Acta Math. Univ. Comenianae
Vol. LXXXVI, 2 (2017), pp. 215–226

ON 3-DIMENSIONAL ALMOST KENMOTSU MANIFOLDS ADMITTING CERTAIN NULLITY DISTRIBUTION

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Abstract. The aim of this paper is to characterize 3-dimensional almost Kenmotsu manifolds with ξ belonging to the $(k,\mu)'$-nullity distribution and $h' \neq 0$ satisfying certain geometric conditions. Finally, we give an example to verify some results.

1. Introduction

The conformal curvature tensor C is invariant under conformal transformations and vanishes identically for 3-dimensional manifolds. Using this fact, many authors [**4, 6, 7, 14**] studied several types of 3-dimensional manifolds.

A Riemannian manifold is called semisymmetric (resp., Ricci semisymmetric) if $R(X,Y)\cdot R = 0$ (resp. $R(X,Y)\cdot S = 0$) [**19**], where $R(X,Y)$ is considered as a field of linear operators acting on R (resp., S).

The notion of k-nullity distribution ($k \in \mathbb{R}$) was introduced by Gray [**11**] and Tanno [**21**] in the study of Riemannian manifolds (M,g), which is defined for any $p \in M$ and $k \in \mathbb{R}$, as follows:

$$N_p(k) = \{Z \in T_pM : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y]\} \tag{1.1}$$

for any $X,Y \in T_pM$, where T_pM denotes the tangent vector space of M at any point $p \in M$ and R denotes the Riemannian curvature tensor of type $(1,3)$.

Recently, Blair, Koufogiorgos and Papantoniou [**3**] introduced the (k,μ)-nullity distribution which is a generalized notion of the k-nullity distribution on a contact metric manifold $(M^{2n+1},\phi,\xi,\eta,g)$ and defined for any $p \in M^{2n+1}$ and $k,\mu \in \mathbb{R}$, as follows:

$$\begin{aligned} N_p(k,\mu) = \{Z \in T_pM^{2n+1} &: R(X,Y)Z \\ &= k[g(Y,Z)X - g(X,Z)Y] + \mu[g(Y,Z)hX - g(X,Z)hY]\} \end{aligned} \tag{1.2}$$

for any $X,Y \in T_pM$ and $h = \frac{1}{2}\pounds_\xi\phi$, where $\pounds$ denotes the Lie differentiation.

Next, Dileo and Pastore [**9**] introduced another generalized notion of the k-nullity distribution which is named the $(k,\mu)'$-nullity distribution on an almost Kenmotsu

Received December 15, 2015; revised February 20, 2017.
2010 *Mathematics Subject Classification.* Primary 53C25, 53D15.
Key words and phrases. Almost Kenmotsu manifold; Ricci semisymmetric; Codazzi type Ricci tensor; Cyclic parallel Ricci tensor; η-parallel Ricci tensor; Locally ϕ-Ricci symmety; Einstein manifold.

manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ and is defined for any $p \in M^{2n+1}$ and $k, \mu \in \mathbb{R}$, as follows:

$$N_p(k,\mu)' = \{Z \in T_pM^{2n+1} : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y] + \mu[g(Y,Z)h'X - g(X,Z)h'Y]\}, \tag{1.3}$$

for any $X, Y \in T_pM$ and $h' = h \circ \phi$.

On the other hand, in 1972, Kenmotsu [**15**] introduced a special class of almost contact metric manifolds known as Kenmotsu manifolds nowadays. Recently, Dileo and Pastore ([**8**], [**9**], [**10**]) and Wang et al. ([**22**], [**23**], [**24**], [**25**], [**26**]) studied almost Kenmotsu manifolds with some nullity distributions and obtained some classification theorems. In [**9**], Dileo and Pastore gave some classifications on 3-dimensional almost Kenmotsu manifolds assuming ξ belongs to the $(k,\mu)'$-nullity distribution. Later, Wang and Liu [**26**] obtained some theorems on 3-dimensional almost Kenmotsu manifolds.

Motivated by these circumstances, in this paper, we study some meaningful geometric conditions in 3-dimensional almost Kenmotsu manifolds such that ξ belongs to the $(k,\mu)'$-nullity distribution and $h' \neq 0$.

