}{2}\sigma(\xi)\equiv\{\Omega(q_1(\xi))-\rho_2 q_2(\xi)\}\rho_2[M] \bmod 2.$$

The same computation as in the previous Remark yields that the left hand side is

$$\{\frac{1}{8}(p_1(\xi)p_1(M)-p_1^2(\xi))-q_2(\xi))\}[M] \equiv \{\frac{1}{2}(q_1(\xi)q_1(M)-q_1^2(\xi))-q_2(\xi)\}[M] \bmod 2.$$

Hence we get

$$\Omega(q_1(\xi)) = \rho_2\frac{1}{2}\{q_1(\xi)q_1(M) - q_1^2(\xi))\}.$$

Since $z = q_1(\xi)$, we obtain the formula from the theorem.

Example 3.6. We shall consider the complex Grassmann manifold $G_{4,2}(\mathbb{C})$. It is a compact real manifold of dimension 8. Let ξ be a spin vector bundle over $G_{4,2}(\mathbb{C})$ (i. e. $w_2(\xi) = 0$). In [CV1], Example 5.5, the existence of two linearly independent sections of the bundle ξ was examined. Here we will deal with the existence of 2-distribution in ξ.

$H^*(G_{4,2}(\mathbb{C});\mathbb{Z}) \cong \mathbb{Z}[x_1,x_2]/(x_1^3-2x_1x_2, x_2^2-x_1^2x_2)$. The isomorphism is given by $x_1 \mapsto c_1$, $x_2 \mapsto c_2$, where c_1 and c_2 are Chern classes of the canonical complex vector bundle γ_2 over $G_{4,2}(\mathbb{C})$.

Let us write

$$p_1(\xi) = 2ac_1^2 + 2bc_2, \quad p_2(\xi) = Cc_1^2c_2, \quad e(\xi) = Dc_1^2c_2.$$

We have $p_1(\xi) = 2q_1(\xi)$ and $w_4(\xi) = \rho_2(ac_1^2 + bc_2)$. (In [CV1] we used $A = 2a$ and $B = 2b$.) Further, we shall denote here $w_i = w_i(\gamma_2)$. Let

$$u = kc_1 \in H^2(G_{4,2}(\mathbb{C});\mathbb{Z}),$$

where $k \in \mathbb{Z}$ is uniquely determined. We are interested in the existence of the 2-distribution with the Euler class u in ξ. So we are looking for $v = lc_1c_2 \in H^6(G_{4,2}(\mathbb{C});\mathbb{Z})$, $l \in \mathbb{Z}$, satisfying the condition (i) of Theorem 3.1.

We have

$$\begin{aligned} w_6(\xi) &= w_2(\xi)w_4(\xi) + Sq^2w_4(\xi) = Sq^2w_4(\xi) \\ &= Sq^2\rho_2(ac_1^2+bc_2) = Sq^2(aw_2^2+bw_4) = bSq^2w_4 \\ &= bw_2w_4 = \rho_2(bc_1c_2). \end{aligned}$$

Hence

$$w_6(\xi) + w_4(\xi)\rho_2u + \rho_2u^3 = \rho_2((k+1)bc_1c_2).$$

Obviously, the condition (i) has the form

$$l \equiv (k+1)b \bmod 2 \quad , \quad kl = D.$$

Now, we must distinguish two cases, namely $b \equiv 0 \bmod 2$ and $b \equiv 1 \bmod 2$.

For $b \equiv 0 \bmod 2$ we find easily that (i) is satisfied if and only if D is even and $\frac{D}{k}$ is even.

For $b \equiv 1 \bmod 2$ we get that (i) is satisfied if and only if D is even and either $\frac{D}{k}$ is odd or k is odd.

Along the same lines as in Example 5.5 in [CV1] or using Theorem 3.5 ($q_1(G_{4,2}) = c_1^2$) it can be proved that (ii) of Theorem 3.1 is satisfied if and only if

$$C \equiv 2a^2 + 6ab + 3b^2 - 2b + 2D \bmod 8.$$

4. Proof of Theorem 3.1. Let γ_n denote the canonical vector bundle over $BSO(n)$. Let $\pi : BSO(6) \times BSO(2) \to BSO(8)$ stand for the map corresponding to the bundle $\gamma_6 \times \gamma_2$ over $BSO(6) \times BSO(2)$. We shall consider the map $p : BSO(6) \times BSO(2) \to BSO(8) \times BSO(2)$, where $p = (\pi, r)$, r being the projection on the right component. Because p need not be a fibration, we extend immediately the total space $BSO(6) \times BSO(2)$ in the usual way in order to obtain a fibration. The extended total space we denote by $B(SO(6) \times SO(2))$, the extension of p we denote by the same letter. The fibre of this fibration is homotopy equivalent to the Stiefel manifold $V_{8,2}$ (see [T4]).

