On 2-spherical Kac-Moody groups and their central extensions

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Abstract

We provide a new presentation for simply connected Kac-Moody groups of 2-spherical type and for their universal central extensions. Under mild local restrictions, these results extend to the more general class of groups of Kac-Moody type (i.e. groups endowed with a root datum).

1 Introduction

Let (W, S) is a Coxeter system and let Φ be the associated root system. A **twin root datum** of type (W, S) is a system $(G, (U_\alpha)_{\alpha\in\Phi})$ consisting of a group and a family of root subgroups U_α indexed by the root system Φ, and satisfying a series of axioms (see Section 3.1.1 below). Groups endowed with a twin root datum include isotropic semi-simple algebraic groups, split and quasi-split Kac-Moody groups, and other more exotic families of groups. Following Tits' terminology [Ti90], a group endowed with a twin root datum is called a group of **Kac-Moody type**. The notion of a twin root datum is the group theoretic counterpart of the geometric notion of a Moufang (twin) building (see Section 3.1.3): every group of Kac-Moody type has a natural action on a Moufang twin building, and conversely, every Moufang twin building admits a group of Kac-Moody type as a group of automorphisms.

We are mainly interested in the case where the Coxeter group (W, S) is **2-spherical** (i.e. all pairs of elements of S generate a finite subgroup of W).

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Let $\mathcal{D} = (G, (U_\alpha)_{\alpha\in\Phi})$ be a twin root datum of type (W, S), let Π be a basis of the root system Φ, let $H := \bigcap_{\alpha\in\Phi} N_G(U_\alpha)$ and, for $\alpha, \beta \in \Pi$, let

$$X_\alpha := \langle U_\alpha \cup U_{-\alpha}\rangle, \quad X_{\alpha,\beta} := \langle X_\alpha \cup X_\beta\rangle$$

and

$$L_\alpha := HX_\alpha, \quad L_{\alpha,\beta} := HX_{\alpha,\beta}.$$

We will need to consider the following condition, which excludes the possibility for $X_{\alpha,\beta}$ to be isomorphic to some very small finite groups:

$$X_{\alpha,\beta}/Z(X_{\alpha,\beta}) \not\cong B_2(2), G_2(2), G_2(3), {}^2F_4(2) \text{ for all pairs } \{\alpha,\beta\} \subset \Pi. \qquad \text{(Co}^*\text{)}$$

The Curtis-Tits theorem asserts that, if the Coxeter group W is finite, then G is the amalgamated sum of its subgroups L_α and $L_{\alpha,\beta}$, where α, β run over Π. By the main result of [AM97], this remains true when the Coxeter system (W, S) is 2-spherical (but W possibly infinite), provided that $\mathcal{D}$ satisfies Condition (Co*). The following theorem, which will be proved in Section 3.3.2 below, is a refinement of the latter fact.

Theorem A. *Suppose that (W, S) is 2-spherical and $\mathcal{D}$ satisfies Condition (Co*). Let $\widetilde{G}$ be the direct limit of the inductive system formed by the X_α and $X_{\alpha,\beta}$ with natural inclusions ($\alpha, \beta \in \Pi$). Then the kernel of the canonical homomorphism $\widetilde{G} \to G$ is central.*

In the special case where W is finite, this is the main result of [Tim04]. However, our approach, which builds upon P. Abramenko's ideas outlined in §2 of [AM97] (see Theorem 3.6 below), is quite different from the one in [Tim04] and provides a considerably shorter proof of that result. We also note that the main result of [AM97] recalled above can be easily deduced from Theorem A (see Corollary 3.8 below).

The kernel of the canonical homomorphism $\widetilde{G} \to G$ might be infinite in general. Actually, in the case where G is a split Kac-Moody group (in the sense of [Ti87]), the group $\widetilde{G}$ is always a quotient of the simply connected central extension of G with finite kernel (see Proposition 3.12 below). Furthermore, if G itself is a simply connected split Kac-Moody group, then G and $\widetilde{G}$ coincide (see Corollary 3.13). As an application of this observation, we now describe an elementary and handy way to define a simply connected split Kac-Moody group of 2-spherical type starting from its Dynkin diagram.

Application. Let $I = \{1, 2, \ldots, n\}$, let $A = (A_{ij})_{i,j\in I}$ be a generalized Cartan matrix. This means that $A_{ii} = 2$, that $A_{ij} \in \mathbb{Z}_{\leq 0}$ and that $A_{ij} = 0$ if and only if $A_{ji} = 0$ for all $i \neq j \in I$. For each subset $J \subset I$, we set $A_J := (A_{ij})_{i,j\in J}$. Assume that the generalized Cartan matrix is 2-spherical. In other words, for each 2-subset J of I we require that A_J is a classical Cartan matrix or, equivalently, that $A_{ij}A_{ji} \leq 3$ for all $i \neq j \in I$. Note that the information contained in such a generalized Cartan matrix is equivalent to the information contained in the associated Dynkin diagram.

Let $\mathbb{K}$ be a field and assume that $\mathbb{K}$ is of cardinality at least 3 (resp. at least 4) if $A_{ij} = -2$ (resp. $A_{ij} = -3$) for some $i, j \in I$. For each $i \in I$, let X_i be a copy of $SL_2(\mathbb{K})$ and for each 2-subset $J = \{i, j\}$ of I, let $X_{i,j}$ be a copy of the universal Chevalley group of type A_J over $\mathbb{K}$. Let also $\varphi_{i,j} : X_i \hookrightarrow X_{i,j}$ be the canonical monomorphism corresponding to the inclusion of Cartan matrices $A_{\{i\}} \hookrightarrow A_{\{i,j\}}$. The direct limit of the inductive system formed by the groups X_i and $X_{i,j}$ along with the monomorphisms $\varphi_{i,j}$ ($i, j \in I$) coincides with the simply connected Kac-Moody group of type A over $\mathbb{K}$.

We now turn to universal central extensions of groups of Kac-Moody type. As before, let $\mathcal{D} = (G, (U_\alpha)_{\alpha\in\Phi})$ be a twin root datum of type (W, S) and let Π, Φ and H be as above. Following Tits [Ti87], we define the **Steinberg group** $\widehat{G}$ to be the direct limit of the inductive system formed by the groups U_α and $U_{[\alpha,\beta]}$ for all prenilpotent pairs $\{\alpha, \beta\} \subset \Phi$ along with natural inclusions (see Section 3.1.1 for the definition of a prenilpotent pair). In other words, $\widehat{G}$ is the group freely generated by the root groups U_α modulo Steinberg commutator relations.

Theorem B. *Assume that $\mathcal{D}$ satisfies Condition (Co*) and that (W, S) is 2-spherical and has no direct factor of type A_1. Let $\widehat{G}$ be the Steinberg group and let $\widehat{U}_\alpha$ be the canonical image of U_α in $\widehat{G}$. Then $(\widehat{G}, (\widehat{U}_\alpha)_{\alpha\in\Phi})$ is a twin root datum of type (W, S) which is isogenic to $\mathcal{D}$ (in the sense of Section 3.2.2). Let $G^\dagger := \langle U_\alpha |\ \alpha \in \Phi\rangle$. Then $G^\dagger$ is perfect and in the 'generic case' $\pi : \widehat{G} \to G^\dagger$ is the universal central extension of $G^\dagger$, where π is the canonical homomorphism.*

We refer to Theorem 3.11 below for a precise statement of what we mean by the 'generic case'. This restriction is a local condition for rank 2 subgroups; in loose terms, Theorem B could be rephrased as follows: *If G has no rank 1 factor, then the Steinberg group $\widehat{G}$ is a universal central extension of $G^\dagger$, provided this is true for all rank 2 subgroups.*

Remark 1. Universal central extensions of simply connected split Kac-Moody groups in the sense of Tits have been studied by J. Morita and U. Rehmann [MR90]. More precisely, it is shown in loc.cit. that, given such a Kac-Moody group G, the universal central extension of G coincides with a quotient of $\widehat{G}$ by certain relations, which they call Relations (B') by analogy with [St68], Page 66. We note that, contrary to Tits' terminology which we have adopted here, it is the aforementioned quotient of $\widehat{G}$ that J. Morita and U. Rehmann call the 'Steinberg group'. Actually, it follows from Theorem B (see also Remark 3.7(a) on p. 530 of [Ti87]) that, in the case of twin root data of 2-spherical type, both definitions of the Steinberg group coincide. This is however false in general, see Remark 2.

Remark 2. The condition that (W, S) is 2-spherical cannot be removed from Theorem B. Let us illustrate this by considering a twin root datum $(G, (U_\alpha)_{\alpha\in\Phi})$ of type $\tilde{A}_1$, namely whose type is the Coxeter system (W, S) with W infinite dihedral. Let T be the translation subgroup of W, let $\Pi = \{\phi_1, \phi_2\}$ be a basis of the root system Φ and for $i = 1, 2$, let

$$\Phi_i := \{\phi \in \Phi |\ t(\phi) \in \{\phi_i, s_{\phi_i}(\phi_{3-i})\} \text{ for some } t \in T\}.$$

Then $\Phi = \Phi_1 \sqcup \Phi_2$ and a pair of roots $\{\alpha, \beta\} \subset \Phi$ is prenilpotent if and only if either $\{\alpha, \beta\} \subset \Phi_1$ or $\{\alpha, \beta\} \subset \Phi_2$. Therefore, setting $U_i := \langle U_\alpha |\ \alpha \in \Phi_i\rangle$ for $i = 1, 2$, the Steinberg group $\widehat{G}$ splits as $\widehat{G} = U_1 * U_2$ and is thus center–free.