The present paper is organized as follows: In Section 2, we give some basic results on almost Kenmotsu manifolds with ξ belonging to the $(k,\mu)'$-nullity distribution. Section 3 is devoted to study 3-dimensional Ricci semisymmetric almost Kenmotsu manifolds with ξ belonging to the $(k,\mu)'$-nullity distribution. Section 4 deals with Codazzi type Ricci tensor with ξ beloning to the $(k,\mu)'$-nullity distribution. Cyclic parallel Ricci tensor with ξ beloning to the $(k,\mu)'$-nullity distribution is studied in Section 5. In the next two sections, we consider η-parallel Ricci tensor and locally ϕ-Ricci symmetric almost Kenmotsu manifolds of dimension 3 assuming ξ belongs to the $(k,\mu)'$-nullity distribution. Finally, we give an example to verify some results.

2. Almost Kenmotsu manifolds

Let M be a $(2n+1)$-dimensional differentiable manifold endowed with an almost contact metric structure (ϕ, ξ, η, g), where ϕ, ξ, η are tensor fields on M of types $(1,1), (1,0), (0,1)$, respectively, and a Riemannian metric g such that

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \qquad g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.1}$$

where I denotes the identity endomorphism ([**1**], [**2**]). Then also $\phi\xi = 0$ and $\eta \circ \phi = 0$; both can be derived from (2.1).

The fundamental 2-form Φ on an almost contact metric manifold is defined by $\Phi(X,Y) = g(X, \phi Y)$ for any vector fields X, Y of T_pM^{2n+1}. An almost Kenmotsu manifold is defined as an almost contact metric manifold such that $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$. An almost contact metric manifold is said to be normal if $(1,2)$-type torsion tensor N_ϕ vanishes, where $N_\phi = [\phi,\phi] + 2d\eta \otimes \xi$ and $[\phi,\phi]$ is the Nijenhuis torsion of ϕ [**1**]. Obviously, a normal almost Kenmotsu manifold is a Kenmotsu manifold. Also Kenmotsu manifolds can be characterized by $(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$ for any vector fields X, Y. It is well known [**15**] that a Kenmotsu manifold M^{2n+1} is locally a warped product $I \times_f N^{2n}$, where N^{2n} is a

Kähler manifold, I is an open interval with coordinate t and the warping function f, defined by $f = ce^t$ for some positive constant c. Let $\mathcal{D}$ be the distribution orthogonal to ξ and defined by $\mathcal{D} = \mathrm{Ker}(\eta) = \mathrm{Im}(\phi)$. In an almost Kenmotsu manifold $\mathcal{D}$ is an integrable distribution as η is closed. Further, on an almost Kenmotsu manifold M^{2n+1}, we let the two tensor fields $h = \frac{1}{2}\pounds_\xi\phi$ and $l = R(\cdot,\xi)\xi$, which are symmetric and satisfy the following relations [**9, 23**]:

$$h\xi = 0, \qquad l\xi = 0, \qquad \mathrm{tr}(h) = 0, \qquad \mathrm{tr}(h') = 0, \qquad h\phi + \phi h = 0, \tag{2.2}$$

$$\nabla_X\xi = -\phi^2X + h'X \qquad (\Rightarrow \nabla_\xi\xi = 0), \tag{2.3}$$

$$\phi l\phi - l = 2(h^2 - \phi^2), \tag{2.4}$$

$$R(X,Y)\xi = \eta(X)(Y - \phi hY) - \eta(Y)(X - \phi hX) + (\nabla_Y\phi h)X - (\nabla_X\phi h)Y \tag{2.5}$$

for any vector fields X, Y.

Now we provide some basic results on almost Kenmotsu manifolds with ξ belonging to the $(k,\mu)'$-nullity distribution. The $(1,1)$-type symmetric tensor field h' satisfies $h'\phi + \phi h' = 0$ and $h'\xi = 0$. Also it is clear that

$$h = 0 \Leftrightarrow h' = 0, \qquad h'^2 = (k+1)\phi^2 \qquad (\Leftrightarrow h^2 = (k+1)\phi^2). \tag{2.6}$$

For an almost Kenmotsu manifold, we have from (1.3)

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)h'X - \eta(X)h'Y], \tag{2.7}$$

$$R(\xi,X)Y = k[g(X,Y)\xi - \eta(Y)X] + \mu[g(h'X,Y)\xi - \eta(Y)h'X], \tag{2.8}$$

where $k, \mu \in \mathbb{R}$. Contracting Y in (2.8), we have

$$S(X,\xi) = 2k\eta(X). \tag{2.9}$$

Let $X \in \mathcal{D}$ be the eigen vector of h' corresponding to the eigen value λ. It follows from (2.6) that $\lambda^2 = -(k+1)$, a constant. Therefore, $k \leq -1$ and $\lambda = \pm\sqrt{-k-1}$. We denote by $[\lambda]'$ and $[-\lambda]'$ the corresponding eigenspaces associated with h' corresponding to the non-zero eigen value λ and $-\lambda$, respectively. We have the following lemmas.