Now, let ξ resp. η be an 8-dimensional resp. a 2-dimensional oriented vector bundle over a connected CW-complex X. We denote by (ξ, η) the corresponding map $(\xi, \eta) : X \to BSO(8) \times BSO(2)$. It can be immediately seen that in the 8-dimensional vector bundle ξ over X there exists a 2-distribution isomorphic with the vector bundle η if and only if the map (ξ, η) can be lifted in the fibration p.

Next, consider the fibration $\nu : BSpin(8) \to BSO(8)$ whose fibre is the Eilenberg-MacLane space $K(\mathbb{Z}_2, 1)$. An oriented 8-dimensional vector bundle ξ over X is a spin vector bundle if and only if the map $\xi : X \to BSO(8)$ can be lifted in the fibration ν.

Finally, let C together with the maps $\bar{\nu}$ and $\bar{p}$ be a coamalgam of the maps p and $\nu \times id$. We obtain the following commutative diagram.

$$
\begin{array}{ccccc}
 & & K(\mathbb{Z}_2,1) & = & K(\mathbb{Z}_2,1) \\
 & & \downarrow & & \downarrow \\
V_{8,2} & \longrightarrow & C & \xrightarrow{\ \bar{p}\ } & BSpin(8) \times BSO(2) \\
\| & & \downarrow{\scriptstyle \bar{\nu}} & & \downarrow{\scriptstyle \nu \times id} \\
V_{8,2} & \longrightarrow & B(SO(6) \times SO(2)) & \xrightarrow{\ p\ } & BSO(8) \times BSO(2)
\end{array}
$$

Hence, in an 8-dimensional oriented vector bundle ξ over X with $w_2(\xi) = 0$ there exists a 2-distribution isomorphic with the vector bundle η if and only if for some lift $\bar{\xi} : X \to BSpin(8)$ the map $(\bar{\xi}, \eta) : X \to BSpin(8) \times BSO(2)$ can be lifted in the fibration $\bar{p}$. We will find the Postnikov resolution for this fibration using the Postnikov resolution built up by E. Thomas in [T1] for the fibration p.

Let $\mu : BSpin(8) \to BSpin(8) \times BSO(2)$ denote the canonical inclusion. We construct a coamalgam of the maps $\bar{p}$ and μ. It is easy to see that this coamalgam is the classifying space $BSpin(6)$. Thus we obtain the following commutative diagram.

$$
\begin{array}{ccccc}
V_{8,2} & \longrightarrow & BSpin(6) & \xrightarrow{\ \tilde{p}\ } & BSpin(8) \\
\| & & \downarrow{\scriptstyle \tilde{\mu}} & & \downarrow{\scriptstyle \mu} \\
V_{8,2} & \longrightarrow & C & \xrightarrow{\ \bar{p}\ } & BSpin(8) \times BSO(2) \\
\| & & \downarrow{\scriptstyle \bar{\nu}} & & \downarrow{\scriptstyle \nu \times \mathrm{id}} \\
V_{8,2} & \longrightarrow & B(SO(6) \times SO(2)) & \xrightarrow{\ p\ } & BSO(8) \times BSO(2)
\end{array}
$$

The first Postnikov invariant for p is $\delta\theta_6 \in H^7(BSO(8) \times BSO(2); \mathbb{Z})$, where

$$\theta_i = w_i\big((\gamma_8 \times 1) - (1 \times \gamma_2)\big),$$

(γ) denotes the stable equivalence class of γ (see [T1]). Consequently, the Postnikov invariant for $\bar{p}$ is $(\nu \times \mathrm{id})^*(\delta\theta_6)$. Since

$$\theta = (\sum_{i=0}^{8} w_i \otimes 1)(\sum_{n=0}^{\infty} 1 \otimes w_2^n),$$

we get

$$\begin{aligned}(\nu \times \mathrm{id})^*(\delta\theta_6) &= \delta(\nu \times \mathrm{id})^*\theta_6 = \delta(\nu \times \mathrm{id})^*(w_6 \otimes 1 + w_4 \otimes w_2 + w_2 \otimes w_2^2 \\ &\quad + 1 \otimes w_2^3) = \delta(Sq^2\rho_2 q_1 \otimes 1 + \rho_2 q_1 \otimes \rho_2 e_2 + 1 \otimes \rho_2 e_2^3) \\ &= \delta Sq^2 \rho_2 q_1 \otimes 1.\end{aligned}$$