As a consequence of Theorem B, we have the following result, which elucidates somewhat Remark 3.7(c) on p. 530 of [Ti87] (see Section 3.5 below for a review of the different notions and notation used here).

Corollary C. *Let A be a generalized Cartan matrix of 2-spherical type without direct factor of type A_1, let $\mathcal{K} = (I, A, \Lambda, (c_i)_{i\in I}, (h_i)_{i\in I})$ be a Kac-Moody root datum of type A such that the h_i's generate the lattice $\Lambda^\vee$ and let $\mathcal{K}_A^{\mathrm{sc}}$ be the simply connected root datum* *of type A. If $\mathbb{K}$ is a field of cardinality at least 5, then the Steinberg group $\mathfrak{St}_A(\mathbb{K})$ is the universal central extension of the Kac-Moody group $\mathcal{G}_{\mathcal{K}}(\mathbb{K})$. If moreover $\mathbb{K} \neq \mathbb{F}_9$ is an algebraic extension of a finite field, then the Steinberg group $\mathfrak{St}_A(\mathbb{K})$ coincides with $\mathcal{G}_{\mathcal{K}_A^{sc}}(\mathbb{K})$.*

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2 Some definitions and facts from the theory of buildings

2.1 Buildings and twinnings

2.1.1 Buildings

Let (W, S) be the Coxeter system. A **building** of type (W, S) is a pair $\mathcal{B} = (\mathcal{C}, \delta)$ where $\mathcal{C}$ is a set and $\delta : \mathcal{C} \times \mathcal{C} \to W$ is a **distance function** satisfying the following axioms where $x, y \in \mathcal{C}$ and $w = \delta(x, y)$:

(Bu 1) $w = 1$ if and only if $x = y$;

(Bu 2) if $z \in \mathcal{C}$ is such that $\delta(y, z) = s \in S$, then $\delta(x, z) = w$ or ws, and if, furthermore, $l(ws) = l(w) + 1$, then $\delta(x, z) = ws$;

(Bu 3) if $s \in S$, there exists $z \in \mathcal{C}$ such that $\delta(y, z) = s$ and $\delta(x, z) = ws$.

Given $s \in S$, chambers $x, y \in \mathcal{C}$ are called s-**adjacent** if $\delta(x, y) \in \{1, s\}$. Two chambers are called **adjacent** if they are s-adjacent for some $s \in S$. A building of type (W, S) is called **thick** if for every chamber x and every $s \in S$, there exist at least three chambers s-adjacent to x.

2.1.2 Twin buildings

Let $\mathcal{B}_+ = (\mathcal{C}_+, \delta_+), \mathcal{B}_- = (\mathcal{C}_-, \delta_-)$ be two buildings of the same type (W, S). A **codistance** (or a **twinning**) between $\mathcal{B}_+$ and $\mathcal{B}_-$ is mapping $\delta_* : (\mathcal{C}_+ \times \mathcal{C}_-) \cup (\mathcal{C}_- \times \mathcal{C}_+) \to W$ satisfying the following axioms, where $\epsilon \in \{+, -\}, x \in \mathcal{C}_\epsilon, y \in \mathcal{C}_{-\epsilon}$ and $w = \delta_*(x, y)$:

(Tw 1) $\delta_*(y, x) = w^{-1}$;

(Tw 2) if $z \in \mathcal{C}_{-\epsilon}$ is such that $\delta_{-\epsilon}(y, z) = s \in S$ and $l(ws) = l(w) - 1$, then $\delta_*(x, z) = ws$;

(Tw 3) if $s \in S$, there exists $z \in \mathcal{C}_{-\epsilon}$ such that $\delta_{-\epsilon}(y, z) = s$ and $\delta_*(x, z) = ws$.

A **twin building of type** (W, S) is a triple $(\mathcal{B}_+, \mathcal{B}_-, \delta_*)$ where $\mathcal{B}_+, \mathcal{B}_-$ are buildings of type (W, S) and where δ_* is a twinning between $\mathcal{B}_+$ and $\mathcal{B}_-$. A crucial feature of twin buildings is that they constitute rather rigid structures. This is made more precise in the following basic but extremely important result.

Proposition 2.1. *Let $\mathcal{B} = (\mathcal{B}_+, \mathcal{B}_-, \delta_*)$ be a thick twin building and let c, d be opposite chambers of $\mathcal{B}$. Let $E_1(c)$ denote the set of all chambers of $\mathcal{B}$ adjacent to c. Any two automorphisms of $\mathcal{B}$ coincide if and only if their restrictions to $E_1(c) \cup \{d\}$ coincide.*

Proof. This is Théorème 1 on Page 87 of [Ti88/89]. □

2.1.3 The Moufang property

Let $\mathcal{B} = ((\mathcal{C}_+, \delta_+), (\mathcal{C}_-, \delta_-), \delta_*)$ be a thick twin building of type (W, S).

Let $x_+ \in \mathcal{C}_+$ and $c_- \in \mathcal{C}_-$ be chambers such that $\delta_*(x_+, x_-) = s \in S$. The set

$$\phi(x_+, x_-) := \{c \in \mathcal{C}_\epsilon | \ \epsilon \in \{+, -\}, \delta_\epsilon(x_\epsilon, c) = s\delta_*(x_{-\epsilon}, c) \text{ and } \ell(\delta_\epsilon(x_\epsilon, c)) < \ell(s\delta_\epsilon(x_\epsilon, c))\}$$

is called a **(twin) root** of $\mathcal{B}$. Given ϕ a twin root of $\mathcal{B}$, the group

$$U_\phi := \{g \in \mathrm{Aut}(\mathcal{B}) | \ g \text{ fixes } \rho \text{ pointwise for each panel } \rho \text{ such that } |\rho \cap \phi| = 2\}$$

is called the **root group** associated with ϕ.

We now assume that (W, S) has no direct factor of type A_1 (i.e. no element of S is central in W). It is then a consequence of Proposition 2.1 that for each twin root ϕ and each panel ρ such that $|\rho \cap \phi| = 1$, the root group U_ϕ acts freely on the set $\rho \backslash \phi$. We say that the twin building $\mathcal{B}$ has the **Moufang property** (or simply is **Moufang**) if U_ϕ is transitive on $\rho \backslash \phi$ for each twin root ϕ and each panel ρ such that $|\rho \cap \phi| = 1$.

Moufang twin buildings provide the geometric counterpart of the algebraic notion of (twin) root data. We refer to Section 3.1.3 below for the exact correspondence.

3 Groups of Kac-Moody type

3.1 Twin root data

3.1.1 Definitions

Let (W, S) be a Coxeter system, let Φ be the associated root system (viewed either as a set of half-spaces in the chamber system associated with (W, S) or as a subset of the real vector space $\mathbb{R}^S$) and let Π be a basis of Φ.

A pair of roots $\{\alpha, \beta\} \subset \Phi$ is called **prenilpotent** if there exist $w, w' \in W$ such that $\{w(\alpha), w(\beta)\} \subset \Phi_+$ and $\{w'(\alpha), w'(\beta)\} \subset \Phi_-$. In that case, we set

$$[\alpha, \beta] := \bigcap_{\substack{w \in W \\ \epsilon \in \{+,-\}}} \{\gamma \in \Phi | \ \{w(\alpha), w(\beta)\} \subset \Phi_\epsilon \Rightarrow w(\gamma) \in \Phi_\epsilon\}$$

and

$$]\alpha, \beta[:= [\alpha, \beta] \backslash \{\alpha, \beta\}.$$

A sequence of roots $\alpha_0, \alpha_1, \dots, \alpha_n$ is called a **nibbling sequence** if for each $i = 1, \dots, n$, the pair $\{\alpha_0, \alpha_i\}$ is prenilpotent and $[\alpha_0, \alpha_i] = [\alpha_0, \alpha_{i-1}] \cup \{\alpha_i\}$ (this terminology is inspired by Section 13.25 of [Ti74]). It is an easy consequence of the definition that for any prenilpotent pair of roots $\{\alpha, \beta\}$, the set $[\alpha, \beta]$ can be ordered in a nibbling sequence of the form $\alpha = \alpha_0, \alpha_1, \dots, \alpha_n = \beta$.

A **twin root datum** of type (W, S) is a system $(G, (U_\alpha)_{\alpha\in\Phi})$ consisting of a group G together with a family of subgroups U_α indexed by the root system Φ, which satisfy the following axioms, where $H := \bigcap_{\alpha\in\Phi} N_G(U_\alpha)$, $U_+ := \langle U_\alpha |\ \alpha \in \Phi_+\rangle$ and $U_- := \langle U_\alpha |\ \alpha \in \Phi_-\rangle$:

(TRD 0) For each $\alpha \in \Phi$, we have $U_\alpha \neq \{1\}$.

(TRD 1) For each prenilpotent pair $\{\alpha, \beta\} \subset \Phi$, the commutator group $[U_\alpha, U_\beta]$ is contained in the group $U_{]\alpha,\beta[} := \langle U_\gamma |\ \gamma \in]\alpha, \beta[\rangle$.

(TRD 2) For each $\alpha \in \Pi$ and each $u \in U_\alpha\backslash\{1\}$, there exists elements $u', u'' \in U_{-\alpha}$ such that the product $\mu(u) := u'uu''$ conjugates U_β onto $U_{s_\alpha(\beta)}$ for each $\beta \in \Phi$.

(TRD 3) For each $\alpha \in \Pi$, the group $U_{-\alpha}$ is not contained in U_+ and the group U_α is not contained in U_-.

(TRD 4) $G = H\langle U_\alpha |\ \alpha \in \Phi\rangle$.

3.1.2 Basic properties

As before, let (W, S) be a Coxeter system, let Φ be the associated root system and let Π be a basis of Φ.