Lemma 2.1. ([**9**, Proposition 4.1]) *Let $(M^{2n+1},\phi,\xi,\eta,g)$ be an almost Kenmotsu manifold such that ξ belongs to the $(k,\mu)'$-nullity distribution and $h' \neq 0$. Then $k < -1$, $\mu = -2$ and* $\mathrm{Spec}(h') = \{0,\lambda,-\lambda\}$ *with* 0 *as simple eigen value and $\lambda = \sqrt{-k-1}$. The distributions $[\xi] \oplus [\lambda]'$ and $[\xi] \oplus [-\lambda]'$ are integrable with totally geodesic leaves. The distributions $[\lambda]'$ and $[-\lambda]'$ are integrable with totally umbilical leaves.*

Lemma 2.2. ([**9**, Lemma 4.1]) *Let $(M^{2n+1},\phi,\xi,\eta,g)$ be an almost Kenmotsu manifold with $h' \neq 0$ and ξ belonging to the $(k,-2)'$-nullity distribution. Then for any $X, Y \in T_pM$,*

$$(\nabla_X h')Y = -g(h'X + h'^2X, Y)\xi - \eta(Y)(h'X + h'^2X). \tag{2.10}$$

Takahashi [**20**] introduced the notion of ϕ-symmetry in the study of Sasakian manifolds. Then De and Sarkar [**5**] introduced a generalized notion of ϕ-symmetry called ϕ-Ricci symmetry in the study of Sasakian manifolds.

Definition 2.1. An almost Kenmotsu manifold is said to be ϕ-Ricci symmetric if it satisfies

$$\phi^2((\nabla_W Q)Y) = 0 \tag{2.11}$$

for any vector fields $W, Y \in T_pM$, where Q is the Ricci operator defined by $S(X,Y) = g(QX,Y)$. In addition, if the vector fields W, Y are orthogonal to ξ, then the manifold is called locally ϕ-Ricci symmetric manifold.

3. Ricci semisymmetric almost Kenmotsu manifolds

In a 3-dimensional Riemannian manifold, we have [**27**]

$$\begin{aligned} R(X,Y)Z = {} & S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY \\ & - \frac{r}{2}\{g(Y,Z)X - g(X,Z)Y\}, \end{aligned} \tag{3.1}$$

where Q is the Ricci operator defined by $g(QX,Y) = S(X,Y)$ for all $X, Y \in T_pM$ and r is the scalar curvature of the manifold.
Putting $Y = Z = \xi$ in (3.1) and using Lemma 2.1 and (2.9), we obtain

$$QX = \left(\frac{r}{2} - k\right) X - \left(\frac{r}{2} - 3k\right)\eta(X)\xi - 2h'X, \tag{3.2}$$

which is equivalent to

$$S(X,Y) = \left(\frac{r}{2} - k\right) g(X,Y) - \left(\frac{r}{2} - 3k\right) \eta(X)\eta(Y) - 2g(h'X,Y) \tag{3.3}$$

for any $X, Y \in T_pM$.

With the help of (3.2) and (3.3), it follows from (3.1) that

$$\begin{aligned} R(X,Y)Z = {} & \left(\frac{r}{2} - 2k\right) [g(Y,Z)X - g(X,Z)Y] - \left(\frac{r}{2} - 3k\right) [g(Y,Z)\eta(X)\xi \\ & - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y] - 2g(Y,Z)h'X \\ & + 2g(X,Z)h'Y - 2g(h'Y,Z)X + 2g(h'X,Z)Y \end{aligned} \tag{3.4}$$

for any $X, Y, Z \in T_pM$.