Denote by $s : E \to BSO(8) \times BSO(2)$ the principal fibration with the classifying map $\delta\theta_6 : BSO(8) \times BSO(2) \to K(\mathbb{Z}, 7)$. There exists a 7-equivalence $t : BSO(6) \times BSO(2) \to E$ such that $st = p$. We can replace the space $B(SO(6) \times SO(2))$ and the map t by their homotopy equivalents in such a way that the new map is a fibration. We will denote the new space and the new map by the same symbols (which is a common procedure in building the Postnikov towers). Having performed this change, we shall reconstruct the previous diagram, but keeping the old notation. The new C in this diagram together with the new $\bar{\nu}$ and the new $\bar{p}$ will be a coamalgam of the new $p = st$ and the old $\nu \times id$. Similarly, instead of the old coamalgam $BSpin(6)$, we create a new coamalgam of the new $\bar{p}$ and the old μ. But it can be easily seen that this new coamalgam is again a classifying space $BSpin(6)$ (homotopy equivalent with the original one).

Further let $\bar{s} : \bar{E} \to BSpin(8) \times BSO(2)$ and $\tilde{s} : \tilde{E} \to BSpin(8)$ denote the fibrations induced from $s : E \to BSO(8) \times BSO(2)$ by the maps $\nu \times \mathrm{id}$ and $(\nu \times \mathrm{id})\mu$, respectively. These fibrations are stages in the Postnikov towers for fibrations $\bar{p}$ and $\tilde{p}$ given by the invariants $\delta w_6 \otimes 1$ and δw_6, respectively. We get thus the following commutative diagram where the spaces in the left upper corners of all squares are coamalgams of mappings given in these squares.

$$\begin{array}{ccccc}
BSpin(6) & \xrightarrow{\tilde{t}} & \tilde{E} & \xrightarrow{\tilde{s}} & BSpin(8) \\
\downarrow{\scriptstyle \tilde{\mu}} & & \downarrow & & \downarrow{\scriptstyle \mu} \\
C & \xrightarrow{\bar{t}} & \bar{E} & \xrightarrow{\bar{s}} & BSpin(8) \times BSO(2) \\
\downarrow{\scriptstyle \bar{\nu}} & & \downarrow & & \downarrow{\scriptstyle \nu \times \mathrm{id}} \\
B(SO(6) \times SO(2)) & \xrightarrow{t} & E & \xrightarrow{s} & BSO(8) \times BSO(2)
\end{array}$$

Since, $\tilde{s} \times \mathrm{id} : \tilde{E} \times BSO(2) \to BSpin(8) \times BSO(2)$ is the principal fibration determined by the same element of $H^7(BSpin(8) \times BSO(2))$ as the fibration $\bar{s}$, there is a fibre homotopy equivalence $\alpha : \tilde{E} \times BSO(2) \to \bar{E}$ over $BSpin(8) \times BSO(2)$.

Denote $\bar{t}' : C' \to \tilde{E} \times BSO(2)$ the fibration induced from the fibration $\bar{t}$ by the map α. (C' is again a coamalgam of α and $\bar{t}$.) One can easily show that the map from X into $BSpin(8) \times BSO(2)$ can be lifted in the fibration $\bar{p} = \bar{s}\bar{t}$ if and only if it can be lifted in the fibration $(\tilde{s} \times \mathrm{id})\bar{t}' : C' \to BSpin(8) \times BSO(2)$. Moreover, one can change the preceding diagram in such a way that the map $\mu_E : \tilde{E} \to \tilde{E} \times BSO(2)$ is a canonical inclusion.

$$\begin{array}{ccccc}
BSpin(6) & \xrightarrow{\tilde{t}} & \tilde{E} & \xrightarrow{\tilde{s}} & BSpin(8) \\
\downarrow{\scriptstyle \tilde{\mu}'} & & \downarrow{\scriptstyle \mu_E} & & \downarrow{\scriptstyle \mu} \\
C' & \xrightarrow{\bar{t}'} & \tilde{E} \times BSO(2) & \xrightarrow{\tilde{s} \times \mathrm{id}} & BSpin(8) \times BSO(2) \\
\downarrow{\scriptstyle \bar{\nu}'} & & \downarrow{\scriptstyle \nu_E} & & \downarrow{\scriptstyle \nu \times id} \\
B(SO(6) \times SO(2)) & \xrightarrow{t} & E & \xrightarrow{s} & BSO(8) \times BSO(2)
\end{array}$$