Lemma 3.1. *Let $(G, (U_\alpha)_{\alpha\in\Phi})$ be a twin root datum and let $\alpha \in \Pi$. Let $X_\alpha := \langle U_\alpha \cup U_{-\alpha}\rangle$. We have*

$$N_{X_\alpha}(U_\alpha) \cap N_{X_\alpha}(U_{-\alpha}) = \langle \mu(u)\mu(v) |\ u, v \in U_\alpha\backslash\{1\}\rangle,$$

where μ is defined as in Axiom (TRD 2).

Proof. This follows from (33.9) in [TW02]. □

Lemma 3.2. *Let $(G, (U_\alpha)_{\alpha\in\Phi})$ be a twin root datum and let $H \leq G$ be a subgroup which normalizes U_α for each $\alpha \in \Phi$ and such that $\mu(u)H = \mu(v)H$ for all $u, v \in U_\beta$ and all $\beta \in \Pi$, where μ is defined as in Axiom (TRD 2). Then $H = \bigcap_{\alpha\in\Phi} N_G(U_\alpha)$.*

Proof. This follows from the corollary in Section 1.5.3 of [Ré02]. □

Lemma 3.3. *Let $(G, (U_\alpha)_{\alpha\in\Phi})$ be a twin root datum such that $G = \langle U_\alpha |\ \alpha \in \Phi\rangle$. For each $\alpha \in \Pi$, let $X_\alpha := \langle U_\alpha \cup U_{-\alpha}\rangle$ and let $H_\alpha := N_{X_\alpha}(U_\alpha) \cap N_{X_\alpha}(U_{-\alpha})$. We have*

$$\bigcap_{\alpha\in\Phi} N_G(U_\alpha) = \prod_{\alpha\in\Pi} H_\alpha.$$

Proof. Let $H := \langle H_\alpha |\ \alpha \in \Pi\rangle$. Combining Lemma 3.1 with Axiom (TRD 2), we see that H_α normalizes U_β for all $\alpha \in \Pi$ and $\beta \in \Phi$. It follows that H normalizes U_β for all $\beta \in \Phi$. Therefore, we have $H \cap X_\alpha \leq H_\alpha$ for all $\alpha \in \Phi$, from which it follows that $H \cap X_\alpha = H_\alpha$ and, hence, that H normalizes H_α. This implies $H = \prod_{\alpha\in\Pi} H_\alpha$. Finally, using Lemma 3.1 again, we obtain $\mu(u)\mu(v) \in H$ for all $u, v \in U_\alpha\backslash\{1\}$ and all $\alpha \in \Pi$. The conclusion now follows from Lemma 3.2. □

3.1.3 Relationship with buildings and twinnings

This section aims to recall the correspondence between twin root data and Moufang twin buildings (see Théorème 3 in [Ti88/89] and Proposition 7 in [Ti90]).

Let $\mathcal{D} = (G, (U_\alpha)_{\alpha\in\Phi})$ be a twin root datum of type (W, S). Let H be the intersection of the normalizers of all U_ϕ's, let N be the subgroup of G generated by H together with all $\mu(u)$ such that $u \in U_\phi\backslash\{1\}$, where $\mu(u)$ is as in (TRD2), and for each $\epsilon \in \{+, -\}$, let $U_\epsilon := \langle U_\alpha |\ \alpha \in \Phi_\epsilon\rangle$ and $B_\epsilon := H.U_\epsilon$. We recall from [Ti90], Proposition 4, that (G, B_+, N) and (G, B_-, N) are both BN-pairs of type (W, S). Thus, we have corresponding Bruhat decompositions of G:

$$G = \coprod_{w\in W} B_+wB_+ \qquad \text{and} \qquad G = \coprod_{w\in W} B_-wB_-.$$

For each $\epsilon \in \{+, -\}$, the set $\mathcal{C}_\epsilon := G/B_\epsilon$ endowed with the map $\delta_\epsilon : \mathcal{C}_\epsilon \times \mathcal{C}_\epsilon \to W$ by

$$\delta_\epsilon(gB_\epsilon, hB_\epsilon) = w \Leftrightarrow B_\epsilon g^{-1}hB_\epsilon = B_\epsilon wB_\epsilon,$$

has a canonical structure of a thick building of type (W, S).

The twin root datum axioms imply that G also admits Birkhoff decompositions (see Lemma 1 in [Ab96]):

$$G = \coprod_{w\in W} B_\epsilon wB_{-\epsilon}$$

for each $\epsilon \in \{+, -\}$. The pair $((\mathcal{C}_+, \delta_+), (\mathcal{C}_-, \delta_-))$ of buildings admits a natural twinning by means of the W-codistance δ^* defined by

$$\delta^*(gB_\epsilon, hB_{-\epsilon}) = w \Leftrightarrow B_\epsilon g^{-1}hB_{-\epsilon} = B_\epsilon wB_{-\epsilon}$$

for each $\epsilon \in \{+, -\}$. The triple $\mathcal{B} := ((\mathcal{C}_+, \delta_+), (\mathcal{C}_-, \delta_-), \delta^*)$ is a twin building of type (W, S). The diagonal action of G on $\mathcal{C}_+ \times \mathcal{C}_-$ by left multiplication is transitive on pairs of opposite chambers.

If (W, S) has no direct factor of type A_1, then $\mathcal{B}$ is a Moufang twin building and the groups U_α's, as well as all their conjugates, are the corresponding root groups.

Conversely, let $\mathcal{B}$ be a Moufang twin building of type (W, S). Let $x_+ \in \mathcal{C}_+$ and $x_- \in \mathcal{C}_-$ be opposite chambers; we set

$$\Sigma := \{c \in \mathcal{C}_\epsilon |\ \epsilon \in \{+, -\} \text{ and } \delta_\epsilon(x_\epsilon, c) = \delta_*(x_{-\epsilon}, c)\}.$$

Let $\Phi(\Sigma)$ be the set of all twin roots of $\mathcal{B}$ such that $\phi \subset \Sigma$. Let $G(\mathcal{B})$ be the group generated by all root groups U_ϕ such that $\phi \in \Phi(\Sigma)$. Then $(G(\mathcal{B}), (U_\phi)_{\phi\in\Phi(\Sigma)})$ is a twin root datum of type (W, S) and the associated twin building is isomorphic to $\mathcal{B}$.

3.2 Isogeny classes of twin root data

3.2.1 A crucial lemma

The following result is straightforward but crucial for our discussion of central extensions.

Lemma 3.4. *Let $\mathcal{D} = (G, (U_\alpha)_{\alpha\in\Phi})$ be a twin root datum such that $G = \langle U_\alpha |\ \alpha \in \Phi\rangle$. The kernel of the action of G on the twin building associated with $\mathcal{D}$ coincides with the center of G.*

Proof. Let $K \trianglelefteq G$ denote this kernel. We have $K \leq \bigcap_{\alpha\in\Phi} N_G(U_\alpha)$. Therefore, it follows from Proposition 2.1 that $K = \bigcap_{\alpha\in\Pi} C_G(U_\alpha)$. Since the latter equality holds for any basis Π of the root system Φ, we deduce $K = \bigcap_{\alpha\in\Phi} C_G(U_\alpha)$ and the conclusion follows from the hypothesis $G = \langle U_\alpha | \ \alpha \in \Phi\rangle$. □

3.2.2 Reduction of twin root data and isogeny

In this section, we introduce an equivalence relation on the collection of all twin root data. We call it the **isogeny** relation by analogy with the theory of algebraic groups. We first need some additional terminology.

A twin root datum $\mathcal{D} = (G, (U_\alpha)_{\alpha\in\Phi})$ of type (W, S) is called **centered** if $G = \langle U_\alpha | \ \alpha \in \Phi\rangle$. It is called **reduced** if it is centered and if moreover G acts faithfully on the twin building associated with $\mathcal{D}$. By Lemma 3.4, if $\mathcal{D}$ is centered, then it is reduced if and only if G is center-free.

Let $\mathcal{D} = (G, (U_\alpha)_{\alpha\in\Phi})$ be a twin root datum of type (W, S), let $H = \bigcap_{\alpha\in\Phi} N_G(U_\alpha)$, let $G^\dagger := \langle U_\alpha | \ \alpha \in \Phi\rangle$ and let $p : G^\dagger \to G_0 := G^\dagger/Z(G^\dagger)$ be the canonical projection. It follows from Lemma 3.4 that $\operatorname{Ker} p \leq H$. Since $H \cap U_\alpha = \{1\}$ for all $\alpha \in \Phi$ (see Section 3.5.4 in [Ré02]), the restriction of p to each U_α is injective. Denoting the image of U_α under p again by U_α, one deduces easily that $\mathcal{D}_0 := (G_0, (U_\alpha)_{\alpha\in\Phi})$ is a reduced twin root datum of type (W, S). This twin root datum is called the **reduction** of $\mathcal{D}$. Two twin root data are called **isogenic** if they have the same reduction.

They key role of Lemma 3.4 is that it suggests a relationship between central extensions of G and the isogeny class of $\mathcal{D}$ when $\mathcal{D}$ is centered. Theorem B of the introduction shows that if (W, S) is 2-spherical and with no direct factor of type A_1, then (in the generic case) the universal central extension of G is canonically endowed with a (centered) twin root datum $\widehat{\mathcal{D}}$ which is isogenic to $\mathcal{D}$. Thus the class $\mathfrak{C}(\mathcal{D})$ of all centered twin root data isogenic to $\mathcal{D}$ possesses a unique 'minimal' element $\mathcal{D}_0$ (the reduction of $\mathcal{D}$) and a unique 'maximal' element $\widehat{\mathcal{D}}$ in the sense that for any twin root datum $(G', (U'_\alpha)_{\alpha\in\Phi}) \in \mathfrak{C}(\mathcal{D})$, the group G' is a central extension of G_0 and a central quotient of $\widehat{G}$.