Now we suppose that the manifold M^3 is Ricci semisymmetric, that is,

$$(R(X,Y) \cdot S)(U,V) = 0 \tag{3.5}$$

for all vector fields X, Y, U, V, which implies

$$S(R(X,Y)U,V) + S(U,R(X,Y)V) = 0. \tag{3.6}$$

Substituting $X = U = \xi$ in (3.6), we get

$$S(R(\xi,Y)\xi,V) + S(\xi,R(\xi,Y)V) = 0. \tag{3.7}$$

Using (2.9), it follows from (3.7) that

$$S(R(\xi,Y)\xi,V) + 2k\eta(R(\xi,Y)V) = 0. \tag{3.8}$$

Making use of (2.8) and (3.8), we have

$$
\begin{aligned}
&2k^2\eta(Y)\eta(V) - kS(Y,V) + 2S(h'Y,V) \\
&+ 2k^2 g(Y,V) - 2k^2\eta(Y)\eta(V) - 4kg(h'Y,V) = 0,
\end{aligned} \tag{3.9}
$$

which implies

$$
kS(Y,V) - 2S(h'Y,V) - 2k^2 g(Y,V) + 4kg(h'Y,V) = 0. \tag{3.10}
$$

Replacing Y by $h'Y$ in (3.10) and using the fact $h'^2 = (k+1)\phi^2$, it yields to

$$
kS(h'Y,V) + 2(k+1)S(Y,V) - 2k^2 g(h'Y,V) - 4k(k+1)g(Y,V) = 0. \tag{3.11}
$$

Adding k times of (3.10) and two times of (3.11), we have

$$
(k+2)^2[S(Y,V) - 2kg(Y,V)] = 0. \tag{3.12}
$$

Now we consider the following two cases:
Case 1. $k \neq -2$. It follows from (3.12) that

$$
S(Y,V) = 2kg(Y,V),
$$

which implies that the manifold is an Einstein manifold.

Case 2. $k = -2$. Then by [**9**, Corollary 4.1], the manifold is an CR-manifold.

From the above discussions, we have the following theorem.

Theorem 3.1. *Let $(M^3, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold such that ξ belongs to the $(k,\mu)'$-nullity distribution with $h' \neq 0$. If M^3 is Ricci semisymmetric, then, either the manifold is*

1. *an Einstein manifold, or*
2. *a CR-manifold.*

Also Ricci symmetric manifold ($\nabla S = 0$) implies Ricci semisymmetric ($R \cdot S = 0$), therefore we can state the following:

Corollary 3.1. *Let $(M^3, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold such that ξ belongs to the $(k,\mu)'$-nullity distribution with $h' \neq 0$. If M^3 is Ricci symmetric, then either the manifold is*

1. *an Einstein manifold or*
2. *a CR-manifold.*

A Riemannian manifold is said to be Ricci-recurrent [**18**] if the Ricci tensor S is non-zero and satisfies the condition

$$
(\nabla_X S)(Y,Z) = A(X)S(Y,Z),
$$

where $X, Y, Z \in T_pM$ and A is a non-zero 1-form.

In [**13**], Jun et al proved that a Ricci-recurrent Riemannian manifold is Ricci semisymmetric.
Hence we can state the following corollary.

Corollary 3.2. *Let $(M^3, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold such that ξ belongs to the $(k, \mu)'$-nullity distribution with $h' \neq 0$. If M^3 is Ricci-recurrent, then either the manifold is*

1. *an Einstein manifold or*
2. *a CR-manifold.*

4. Codazzi type Ricci tensor

In this section, we assume that the manifold under consideration satisfies Codazzi type [**12**] of Ricci tensor, then the Ricci tensor S satisfies

$$(\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z). \tag{4.1}$$

Taking the covariant derivative of (3.3) along arbitrary vector field Y and using (2.3), we have

$$\begin{aligned}(\nabla_Y S)(X, Z) = {} & \frac{\mathrm{dr}(Y)}{2}[g(X, Z) - \eta(X)\eta(Z)] - \Big(\frac{r}{2} - 3k\Big)[g(X, Y)\eta(Z) \\ & + g(h'Y, X)\eta(Z) + g(Y, Z)\eta(X) + g(h'Y, Z)\eta(X) \\ & - 2\eta(X)\eta(Y)\eta(Z)] - 2g((\nabla_Y h')X, Z).\end{aligned} \tag{4.2}$$

Interchanging X and Y in (4.2), we get

$$\begin{aligned}(\nabla_X S)(Y, Z) = {} & \frac{\mathrm{dr}(X)}{2}[g(Y, Z) - \eta(Y)\eta(Z)] - \Big(\frac{r}{2} - 3k\Big)[g(X, Y)\eta(Z) \\ & + g(h'X, Y)\eta(Z) + g(X, Z)\eta(Y) + g(h'X, Z)\eta(Y) \\ & - 2\eta(Y)\eta(X)\eta(Z)] - 2g((\nabla_X h')Y, Z).\end{aligned} \tag{4.3}$$