The Postnikov invariants $\bar{\varphi}$ and $\bar{\psi}$ for $\bar{t}'$ are ν_E^*-images of the Postnikov invariants $\varphi \in H^8(E;\mathbb{Z})$ and $\psi \in H^8(E;\mathbb{Z}_2)$ computed by Thomas in [T1]. In this paper Thomas showed that the set of cohomology classes $\{g^*\varphi\}$ as $g : X \to E$ runs over all liftings of $(\xi,\eta) : X \to BSO(8) \times BSO(2)$ (with $(\xi,\eta)^*(\delta\theta_6) = 0$) is the set of classes $\{e(\xi) - e(\eta)v\}$ where v runs over all classes in $H^6(X;\mathbb{Z})$ such that $\rho_2 v = w_6(\xi)$.

For our purposes it is sufficient to find the set

$$k(\bar{\xi},\eta) = \{\bar{g}^*\bar{\psi}; (\tilde{s} \times \mathrm{id})\bar{g} = (\bar{\xi},\eta)\} \tag{4.1}$$

where $\bar{g} : X \to \tilde{E} \times BSO(2)$ and $(\bar{\xi},\eta) : X \to BSpin(8) \times BSO(2)$ are the liftings of $(\xi,\eta) : X \to BSO(8) \times BSO(2)$ with $w_2(\xi) = 0$.

Thomas in [T1] proved that

$$t^*\psi = 0 \quad , \quad j^*\psi = Sq^2\rho_2\iota_6$$

where $j : K(\mathbb{Z},6) \hookrightarrow E$ is the inclusion of the fibre of s. Let $\bar{j} : K(\mathbb{Z},6) \hookrightarrow \tilde{E} \times BSO(2)$ be the inclusion of the fibre of $\tilde{s} \times \mathrm{id}$. Then $\bar{\psi}$ is uniquely determined by the relations

$$\bar{t}'^*\bar{\psi} = 0 \quad , \quad \bar{j}^*\bar{\psi} = Sq^2\rho_2\iota_6.$$

Further, we proceed in a similar way as in the proof of Theorem 5.1 in [CV1] .

The class (4.1) is the coset of $Sq^2\rho_2 H^6(X;\mathbb{Z})$ which is the same as the indeterminacy of the secondary operation Ω. Theorem 3.1 will be proved when we show

$$\bar{\psi} + \tilde{s}^*\rho_2 q_2 \otimes 1 + a\tilde{w}_8 \otimes 1 \in \Omega(\tilde{s}^* q_1 \otimes 1) \tag{4.2}$$

where $a = 0$ or 1. Applying $\bar{g}^*$ to (4.2) we get

$$k(\bar{\xi},\eta) + \rho_2 q_2(\xi) + a w_8(\xi) = \Omega(q_1(\xi)).$$

It means that $(\bar{\xi}, \eta) : X \to BSpin(8) \times BSO(2)$ can be lifted into C' if and only if (i) of Theorem 3.1 is satisfied and $0 \in k(\bar{\xi}, \eta)$, i. e.

$$\rho_2 q_2(\xi) + a w_8(\xi) \in \Omega(q_1(\xi)). \tag{4.3}$$

But if (i) holds, we get

$$\begin{aligned} w_8(\xi) &= \rho_2(uv) = w_6(\xi)\rho_2 u + w_4(\xi)\rho_2 u^2 + \rho_2 u^4 \\ &= Sq^2 \rho_2(q_1(\xi)u + u^3) \in \text{Indet } (\Omega, X). \end{aligned}$$

Hence under (i) the formula (4.3) is equivalent to (ii).

Let us return to the proof of (4.2). Consider the following diagram

$$\begin{array}{ccccccc}
 & & K(\mathbb{Z},6) & = & K(\mathbb{Z},6) & & \\
 & & \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle \bar{j}} & & \\
 & & Y & \xleftarrow{\ f\ } & \tilde{E} \times BSO(2) & \xleftarrow{\ \bar{t}'\ } & C' \\
 & & \downarrow & & \downarrow{\scriptstyle \tilde{s} \times \mathrm{id}} & & \downarrow{\scriptstyle \bar{p}'} \\
K(\mathbb{Z},7) & \xleftarrow{\ \delta Sq^2 \rho_2 \iota_4\ } & K(\mathbb{Z},4) & \xleftarrow{\ q_1 \otimes 1\ } & BSpin(8) \times BSO(2) & = & BSpin(8) \times BSO(2)
\end{array}$$