We end by noting that for non-centered twin root data, the kernel of the action of G on the associated twin building can be arbitrary (G can be replaced by a direct product $G \times K$ where K is any group). This remark, together with Lemma 3.4, provides a justification for the choice of the term 'centered'.

3.3 Amalgams

Let (W, S) be a Coxeter system, let Φ be the associated root system and let Π be a basis of Φ. Let $\mathcal{D} = (G, (U_\alpha)_{\alpha\in\Phi})$ be a twin root datum of type (W, S), let $H := \bigcap_{\alpha\in\Phi} N_G(U_\alpha)$ and, for $\alpha, \beta \in \Pi$, let

$$X_\alpha := \langle U_\alpha \cup U_{-\alpha}\rangle, \quad X_{\alpha,\beta} := \langle X_\alpha \cup X_\beta\rangle$$

and

$$L_\alpha := HX_\alpha, \quad L_{\alpha,\beta} := HX_{\alpha,\beta}.$$

We keep this notation throughout this section.

3.3.1 Presentations for U_+ and U_-

It is known (see [Ré02], Théorème of Section 3.5.3) that for $\epsilon \in \{+,-\}$, the group U_ϵ is the amalgam of its subgroups U_α for all roots $\alpha \in \Phi_\epsilon$ and $U_{[\alpha,\beta]}$ for all prenilpotent pairs $\{\alpha,\beta\} \subset \Phi_\epsilon$. The main result if this section shows that this presentation can be made much more economical under the hypotheses of the introduction. Before stating that result, we first recall an auxiliary lemma which we shall need several times later.

Lemma 3.5. *Suppose that $\mathcal{D}$ is of 2-spherical type and satisfies Condition (Co*). For each pair of roots $\{\alpha,\beta\} \subset \Phi$ which is the basis of a finite root subsystem, we have*

$$U_{[\alpha,\beta]} = \langle U_\alpha \cup U_\beta \rangle$$

or equivalently

$$U_{]\alpha,\beta[} = [U_\alpha, U_\beta].$$

Proof. This follows from Lemma 18 and Proposition 7 on Page 60 of [Ab96]. □

The next theorem is trivial in the case where $\mathcal{D}$ is of spherical type. It can be viewed as a 'unipotent version' of Théorème A of [AM97], and its proof is similar to the proof of Théorème A which is sketched in loc.cit. Condition (Co*) was defined in the introduction (see Page 2).

Theorem 3.6. *Suppose that $\mathcal{D}$ is of 2-spherical type and satisfies Condition (Co*). Let $\epsilon \in \{+,-\}$. Then U_ϵ is isomorphic to the direct limit of the inductive system formed by the groups U_α for all $\alpha \in \Phi_+$ and $U_{[\alpha,\beta]}$ for all prenilpotent pair $\{\alpha,\beta\} \subset \Phi_+$ such that $s_\alpha s_\beta$ has finite order.*

Proof. Let $\widehat{U}_\epsilon$ be the direct limit of the inductive system formed by the groups U_α for all $\alpha \in \Phi_+$ and $U_{[\alpha,\beta]}$ for all prenilpotent pair $\{\alpha,\beta\} \subset \Phi_+$ such that $s_\alpha s_\beta$ has finite order, and let $\widehat{\pi} : \widehat{U}_\epsilon \longrightarrow U_\epsilon$ be the canonical homomorphism. We need to prove that $\widehat{\pi}$ is injective.

For each $\alpha \in \Phi_\epsilon$, let $\widehat{U}_\alpha$ be the canonical image of U_α in $\widehat{U}_\epsilon$. As mentioned above, we already know that U_ϵ is the amalgam of its subgroups U_α for all roots $\alpha \in \Phi_\epsilon$ and $U_{[\alpha,\beta]}$ for all prenilpotent pairs $\{\alpha,\beta\} \subset \Phi_\epsilon$. Thus, it suffices to prove that, for each prenilpotent pair of roots $\{\alpha,\beta\} \subset \Phi_\epsilon$, the restriction of $\widehat{\pi}$ to the subgroup $\widehat{U}_{[\alpha,\beta]} := \langle \widehat{U}_\gamma | \ \gamma \in [\alpha,\beta] \rangle \leq \widehat{U}_\epsilon$ is an isomorphism onto $U_{[\alpha,\beta]}$. We do this by induction on the cardinality of the set $[\alpha,\beta]$ (which is always finite), the result being trivial when $s_\alpha s_\beta$ has finite order.

Let us order the elements of $[\alpha,\beta]$ in a nibbling sequence $\gamma_0 = \alpha, \gamma_1, \dots, \gamma_n = \beta$ (see Section 3.1.1). Since $\mathcal{D}$ is of 2-spherical type, there exists a finite rank 2 root subsystem $\Phi_n \subset \Phi$ which contains the roots γ_{n-1} and γ_n. Let $j \in \{0,1,\dots,n-1\}$ be minimal with respect to the property that γ_j belongs to Φ_n. Let also $\gamma \in \Phi_n$ be such that $\{\gamma,\gamma_j\}$ is a basis of Φ_n. Now the pair $\{\alpha,\gamma_j\}$ (resp. $\{\alpha,\gamma\}$) is prenilpotent and the cardinality of $[\alpha,\gamma_j]$ (resp. $[\alpha,\gamma]$) is at most $n-1$. By induction, the restriction of $\widehat{\pi}$ to $\widehat{U}_{[\alpha,\gamma_j]}$ (resp. $\widehat{U}_{[\alpha,\gamma]}$) is an isomorphism. In particular, we have

$$\widehat{U}_{[\alpha,\gamma_j]} = \widehat{U}_\alpha \widehat{U}_{\gamma_1} \dots \widehat{U}_{\gamma_{j-1}} \widehat{U}_{\gamma_j}$$

and $\widehat{U}_{\gamma_j}$ normalizes $\widehat{U}_{[\alpha,\gamma_j[} = \widehat{U}_{[\alpha,\gamma_{j-1}]}$. Similarly, the group $\widehat{U}_\gamma$ normalizes $\widehat{U}_{[\alpha,\gamma[} = \widehat{U}_{[\alpha,\gamma_{j-1}]}$. Therefore, we have

$$\begin{aligned} \langle \widehat{U}_\phi | \ \phi \in \{\gamma_0,\gamma_1,\dots,\gamma_j,\gamma\} \rangle &= \widehat{U}_{[\alpha,\gamma_{j-1}]} \langle \widehat{U}_{\gamma_j} \cup \widehat{U}_\gamma \rangle \\ &\simeq \widehat{U}_{[\alpha,\gamma_{j-1}]} \rtimes \langle \widehat{U}_{\gamma_j} \cup \widehat{U}_\gamma \rangle \end{aligned}$$

because, by Lemma 3.5, we have $\langle \widehat{U}_{\gamma_j} \cup \widehat{U}_\gamma \rangle = U_{[\gamma_j,\gamma]}$ and hence, the restriction of $\widehat{\pi}$ to $\langle \widehat{U}_{\gamma_j} \cup \widehat{U}_\gamma \rangle$ is an isomorphism. It follows that the restriction of $\widehat{\pi}$ to $\langle \widehat{U}_\phi | \ \phi \in \{\gamma_0, \gamma_1, \dots, \gamma_j, \gamma\} \rangle$ is an isomorphism. But Lemma 3.5 also implies that $\widehat{U}_{\gamma_{j+1}}, \dots, \widehat{U}_{\gamma_{n-1}}, \widehat{U}_{\gamma_n}$ are all contained in $\langle \widehat{U}_{\gamma_j} \cup \widehat{U}_\gamma \rangle$, which yields the desired result. □

3.3.2 The 'simply connected' central extension

Theorem A of the introduction is contained in the following.

Theorem 3.7. *Suppose that $\mathcal{D}$ satisfies Condition (Co*) and that $G = \langle U_\alpha | \ \alpha \in \Phi \rangle$. Let $\widetilde{G}$ be the direct limit of the inductive system formed by the X_α and $X_{\alpha,\beta}$ with natural inclusions ($\alpha, \beta \in \Pi$) and let $\widetilde{\pi} : \widetilde{G} \to G$ be the canonical homomorphism. Then $\widetilde{G}$ is naturally endowed with a twin root datum which is isogenic to $\mathcal{D}$. In particular, the kernel of $\widetilde{\pi}$ is central.*

Proof. Let $\widetilde{G}$ denote the direct limit of the inductive system formed by the X_α and $X_{\alpha,\beta}$ with natural inclusions ($\alpha, \beta \in \Pi$) and let $\pi : \widetilde{G} \to G$ be the canonical homomorphism. For each $\alpha, \beta \in \Pi$, let $H_\alpha := N_{X_\alpha}(U_\alpha) \cap N_{X_\alpha}(U_{-\alpha})$ and let $\widetilde{H}_\alpha$ (resp. $\widetilde{U}_\alpha$, $\widetilde{X}_\alpha$, $\widetilde{X}_{\alpha,\beta}$) denote the canonical image of H_α (resp. U_α, X_α, $X_{\alpha,\beta}$) in $\widetilde{G}$. We also denote by $\widetilde{\mu(u)}$ the canonical image of $\mu(u)$ in $\widetilde{X}_\alpha$, for $u \in U_\alpha \backslash \{1\}$. Clearly, the group

$$\widetilde{W} := \langle \widetilde{\mu(u)} | \ u \in U_\alpha \backslash \{1\}, \alpha \in \Pi \rangle / \widetilde{H},$$

where $\widetilde{H} := \langle \widetilde{H}_\alpha | \ \alpha \in \Pi \rangle$, is isomorphic to the Coxeter group W. By definition (see also Lemma 3.1), the group $\widetilde{H}$ normalizes $\widetilde{U}_\alpha$ for each $\alpha \in \Pi$. Therefore, the group $\widetilde{W}$ has a well-defined action on the conjugacy class of $\widetilde{U}_\alpha$ in $\widetilde{G}$, for each $\alpha \in \Pi$. For each $\phi \in \Phi$, we set $\widetilde{U}_\phi := w \widetilde{U}_\alpha w^{-1}$, where $\alpha \in \Pi$ and $w \in \widetilde{W}$ is chosen so that $w(\alpha) = \phi$. Clearly, the group $\widetilde{U}_\phi$ is independent of the choice of w.