Making use of (4.2) and (4.3) in (4.1) yields to

$$\begin{aligned}& \frac{\mathrm{dr}(X)}{2}[g(Y, Z) - \eta(Y)\eta(Z)] - \frac{\mathrm{dr}(Y)}{2}[g(X, Z) - \eta(X)\eta(Z)] \\ & -2g((\nabla_X h')Y, Z) + 2g((\nabla_Y h')X, Z) - \Big(\frac{r}{2} - 3k\Big)[g(X, Z)\eta(Y) \\ & + g(h'X, Z)\eta(Y) - g(Y, Z)\eta(X) - g(h'Y, Z)\eta(X)] = 0.\end{aligned} \tag{4.4}$$

It is known [**16**] that Cartan hypersurfaces are manifolds with non-parallel Ricci tensor satisfying (4.1). From (4.1), it follows that $r = \text{constant}$. Then (4.4) implies

$$\begin{aligned}& \Big(\frac{r}{2} - 3k\Big)[g(X, Z)\eta(Y) + g(h'X, Z)\eta(Y) - g(Y, Z)\eta(X) \\ & -g(h'Y, Z)\eta(X)] + 2g((\nabla_X h')Y, Z) - 2g[(\nabla_Y h')X, Z] = 0.\end{aligned} \tag{4.5}$$

Making use of (2.10) and (2.3), we have

$$(\nabla_Y h')X - (\nabla_X h')Y = \eta(Y)h'X - \eta(X)h'Y - (k+1)\eta(Y)X + (k+1)\eta(X)Y. \tag{4.6}$$

In view of (4.5) and (4.6), it follows that

$$\begin{aligned}& \Big(\frac{r}{2} - 3k\Big)[g(X, Z)\eta(Y) + g(h'X, Z)\eta(Y) - g(Y, Z)\eta(X) \\ & -g(h'Y, Z)\eta(X)] + 2[(k+1)[g(X, Z)\eta(Y) - g(Y, Z)\eta(X)] \\ & -g(h'X, Z)\eta(Y) + g(h'Y, Z)\eta(X)] = 0.\end{aligned} \tag{4.7}$$

Substituting $X = \xi$ in (4.7) gives

$$\begin{aligned}(r-6k)\,[g(Y,Z)+g(h'Y,Z)-\eta(Y)\eta(Z)]\\ +4[(k+1)g(Y,Z)-(k+1)\eta(Y)\eta(Z)-g(h'Y,Z)]=0.\end{aligned} \tag{4.8}$$

Putting $Y = h'Y$ in (4.8) and applying $h'^2 = (k+1)\phi^2$ yield to

$$\begin{aligned}(r-6k)[g(h'Y,Z)-(k+1)g(Y,Z)+(k+1)\eta(Y)\eta(Z)]\\ +4(k+1)[g(Y,Z)-\eta(Y)\eta(Z)+g(h'Y,Z)]=0.\end{aligned} \tag{4.9}$$

Subtracting (4.9) from (4.8), we have

$$(r-6k)(k+2)[g(Y,Z)-\eta(Y)\eta(Z)]-4(k+2)g(h'Y,Z)=0. \tag{4.10}$$

From (4.10), it follows that either $k = -2$ or

$$g(h'Y,Z)=\frac{r-6k}{4}[g(Y,Z)-\eta(Y)\eta(Z)]. \tag{4.11}$$

Making use of (3.3) and (4.11), we obtain

$$S(Y,Z)=2kg(Y,Z),$$

that is, the manifold is an Einstein manifold.

Hence by the similar argument as in Section 3, we can state the following.