where Y is the universal example for the operation Ω and $f : \tilde{E} \times BSO(2) \to Y$ is a lifting of the map $\tilde{s}^*(q_1) \otimes 1 : \tilde{E} \times BSO(2) \to K(\mathbb{Z}, 4)$. Let $\omega \in H^8(Y; \mathbb{Z}_2)$ define the operation Ω. We have

$$\bar{j}^*(f^*(\omega)) = l^*(\omega) = Sq^2 \rho_2 \iota_6.$$

Since we know $H^8(\tilde{E}; \mathbb{Z}_2)$ from the Serre exact sequence of the fibration $\tilde{s}$, we get

$$\begin{aligned} \Omega(\tilde{s}_1^* q_1 \otimes 1) &= \bar{\psi} + a\tilde{s}^* w_8 \otimes 1 + b\tilde{s}^* \rho_2 q_2 \otimes 1 + c\tilde{s}^* \rho_2 q_1^2 \otimes 1 \\ &+ d(\tilde{s}^* \rho_2 q_1 \otimes \rho_2 e_2^2) + A Sq^2(\rho_2 q_1 \otimes \rho_2 e_2) \\ &+ B(1 \otimes \rho_2 e_2^4) + \text{Indet } (\Omega, \tilde{E} \times BSO(2)) \\ &= \bar{\psi} + a\tilde{s}^* w_8 \otimes 1 + b\tilde{s}^* \rho_2 q_2 \otimes 1 + c\tilde{s}^* w_4^2 \otimes 1 \\ &+ d\tilde{s}^* \rho_2 q_1 \otimes \rho_2 e_2^2 + \text{Indet } (\Omega, \tilde{E} \times BSO(2)). \end{aligned}$$

where $a, b, c, d \in \{0, 1\}$. We will show that $b = 1$ and $c = d = 0$.

The application of $\tilde{\mu'}^* \bar{t'}^* = \tilde{t}^* \mu_E^*$ to $\Omega(\tilde{s}^* q_1 \otimes 1)$ yields in $H^8(BSpin(6); \mathbb{Z}_2)$

$$\begin{aligned} \Omega(q_1) &= (\tilde{t}^* \mu_E^*)\Omega(\tilde{s}^*(q_1) \otimes 1) = (\tilde{t}^* \mu_E^*)(\bar{\psi}) + a(\tilde{t}^* \mu_E^*)(w_8 \otimes 1) \\ &+ b(\tilde{t}^* \mu_E^*)(\tilde{s}^* \rho_2 q_2 \otimes 1) + c(\tilde{t}^* \mu_E^*)(\tilde{s}^* \rho_2 q_1^2 \otimes 1) + d(\tilde{t}^* \mu_E^*)(\rho_2 q_1 \otimes \rho_2 e_2^2) \\ &= b\rho_2 q_2 + c\rho_2 q_1^2 \end{aligned}$$

According to Lemma 2.10, $b = 1$ and $c = 0$.

Next consider the vector bundle β over $BU(3)$ defined in Section 2. In this 8-dimensional spin vector bundle there is the 2-distribution β_1 with the Euler class

c_1. Hence there exists a map $(\bar\beta,\beta_1): BU(3)\to \tilde E\times BSO(2)$ which is a lifting of $(\beta,\beta_1): BU(3)\to BSpin(8)\times BSO(2)$. The application of $(\bar\beta,\beta_1)^*$ to $\Omega(\tilde s^*(q_1)\otimes 1)$, Lemma 2.12 and (2.11) give

$$\begin{aligned}Sq^2\rho_2H^6(BU(3);\mathbb{Z}) &= \Omega(q_1(\beta)) \supseteq (\bar\beta,\beta_1)^*\Omega(\tilde s^* q_1\otimes 1)\\ &\ni (\bar\beta,\beta_1)^*(\bar\psi + a\tilde s^* w_8\otimes 1 + \tilde s^*\rho_2 q_2\otimes 1 + d\tilde s^*\rho_2 q_1\otimes \rho_2 e_2^2)\\ &= aw_8(\beta)+\rho_2q_2(\beta)+d\rho_2q_1(\beta)\rho_2e^2(\beta_1)\\ &= a\rho_2(c_3c_1)+\rho_2(c_3c_1)+d\rho_2(c_1^2-c_2)\rho_2c_1^2\\ &= (a+1)Sq^2\rho_2c_3 + dSq^2\rho_2c_1^2 + d\rho_2(c_2c_1^2).\end{aligned}$$

That is why $d=0$. This completes the proof of Theorem 3.1.

Remark. q_1 is a generating class for the invariant $\bar\psi$ in the sense of [T3].

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