Let us consider the system $\widetilde{\mathcal{D}} := (\widetilde{G}, (\widetilde{U}_\phi)_{\phi \in \Phi})$. Clearly, it satisfies Axioms (TRD 0), (TRD 3) and (TRD 4) (because so does $\mathcal{D}$) while Axiom (TRD 2) follows from the way the groups $\widetilde{U}_\phi$ have been defined. We now check that Axiom (TRD 1) also holds.

Let $\widetilde{U}_+ := \langle \widetilde{U}_\gamma | \ \gamma \in \Phi_+ \rangle$. For any prenilpotent pair $\{\phi, \psi\} \subset \Phi_+$ such that $s_\phi s_\psi$ has finite order, there exists a pair $\{\phi_0, \psi_0\} \subset \Pi$ such that some $\widetilde{W}$-conjugate of $\widetilde{X}_{\phi_0,\psi_0}$ contains $\widetilde{U}_{\phi_0}$ and $\widetilde{U}_{\psi_0}$. It follows that the restriction of $\widetilde{\pi}$ to $\langle \widetilde{U}_{\phi_0} \cup \widetilde{U}_{\psi_0} \rangle$ is an isomorphism. In view of Theorem 3.6, this fact implies that the restriction of $\widetilde{\pi}$ to $\widetilde{U}_+$ is an isomorphism. This in turn implies that any prenilpotent pair of positive roots satisfies Axiom (TRD 1). Using the $\widetilde{W}$-action on the $\widetilde{U}_\phi$'s, this implies that (TRD 1) holds for all prenilpotent pairs of positive roots.

Now, the claim is a direct consequence Lemma 3.4. □

3.3.3 The Curtis-Tits theorem for groups of Kac-Moody type

The purpose of this section is the recover the Curtis-Tits theorem for groups of Kac-Moody type as a consequence of Theorem 3.7.

Corollary 3.8. ***([AM97])*** *Suppose that $\mathcal{D}$ is of 2-spherical type and satisfies Condition (Co*). Then G is isomorphic to the direct limit of the inductive system formed by the groups L_α and $L_{\alpha,\beta}$ with natural inclusions ($\alpha, \beta \in \Pi$).*

Proof. Let $\overline{G}$ (resp. $\widetilde{G}$) be the direct limit of the inductive system formed by the groups L_α and $L_{\alpha,\beta}$ (resp. X_α and $X_{\alpha,\beta}$) with natural inclusions ($\alpha, \beta \in \Pi$). Let $\overline{H} < \overline{G}$ be the canonical image of H in $\overline{G}$ and let $\widetilde{H} < \widetilde{G}$ be as in the preceding subsection. Let $\overline{\pi} : \overline{G} \to G$, $\widetilde{\pi} : \widetilde{G} \to G$ and $\varphi : \widetilde{G} \to \overline{G}$ be the canonical homomorphisms.

By the universal property of $\widetilde{G}$ one has $\widetilde{\pi} = \overline{\pi} \circ \varphi$. It follows that $\operatorname{Ker} \overline{\pi} \leq \varphi(\operatorname{Ker} \widetilde{\pi})$. Since, by Lemma 3.4, one has $\operatorname{Ker} \widetilde{\pi} \leq \widetilde{H}$, we deduce $\operatorname{Ker} \overline{\pi} \leq \varphi(\widetilde{H}) = \overline{H}$. But by definition, the restriction of $\overline{\pi}$ to $\overline{H}$ is injective, whence $\operatorname{Ker} \overline{\pi} = \{1\}$. □

3.4 Universal central extensions

Let (W, S) be a Coxeter system and let Φ be the associated root system. Let $\mathcal{D} = (G, (U_\alpha)_{\alpha \in \Phi})$ be a twin root datum of type (W, S).

3.4.1 The Steinberg group

By definition, the **Steinberg group** associated with the twin root datum $\mathcal{D}$ is the group $\widehat{G}$ which is the direct limit of the inductive system formed by the groups U_α and $U_{[\alpha,\beta]} := \langle U_\gamma |\ \gamma \in [\alpha, \beta] \rangle$ for all prenilpotent pairs $\{\alpha, \beta\} \subset \Phi$. For each $\alpha \in \Phi$, let $\widehat{U}_\alpha$ denote the canonical image of U_α in $\widehat{G}$.

In the sequel, we will need a refined presentation of the Steinberg group, which we now deduce from the presentations we have obtained for the groups U_+ and U_- in Section 3.3.1. We also note that this result is trivial when $\mathcal{D}$ is of spherical type.

Proposition 3.9. *Suppose that $\mathcal{D}$ is of 2-spherical type and satisfies Condition (Co*). Then the Steinberg group $\widehat{G}$ is isomorphic to the direct limit of the inductive system formed by the groups U_α for all $\alpha \in \Phi$ and $U_{[\alpha,\beta]} := \langle U_\gamma |\ \gamma \in [\alpha, \beta] \rangle$ for all prenilpotent pairs $\{\alpha, \beta\} \subset \Phi$ such that $s_\alpha s_\beta$ has finite order.*

Proof. This is a straightforward consequence of Theorem 3.6. □

We now prove that the Steinberg group is a central extension of the group G, when G is of 2-spherical type. This is a first step towards the proof of Theorem B of the introduction. The arguments we use here are already contained in Remark 3.7(a) on p.530 in [Ti87].

Theorem 3.10. *Assume that the Coxeter system (W, S) is 2-spherical with no direct factor of type A_1. Then the system $\widehat{\mathcal{D}} = (\widehat{G}, (\widehat{U}_\alpha)_{\alpha \in \Phi})$ is a twin root datum of type $\mathcal{D}$. Moreover, the kernel of the canonical homomorphism $\widehat{G} \to G$ is central.*

Proof. Since $\mathcal{D}$ satisfies (TRD 0) and (TRD 3), so does $\widehat{D}$ in view of the existence of a canonical homomorphism $\pi : \widehat{G} \to G$. Moreover, it is obvious by the definition of $\widehat{G}$ that $\widehat{D}$ satisfies (TRD 1) and (TRD 4). Let us check that (TRD 2) also holds; the argument is as in Remark 3.7(a) on p. 530 in [Ti87] and we give it now.

For each $\phi \in \Phi$, let π_ϕ be the canonical monomorphism $U_\phi \to \widehat{G}$ (thus $\widehat{U}_\phi = \pi_\phi(U_\phi)$). Let $\alpha \in \Pi$, let $u \in U_\alpha \backslash \{1\}$ and let $u', u'' \in U_{-\alpha}$ be as in Axiom (TRD 2). We set $\widehat{u} := \pi_\alpha(u)$, $\widehat{u}' := \pi_{-\alpha}(u')$, $\widehat{u}'' := \pi_{-\alpha}(u'')$ and $\mu(\widehat{u}) := \widehat{u}'\widehat{u}\widehat{u}''$. Let Φ_α be the set of all $\phi \in \Phi$ such that $\mu(\widehat{u})$ conjugates U_ϕ onto $U_{s_\alpha(\phi)}$. We need to check that $\Phi_\alpha = \Phi$ for all $\alpha \in \Pi$.

Claim 1. *If $\phi \neq \pm\alpha$ and the order of $s_\alpha s_\phi$ is finite, then $\phi \in \Phi_\alpha$.*

Proof of Claim 1. Let $\beta \in \Phi$ be a root such that $\{\alpha, \beta\}$ is a basis of a root subsystem $\Phi_0 \subset \Phi$ which contains ϕ. By Axiom (TRD 1), the group U_α normalizes $U_{]\alpha,\beta]}$ and the group $U_{-\alpha}$ normalizes $U_{]-\alpha,s_\alpha(\beta)]}$. Since $]-\alpha, s_\alpha(\beta)] =]\alpha, \beta]$, it follows from the definitions that $\mu(\widehat{u})$ normalizes $\widehat{U}_{]\alpha,\beta]} := \langle \widehat{U}_\gamma | \ \gamma \in]\alpha, \beta] \rangle$ and that the diagram

$$\begin{array}{ccc} \widehat{U}_{]\alpha,\beta]} & \xrightarrow{\text{Ad } \mu(\widehat{u})} & \widehat{U}_{]\alpha,\beta]} \\ \pi \downarrow & & \downarrow \pi \\ U_{]\alpha,\beta]} & \xrightarrow{\text{Ad } \mu(u)} & U_{]\alpha,\beta]} \end{array}$$

is commutative, where Ad x denotes the conjugation by x. Since $\phi \in]\alpha, \beta]$, it follows from Axiom (TRD 2) for $\mathcal{D}$ that $\mu(\widehat{u})\widehat{U}_\phi\mu(\widehat{u})^{-1} = \widehat{U}_{s_\alpha(\phi)}$. □