Theorem 4.1. *Let (M^3,ϕ,ξ,η,g) be an almost Kenmotsu manifold such that ξ belongs to the $(k,\mu)'$-nullity distribution with $h' \neq 0$. If M^3 admits Codazzi type Ricci tensor, then either the manifold is*

1. *an Einstein manifold or*
2. *a CR-manifold.*

5. Cyclic parallel Ricci tensor

This section is devoted to study cyclic parallel Ricci tensor in almost Kenmotsu manifolds with ξ belonging to the $(k,\mu)'$-nullity distribution and $h' \neq 0$ of dimension 3. Suppose the manifold under consideration satisfies cyclic parallel Ricci tensor [**12**], then the Ricci tensor S satisfies

$$(\nabla_X S)(Y,Z)+(\nabla_Y S)(Z,X)+(\nabla_Z S)(X,Y)=0. \tag{5.1}$$

Taking the covariant derivative of (3.3) along arbitrary vector field Z and using (2.3), we have

$$\begin{aligned}(\nabla_Z S)(X,Y)=\frac{\mathrm{dr}(Z)}{2}[g(X,Y)-\eta(X)\eta(Y)]-\Big(\frac{r}{2}-3k\Big)\,[g(X,Z)\eta(Y)\\ +g(Y,Z)\eta(X)+g(h'X,Z)\eta(Y)+g(h'Y,Z)\eta(X)\\ -2\eta(X)\eta(Y)\eta(Z)[-2g((\nabla_Z h')X,Y).\end{aligned} \tag{5.2}$$

Similarly,

$$\begin{aligned}(\nabla_X S)(Y,Z)=\frac{\mathrm{dr}(X)}{2}[g(Y,Z)-\eta(Y)\eta(Z)]-\Big(\frac{r}{2}-3k\Big)\,[g(X,Y)\eta(Z)\\ +g(X,Z)\eta(Y)+g(h'X,Y)\eta(Z)+g(h'X,Z)\eta(Y)\\ -2\eta(X)\eta(Y)\eta(Z)]-2g((\nabla_X h')Y,Z),\end{aligned} \tag{5.3}$$

and

$$
\begin{aligned}
(\nabla_Y S)(Z,X) = {} & \frac{dr(Y)}{2}[g(Z,X) - \eta(Z)\eta(X)] - \Big(\frac{r}{2} - 3k\Big)[g(Y,Z)\eta(X) \\
& + g(Y,X)\eta(Z) + g(h'Y,Z)\eta(X) + g(h'Y,X)\eta(Z) \\
& - 2\eta(X)\eta(Y)\eta(Z)] - 2g((\nabla_Y h')Z,X).
\end{aligned} \tag{5.4}
$$

It is known [**16**] that Cartan hypersurfaces are manifolds with non-parallel Ricci tensor satisfying (5.1). From (5.1), it follows that $r =$ constant. Making use of (5.2)–(5.4) in (5.1), we have

$$
\begin{aligned}
(r-6k)\,[g(X,Y)\eta(Z) + g(Y,Z)\eta(X) + g(X,Z)\eta(Y) + g(h'X,Y)\eta(Z) \\
+g(h'Y,Z)\eta(X) + g(h'X,Z)\eta(Y) - 3\eta(X)\eta(Y)\eta(Z)] \\
+2g((\nabla_Z h')X,Y) + 2g((\nabla_X h')Y,Z) + 2g((\nabla_Y h')Z,X) = 0.
\end{aligned} \tag{5.5}
$$

Using (2.10) and (2.3), we obtain

$$
\begin{aligned}
& g((\nabla_Z h')X,Y) + g((\nabla_X h')Y,Z) + g((\nabla_Y h')Z,X) \\
& = 2[(k+1)[g(X,Y)\eta(Z) + g(Y,Z)\eta(X) + g(X,Z)\eta(Y)] \\
& \quad - 3\eta(X)\eta(Y)\eta(Z) - g(h'X,Y)\eta(Z) - g(h'Y,Z)\eta(X) - g(h'X,Z)\eta(Y)].
\end{aligned} \tag{5.6}
$$

In account of (5.5) and (5.6), we get

$$
\begin{aligned}
(r-6k)\,[g(X,Y)\eta(Z) + g(Y,Z)\eta(X) + g(X,Z)\eta(Y) \\
+g(h'X,Y)\eta(Z) + g(h'Y,Z)\eta(X) + g(h'X,Z)\eta(Y) - 3\eta(X)\eta(Y)\eta(Z)] \\
+4[(k+1)[g(X,Y)\eta(Z) + g(Y,Z)\eta(X) + g(X,Z)\eta(Y) - 3\eta(X)\eta(Y)\eta(Z)] \\
-g(h'X,Y)\eta(Z) - g(h'Y,Z)\eta(X) - g(h'X,Z)\eta(Y)] = 0.
\end{aligned} \tag{5.7}
$$