Claim 2. *If $\pm\phi, \psi \in \Phi_\alpha$, $\psi \in \Phi_\phi$ and $s_\alpha(\psi) \in \Phi_{s_\alpha(\phi)}$ then $s_\phi(\psi) \in \Phi_\alpha$.*

Proof of Claim 2. Let $\mu_\phi \in \widehat{U}_{-\phi}\widehat{U}_\phi\widehat{U}_{-\phi}$ be such that $\mu_\phi\widehat{U}_\psi\mu_\phi^{-1} = \widehat{U}_{s_\phi(\psi)}$ and let $\mu_{s_\alpha(\phi)} := \mu(\widehat{u})\mu_\phi\mu(\widehat{u})^{-1}$. Since $\pm\phi \in \Phi_\alpha$, it follows that $\mu_{s_\alpha(\phi)}$ conjugates $\widehat{U}_\gamma$ onto $\widehat{U}_{s_{s_\alpha(\phi)}} = \widehat{U}_{s_\alpha s_\phi s_\alpha}$ for all $\gamma \in \Phi_{s_\alpha(\phi)}$. Therefore, we have

$$\begin{aligned} \mu(\widehat{u})\widehat{U}_{s_\phi(\psi)}\mu(\widehat{u})^{-1} &= \mu(\widehat{u})\mu_\phi\widehat{U}_\psi\mu_\phi^{-1}\mu(\widehat{u})^{-1} \\ &= (\mu(\widehat{u})\mu_\phi\mu(\widehat{u})^{-1})\widehat{U}_{s_\alpha(\psi)}(\mu(\widehat{u})\mu_\phi^{-1}\mu(\widehat{u})^{-1}) \\ &= \mu_{s_\alpha(\phi)}\widehat{U}_{s_\alpha(\psi)}\mu_{s_\alpha}(\phi)^{-1} \\ &= \widehat{U}_{s_\alpha s_\phi s_\alpha s_\alpha(\psi)} \\ &= \widehat{U}_{s_\alpha(s_\phi(\psi))}. \end{aligned}$$

□

Claim 3. *If $\{\phi, \psi\}$ is a basis of a root subsystem $\Phi_0 \subset \Phi$ and if $\{\pm\phi, \pm\psi\} \subset \Phi_\alpha$, then $\Phi_0 \subset \Phi_\alpha$.*

Proof of Claim 3. By Claim 1, we have $\Phi_0 \backslash \{\pm\gamma\} \subset \Phi_\gamma$ (resp. $s_\alpha(\Phi_0) \backslash \{\pm\gamma\} \subset \Phi_\gamma$) for each $\gamma \in \Phi_0$ (resp. $\gamma \in s_\alpha(\Phi_0)$). Since any element $\gamma \in \Phi_0$ can be written as $\gamma = \ldots s_\phi s_\psi(\pm\phi)$ or $\gamma = \ldots s_\psi s_\phi(\pm\psi)$, a repeated application of Claim 2 yields the claim. □

Claim 4. *For all $\alpha, \beta \in \Pi$, we have $\pm\beta \in \Phi_\alpha$.*

Proof of Claim 4. If $\beta \neq \pm\alpha$, this is a direct consequence of Claim 1 since (W, S) is 2-spherical. If $\beta = \pm\alpha$, one chooses $\gamma \in \Pi$ such that s_γ does not commute with s_α; this is possible because (W, S) has no direct factor of type A_1. Let Φ_0 be the root subsystem generated by α and γ. By Claim 1, we have $\Phi_0 \backslash \{\pm\alpha\} \subset \Phi_\alpha$. By Claim 3, this implies $\Phi_0 \subset \Phi_\alpha$. □

Claim 5. *For all $\alpha \in \Pi$, we have $\Phi_\alpha = \Phi$.*

Proof of Claim 5. We prove by induction on the length $\ell(w)$ of w that $w(\beta) \in \Phi(\alpha)$ for all $\alpha, \beta \in \Pi$ and all $w \in W$.

For $\ell(w) = 0$, this follows from Claim 4.

Let now $\gamma \in \Pi$ and $s_1, \dots, s_{n-1} \in S$ be such that $w = s_\gamma s_{n-1} \dots s_1$ and that $\ell(w) = n > 0$. By Claim 4, we have $\pm\gamma \in \Phi_\alpha$. By induction, we have $s_{n-1} \dots s_1(\beta) \in \Phi_\alpha \cap \Phi_\gamma$. Moreover, applying the induction hypothesis to the basis $s_\alpha(\Pi)$ of Φ, we also get $s_\alpha s_{n-1} \dots s_1(\beta) \in \Phi_{s_\alpha(\gamma)}$. By Claim 2, this implies $s_\gamma s_{n-1} \dots s_1(\beta) \in \Phi_\alpha$. □

It follows from Claim 5 that $\widehat{D}$ satisfies (TRD 2). Thus $\widehat{D}$ is a twin root datum. Moreover, it follows from the definition that the action of $\widehat{G}$ on the twin building associated with $\widehat{D}$ factors through $\widehat{G}$ via π. Thus the kernel of π acts trivially on the building associated with $\widehat{D}$. Therefore, this kernel is central in $\widehat{G}$ by Lemma 3.4. □

3.4.2 Universality: reduction to the rank 2 case

In this section, we prove a reduction theorem asserting that (under mild restrictions on the rank 1 tori) if the irreducible rank 2 Steinberg groups provide universal central extension of the rank 2 subgroups of G, then the global Steinberg group is a universal central extension of G. We postpone to the next section a discussion of the result in rank 2.

As before, for each $\alpha \in \Pi$, let $X_\alpha := \langle U_\alpha \cup U_{-\alpha} \rangle$ and $H_\alpha := N_{X_\alpha}(U_\alpha) \cap N_{X_\alpha}(U_{-\alpha})$, and for each pair $\{\alpha, \beta\} \subset \Pi$, let $X_{\alpha,\beta} := \langle X_\alpha \cup X_\beta \rangle$ and let $\Phi_{\alpha,\beta} \subset \Phi$ be the rank 2 root subsystem generated by α and β. We recall that $(X_{\alpha,\beta}, (U_\gamma)_{\gamma \in \Phi_{\alpha,\beta}})$ is a twin root datum of rank 2. The corresponding Steinberg group is denoted by $\widehat{X}_{\alpha,\beta}$.

Theorem 3.11. *Assume that (W, S) is 2-spherical and has no direct factor of type A_1. Suppose also that:*

(1) For each $\alpha \in \Pi$ we have $[H_\alpha, U_\alpha] = U_\alpha$;

(2) For each pair $\{\alpha, \beta\} \subset \Pi$ such that s_α and s_β do not commute, the Steinberg group $\widehat{X}_{\alpha,\beta}$ is a universal central extension of $X_{\alpha,\beta}$.

Then $\pi : \widehat{G} \to G_0$ is a universal central extension of the group $G_0 = \langle U_\alpha | \ \alpha \in \Phi \rangle$, where $\widehat{G}$ is the Steinberg group and π is the canonical homomorphism.

Proof. Note that (1) implies that X_α is perfect for each $\alpha \in \Pi$ (and hence for each $\alpha \in \Phi$), and furthermore that $\mathcal{D}$ satisfies Condition (Co*). In particular, the group $X_{\alpha,\beta}$ is perfect for each pair $\{\alpha, \beta\} \subset \Pi$ and so is the group G_0 (as a group generated by perfect subgroups).

Let $\{\alpha, \beta\} \subset \Pi$ be such that s_α and s_β do not commute. By Lemma 3.5, the fact that $\mathcal{D}$ satisfies Condition (Co*) implies that $[U_\phi, U_\psi] = U_{]\phi,\psi[}$ for each basis $\{\phi, \psi\}$ of the root subsystem $\Phi_{\alpha,\beta}$. By the definition of the Steinberg group $\widehat{G}$, this implies that $\widehat{G}$ is perfect.

Now, in order to prove that $\pi : \widehat{G} \to G_0$ is a universal central extension, it suffices to prove that $\widehat{G}$ covers every central extension of G_0 (see Assertion (iv′) on Page 76 in [St68]).

Let $\varphi : E \to G_0$ be a central extension. We need to construct a homomorphism $\pi_E : \widehat{G} \to E$ such that $\varphi \circ \pi_E = \pi$.

Let $\Psi \subset \Phi$ be a finite irreducible root subsystem of rank 2. Let $X_\Psi := \langle U_\alpha | \ \alpha \in \Psi \rangle$. Assumption (2) implies that $\pi_\Psi : \widehat{X}_\Psi \to X_\Psi$ is a universal central extension of $\widehat{X}_\Psi$, where $\widehat{X}_\Psi$ is the Steinberg group associated with the twin root datum $(X_\Psi, (U_\alpha)_{\alpha \in \Psi})$ and π_Ψ is the canonical homomorphism. Therefore, since $\varphi : \varphi^{-1}(X_\Psi) \to X_\Psi$ and $\pi : \pi^{-1}(X_\Psi) \to X_\Psi$ are central extensions of X_Ψ (see Theorem 3.10), there exist unique homomorphisms $\widehat{\varphi}_\Psi : \widehat{X}_\Psi \to E$ and $\widehat{\pi}_\Psi : \widehat{X}_\Psi \to \widehat{G}$ such that

$$\varphi \circ \widehat{\varphi}_\Psi = \pi_\Psi = \pi \circ \widehat{\pi}_\Psi. \tag{$*$}$$

For each $\alpha \in \Psi$, let us denote by $\widehat{U}_\alpha$ the canonical image of U_α in $\widehat{X}_\Psi$. Let also $\widehat{X}_\alpha := \langle \widehat{U}_\alpha \cup \widehat{U}_{-\alpha} \rangle$ and $\widehat{H}_\alpha := N_{\widehat{X}_\alpha}(\widehat{U}_\alpha) \cap N_{\widehat{X}_\alpha}(\widehat{U}_{-\alpha})$. We need the following auxiliary results.

Claim 1. *For each $\alpha \in \Psi$, we have $[\widehat{H}_\alpha, \widehat{U}_\alpha] = \widehat{U}_\alpha$. In particular, the group $\widehat{X}_\alpha$ is perfect.*

Proof of Claim 1. By Theorem 3.10, we know that $(\widehat{X}_\Psi, (\widehat{U}_\alpha)_{\alpha \in \Psi})$ is a twin root datum, which implies that $[\widehat{H}_\alpha, \widehat{U}_\alpha] \leq \widehat{U}_\alpha$ by (TRD 2) and Lemma 3.1. Since $\pi_\Psi(\widehat{H}_\alpha) = H_\alpha$ (resp. $\pi_\Psi(\widehat{U}_\alpha) = U_\alpha$), we deduce from Assumption (1) that $[\widehat{H}_\alpha, \widehat{U}_\alpha] = \widehat{U}_\alpha$. Similarly, we have $[\widehat{H}_\alpha, \widehat{U}_{-\alpha}] = \widehat{U}_{-\alpha}$ and, hence, $\widehat{X}_\alpha$ is perfect. □

Claim 2. *For each $\alpha \in \Psi$, we have $[\varphi^{-1}(H_\alpha), \varphi^{-1}(U_\alpha)] = \widehat{\varphi}_\Psi(\widehat{U}_\alpha)$.*

Proof of Claim 2. Transforming the equality of Claim 1 by $\widehat{\varphi}_\Psi$ we obtain:

$$\begin{aligned} \widehat{\varphi}_\Psi(\widehat{U}_\alpha) &= [\widehat{\varphi}_\Psi(\widehat{H}_\alpha), \widehat{\varphi}_\Psi(\widehat{U}_\alpha)] \\ &= [\widehat{\varphi}_\Psi(\widehat{H}_\alpha). \operatorname{Ker}\varphi, \ \widehat{\varphi}_\Psi(\widehat{U}_\alpha). \operatorname{Ker}\varphi] \\ &= [\varphi^{-1}(H_\alpha), \varphi^{-1}(U_\alpha)], \end{aligned}$$

where the second equality follows from the fact that $\operatorname{Ker}\varphi$ is central and the last equality from $(*)$. □

For each $\alpha \in \Phi$, let us set $U_\alpha^E := [\varphi^{-1}(H_\alpha), \varphi^{-1}(U_\alpha)] \leq E$.

For every basis $\{\alpha, \beta\}$ of Ψ, the restriction of φ to $U_{[\alpha,\beta]}^E := \langle U_\gamma^E | \ \gamma \in [\alpha, \beta] \rangle$ induces an isomorphism onto $U_{[\alpha,\beta]}$. This is a consequence of the claim, together with Equation $(*)$.

On the other hand, given a finite reducible root subsystem $\Psi \subset \Phi$ of rank 2 and a basis $\{\alpha, \beta\}$ of Ψ, we have $[U_{\pm\alpha}^E, y] \leq \operatorname{Ker}\varphi \leq Z(E)$ for each $y \in U_\beta^E$ because $[U_{\pm\alpha}, U_\beta] = \{1\}$. Therefore, the map $\tau_y : x \mapsto [x, y]$ is a homomorphism of $\langle U_\alpha^E \cup U_{-\alpha}^E \rangle$ to an abelian group. Now, since (W, S) has no direct factor of type A_1, there exists a finite irreducible root subsystem $\Psi' \subset \Phi$ of rank 2 such that $\pm\alpha \in \Psi'$. As a consequence of Claim 2, we have $\langle U_\alpha^E \cup U_{-\alpha}^E \rangle = \varphi_{\Psi'}(\widehat{X}_\alpha)$. Furthermore $\widehat{X}_\alpha \leq \widehat{X}_{\Psi'}$ is perfect by Claim 1. Therefore $\langle U_\alpha^E \cup U_{-\alpha}^E \rangle$ is perfect and the homomorphism τ_y is trivial for all $y \in U_\beta^E$. We finally deduce that $[U_\alpha^E, U_\beta^E] = \{1\}$ and, hence, the restriction of φ to $U_{[\alpha,\beta]}^E := \langle U_\gamma^E | \ \gamma \in [\alpha, \beta] \rangle$ induces an isomorphism onto $U_{[\alpha,\beta]}$.

Now, we have proved that for each prenilpotent pair $\{\alpha, \beta\} \subset \Phi$ such that $s_\alpha s_\beta$ is of finite order, the restriction of φ to $U_{[\alpha,\beta]}^E := \langle U_\gamma^E | \ \gamma \in [\alpha, \beta] \rangle$ induces an isomorphism onto $U_{[\alpha,\beta]}$. From this and from Proposition 3.9, we deduce the existence of a homomorphism $\pi_E : \widehat{G} \to E$ which is such that

$$\pi_E \circ \widehat{\pi}_\Psi = \widehat{\varphi}_\Psi \tag{$**$}$$

for all finite irreducible root subsystems $\Psi \subset \Phi$ of rank 2.

Combining $(*)$ and $(**)$, we obtain

$$\varphi \circ \pi_E \circ \widehat{\pi}_\Psi = \pi \circ \widehat{\pi}_\Psi.$$

Since the latter equation holds for all finite irreducible root subsystems $\Psi \subset \Phi$ of rank 2, and since $\widehat{G}$ is generated by its subgroups of the form $\widehat{\pi}_\Psi(\widehat{X}_\Psi)$ when Ψ varies, we finally obtain $\varphi \circ \pi_E = \pi$, as desired. □

3.4.3 Universality: some remarks on the rank 2 case

The reduced twin root data $(G, (U_\alpha)_{\alpha\in\Phi})$ whose type is an irreducible spherical Coxeter system of rank 2 have been classified by Tits and Weiss [TW02]. Based on the results of loc.cit., we believe that it is possible to establish a list of all those twin root data of the type above for which the Steinberg group $\widehat{G}$ does not provide a universal central extension. This would of course make Assumption (2) of Theorem 3.11 more explicit. However, we do not want to go into details here. We simply note that for the case of twin root data arising from Chevalley groups, the work has been done by Steinberg in [St68]. The result is that the only exceptions are over ground fields of cardinality ≤ 4. Actually, over ground fields of cardinality ≤ 3, even Assumption (1) of Theorem 3.11 fails.

3.5 Split Kac-Moody groups

In this section we specialize the results of this paper to the case of Kac-Moody groups in the sense of [Ti87]. We first recall some definitions from [Ré02] and [Ti87].

Let I be a finite set. A **generalized Cartan matrix** is a matrix $A = (A_{ij})_{i,j\in I}$ with integral coefficients such that $A_{ii} = 2$, $A_{ij} \leq 0$ if $i \neq j$ and $A_{ij} = 0 \Leftrightarrow A_{ji} = 0$. Note that for every subset $J \subset I$, the matrix $A_J := (A_{ij})_{i,j\in J}$ is generalized Cartan. A generalized Cartan matrix is said to be of **finite type** if it is the product of a diagonal and a positive definite matrix. It is called 2-**spherical** if A_J is of finite type for every 2-subset $J \subset I$.

A **Kac-Moody root datum** is a system $\mathcal{K} = (I, A, \Lambda, (c_i)_{i\in I}, (h_i)_{i\in I})$ where I is a finite set, A is a generalized Cartan matrix indexed by I, Λ is a free $\mathbb{Z}$-module whose $\mathbb{Z}$-dual is denoted by $\Lambda^\vee$ and where the elements c_i of Λ and h_i of $\Lambda^\vee$ satisfy the relation $\langle c_i | h_j \rangle = A_{ji}$ for all $i, j \in I$. The Kac-Moody root datum $\mathcal{K}_A^{\mathrm{sc}} := (I, A, \bigoplus_{i\in I} \mathbb{Z}e_i, (c_i)_{i\in I}, (h_i)_{i\in I})$, where $c_i := \sum_{j\in I} A_{ji} e_j$ and $\langle e_i | h_j \rangle = \delta_{ij}$ for all $i, j \in I$, is called **simply connected**. Let $M := \bigoplus \mathbb{Z}e_i$, $c_i := \sum_{j\in I} A_{ji} e_j$ and $h_i := e_i^\vee|_\Lambda$, where $\langle e_i | e_j^\vee \rangle = \delta_{ij}$. The Kac-Moody root datum $\mathcal{K}_A^{\mathrm{adj}} := (I, A, \Lambda, (c_i)_{i\in I}, (h_i)_{i\in I})$ is called **minimal adjoint**.

Given a Kac-Moody root datum, there is a group functor $\mathcal{G}_\mathcal{K}$ on the category of $\mathbb{Z}$-modules whose value on fields is uniquely determined by a series of five axioms (noted (KMG 1)–(KMG 5) in [Ti87]). This functor is called the **Tits functor**; its value on a field $\mathbb{K}$ is called a **split Kac-Moody group** of type A over $\mathbb{K}$. Instead of recalling the (KMG) axioms here, we just mention the properties of Kac-Moody groups which are relevant to our purposes.

Let $\mathcal{K} = (I, A, \Lambda, (c_i)_{i\in I}, (h_i)_{i\in I})$ be a Kac-Moody root datum, let $\mathcal{G}_\mathcal{K}$ be the corresponding Tits functor and let $\mathbb{K}$ be a field. Let M be the Coxeter matrix over I defined by $m_{ij} = 2, 3, 4, 6$ or ∞ according as $A_{ij}A_{ji} = 0, 1, 2, 3$ or ≥ 4. Let (W, S) be the Coxeter system of type M.

(1) The group W has a canonical action on Λ defined by $s_i(\lambda) = \lambda - \langle\lambda|h_i\rangle c_i$ for all $i, j \in I$. Moreover, the root system Φ of (W, S) can be identified W-equivariantly to $\{w(c_i)|\ w \in W, i \in I\}$ so that $\Pi := \{c_i|\ i \in I\}$ corresponds to a basis of Φ.

(2) The split Kac-Moody group possesses a family of subgroups U_α indexed by $\Phi \subset \Lambda$ such that $\mathcal{D}_\mathcal{K} = (G := \mathcal{G}_\mathcal{K}(\mathbb{K}), (U_\alpha)_{\alpha\in\Phi})$ is a twin root datum of type (W, S). Its reduction is the twin root datum $\mathcal{D}_{\mathcal{K}_A^{\mathrm{adj}}}$ associated with the minimal adjoint Kac-Moody group $\mathcal{G}_{\mathcal{K}_A^{\mathrm{adj}}}(\mathbb{K})$. In particular, any two split Kac-Moody groups of type A over the same field $\mathbb{K}$ are isogenic.

(3) There exists an isomorphism $\tau : H := \bigcap_{\alpha\in\Phi} N_G(U_\alpha) \to \mathrm{Hom}(\Lambda, \mathbb{K}^\times)$ such that, for each $c_i \in \Pi$, we have $\tau(H_{c_i}) = \mathrm{Hom}(c_i\mathbb{Z}, \mathbb{K}^\times)$, where H_{c_i} is as in Lemma 3.3.

(4) Given $\alpha \in \Phi \subset \Lambda$, there is an isomorphism $\tau_\alpha : (\mathbb{K}, +) \to U_\alpha$ such that for all $h \in H$ and $x \in \mathbb{K}$, one has $h\tau_\alpha(x)h^{-1} = \tau_\alpha(\tau(h)(\alpha).x)$.

(5) For each subset $J \subset I$, the subgroup $G_J := \langle U_{\pm c_i}|\ i \in J\rangle \leq G$ is a Kac-Moody group of type A_J over $\mathbb{K}$.

In particular, the twin root datum $\mathcal{D}_\mathcal{K}$ satisfies the hypotheses of Theorem A whenever A is 2-spherical and $\mathbb{K}$ has cardinality at least 4. The following proposition makes this result more precise in that case.

Proposition 3.12. *Let A be a generalized Cartan matrix of 2-spherical type. Let $\mathcal{K} = (I, A, \Lambda, (c_i)_{i\in I}, (h_i)_{i\in I})$ be a Kac-Moody root datum, let $\mathbb{K}$ be a field of cardinality at least 4 and let $G := \mathcal{G}_\mathcal{K}(\mathbb{K})$ be the corresponding Kac-Moody group over $\mathbb{K}$. Let $\widetilde{G}$ be the direct limit of the inductive system formed by the subgroups $G_J \leq G$, where J runs over the subsets of I of cardinality 1 or 2 and G_J is defined as in (5). Then there is a homomorphism $p : \mathcal{G}_{\mathcal{K}_A^{\mathrm{sc}}}(\mathbb{K}) \to \widetilde{G}$ such that $\mathrm{Ker}\, p$ is finite and central.*

Proof. Let $\mathcal{K}_A^{\mathrm{sc}} := (I, A, \Lambda^{\mathrm{sc}}, (c_i^{\mathrm{sc}})_{i\in I}, (h_i^{\mathrm{sc}})_{i\in I})$ and let $G^{\mathrm{sc}} := \mathcal{G}_{\mathcal{K}_A^{\mathrm{sc}}}(\mathbb{K})$. There is a unique morphism $(\Lambda^\vee)^{\mathrm{sc}} \to \Lambda^\vee : h_i^{\mathrm{sc}} \mapsto h_i$ which induces a morphism $\varphi : \mathrm{Hom}(\Lambda^{\mathrm{sc}}, \mathbb{K}^\times) \to \mathrm{Hom}(\Lambda, \mathbb{K}^\times)$ since $\mathrm{Hom}(\Lambda^{\mathrm{sc}}, \mathbb{K}^\times) = \langle h_i^{\mathrm{sc}}|\ i \in I\rangle$. This homomorphism extends to a homomorphism $G^{\mathrm{sc}} \to G$, again denoted by φ, such that $\mathrm{Ker}\,\varphi$ is central in G^{sc} (this is easy to deduce from the definition of Kac-Moody groups; see the last lemma of Section 8.4.1 in [Ré02]). Similarly, it follows from Lemma 3.3 and Theorem 3.7 that there exists a unique central homomorphism $\widetilde{\varphi} : G^{\mathrm{sc}} \to \widetilde{G}$ such that $\widetilde{\pi} \circ \widetilde{\varphi} = \varphi$, where $\widetilde{\pi}$ is the canonical homomorphism. We have to show that $\mathrm{Ker}\,\widetilde{\varphi}$ is finite.

For every $J \subset I$ such that A_J is of finite type, the group G_J (resp. G_J^{sc}) is a Chevalley group (resp. simply connected Chevalley group) of type A_J over $\mathbb{K}$. The morphism φ induces by restriction a surjective homomorphism $G_J^{\mathrm{sc}} \to G_J$. Therefore $\mathrm{Ker}\,\varphi \cap G_J^{\mathrm{sc}}$ is finite and central in G_J^{sc} (see §4 in [St68]). Let now

$$K := \langle \mathrm{Ker}\,\varphi \cap G_J^{\mathrm{sc}}|\ J \subset I, |J| = 1 \text{ or } 2\rangle.$$

By definition (since I is finite), the group K is a finite central subgroup of G^{sc}. Furthermore, we have clearly

$$K \cap G_J^{\mathrm{sc}} = \mathrm{Ker}\,\varphi \cap G_J^{\mathrm{sc}} = \mathrm{Ker}\,\widetilde{\varphi} \cap G_J^{\mathrm{sc}}$$

for every $J \subset I$ such that $|J| = 1$ or 2. Therefore, we have $K = \mathrm{Ker}\,\widetilde{\varphi}$. □

In view of Proposition 3.12, the group $\widetilde{G}$ is in general not far from being the simply connected Kac-Moody group of type A over $\mathbb{K}$, but in general the cardinality of $\operatorname{Ker} p$ depends on the Kac-Moody root datum $\mathcal{K}$. However, if A is 3-spherical and simply laced, and if each component of A is of rank at least 3, then the kernel $\operatorname{Ker} p$ is actually trivial for all Kac-Moody root data $\mathcal{K}$ and, hence, the group $\widetilde{G}$ is then isomorphic to $\mathcal{G}_{\mathcal{K}_A^{\mathrm{sc}}}(\mathbb{K})$.

Another immediate consequence of Proposition 3.12 is the following.

Corollary 3.13. *If $\mathcal{K} = \mathcal{K}_A^{\mathrm{sc}}$ is simply connected, then $G = \mathcal{G}_{\mathcal{K}}(\mathbb{K})$ is isomorphic to $\widetilde{G}$ (with the notation of Proposition 3.12).*

As it is the case for any twin root datum, the Steinberg group $\mathfrak{St}_A(\mathbb{K})$ associated with the twin root datum $\mathcal{D}_{\mathcal{K}}$ of a split Kac-Moody group over $\mathbb{K}$ depends only on the isogeny class of $\mathcal{D}_{\mathcal{K}}$, or equivalently, on the generalized Cartan matrix A. As in the case of Chevalley groups, this Steinberg group is in general bigger than the simply connected Kac-Moody group, but both coincide over small fields. The precise statement is the following.

Theorem 3.14. *Let A be a generalized Cartan matrix of 2-spherical type without direct factor of type A_1. Let $\mathcal{K}_A^{\mathrm{sc}} = (I, A, \Lambda, (c_i)_{i\in I}, (h_i)_{i\in I})$ be the simply connected Kac-Moody root datum and let $\mathbb{K} \neq \mathbb{F}_9$ be an algebraic extension of a finite field of cardinality at least 5. The Kac-Moody group $G := \mathcal{G}_{\mathcal{K}_A^{\mathrm{sc}}}(\mathbb{K})$ over $\mathbb{K}$ is centrally closed.*

Proof. Since $|\mathbb{K}| \geq 5$, the group G is perfect. Let $\varphi : \widehat{G} \to G$ be a central extension of G.

For each $J \subset I$ with $|J| = 1$ or 2, the group G_J is a simply connected Kac-Moody group of type J over $\mathbb{K}$. By Corollary 2(b) on page 82 in [St68], it is centrally closed. Thus there is a unique homomorphism $\psi_J : G_J \to \widehat{G}$ such that $\varphi \circ \psi_J = \mathrm{id}_{G_J}$. Hence ψ_J is injective. Moreover, the uniqueness of ψ_J implies that $\psi_J|_{G_{J\cap J'}} = \psi_{J'}|_{G_{J\cap J'}}$ for all subsets $J, J' \subset I$ such that $|J|, |J'| \in \{1, 2\}$ and $J \cap J' \neq \emptyset$. On the other hand, Corollary 3.13 implies that G is isomorphic to the direct limit of the inductive system formed by all G_J's. Thus there is a unique homomorphism $\psi : G \to \widehat{G}$ such that $\psi|_{G_J} = \psi_J$ for all $J \subset I$ with $|J| \in \{1, 2\}$. Thus we have $\varphi \circ \psi = \mathrm{id}_G$ since G is generated by the G_J's.

We have shown that G is perfect and that every central extension of G splits. Thus G is centrally closed. □

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