Setting $Z = \xi$ in (5.7) yields to

$$
\begin{aligned}
(r-6k)[g(X,Y) + g(h'X,Y) - \eta(X)\eta(Y)] \\
+4[(k+1)g(X,Y) - (k+1)\eta(X)\eta(Y) - g(h'X,Y)] = 0.
\end{aligned} \tag{5.8}
$$

Replacing X by $h'X$ in (5.8) and applying $h'^2 = (k+1)\phi^2$, it implies

$$
\begin{aligned}
(r-6k)[g(h'X,Y) - (k+1)g(X,Y) + (k+1)\eta(X)\eta(Y)] \\
+4(k+1)\{g(X,Y) - \eta(X)\eta(Y) + g(h'X,Y)\} = 0.
\end{aligned} \tag{5.9}
$$

Subtracting (5.9) from (5.8), we have

$$
(r-6k)(k+2)[g(X,Y) - \eta(X)\eta(Y)] - 4(k+2)g(h'X,Y) = 0. \tag{5.10}
$$

From (5.10), we see that either $k = -2$ or

$$
g(h'X,Y) = \frac{r-6k}{4}[g(X,Y) - \eta(X)\eta(Y)]. \tag{5.11}
$$

With the help of (3.3) and (5.11), we get

$$
S(X,Y) = 2kg(X,Y),
$$

that is, the manifold is an Einstein manifold.

Therefore, by the similar argument as in Section 3, we have the following theorem.

Theorem 5.1. *Let $(M^3, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold such that ξ belongs to the $(k, \mu)'$-nullity distribution with $h' \neq 0$. If M^3 admits cyclic parallel Ricci tensor, then, either the manifold is*

1. *an Einstein manifold, or*
2. *a CR-manifold.*

6. η-parallel Ricci tensor

Definition 6.1. The Ricci tensor S of an almost Kenmotsu manifold M is called η-parallel if it satisfies

$$(\nabla_X S)(\phi Y, \phi Z) = 0 \tag{6.1}$$

for all vector fields X, Y and Z.

The notion of η-parallel Ricci tensor for Sasakian manifolds was given by Kon [**17**]. From (3.3), we have

$$S(\phi X, \phi Y) = \left(\frac{r}{2} - k\right) g(\phi X, \phi Y) - 2g(h'\phi X, \phi Y). \tag{6.2}$$

Taking covariant derivative of (6.2) along any vector field Z we get

$$(\nabla_Z S)(\phi X, \phi Y) = \frac{\mathrm{dr}(Z)}{2} g(\phi X, \phi Y) - 2g((\nabla_Z h')\phi X, \phi Y). \tag{6.3}$$

Using (2.10), we obtain

$$g((\nabla_Z h')\phi X, \phi Y) = 0. \tag{6.4}$$

Taking account of (6.4), from (6.3), we get

$$(\nabla_Z S)(\phi X, \phi Y) = \frac{\mathrm{dr}(Z)}{2} g(\phi X, \phi Y). \tag{6.5}$$

In view of (6.1) and (6.5), we have

$$\frac{\mathrm{dr}(Z)}{2} g(\phi X, \phi Y) = 0, \tag{6.6}$$

that is, $r =$ constant.
Conversely, if $r=$ constant, then it can be easily shown that

$$(\nabla_X S)(\phi Y, \phi Z) = 0$$

for all vector fields X, Y and Z.

Hence we can state the following theorem.

Theorem 6.1. *The Ricci tensor of an almost Kenmotsu manifold M of dimension 3 with ξ belonging to the $(k, \mu)'$-nullity distribution and $h' \neq 0$ is η-parallel if and only if the scalar curvature r is constant.*

7. Locally ϕ-Ricci symmetric almost Kenmotsu manifolds

In this section, we study locally ϕ-Ricci symmetric almost Kenmotsu manifolds of dimension 3 with ξ belonging to the $(k, \mu)'$-nullity distribution and $h' \neq 0$.

Taking covariant derivative of (3.2) along any vector field X, we have

$$(\nabla_X Q)Y = \frac{\mathrm{dr}(X)}{2}[Y - \eta(Y)\xi] - \left(\frac{r}{2} - 3k\right)[(\nabla_X\eta)Y\xi + \eta(Y)\nabla_X\xi] - 2(\nabla_X h')Y. \tag{7.1}$$

Applying ϕ^2 on both sides of (7.1) and using (2.3) yield to

$$\phi^2((\nabla_X Q)Y) = \frac{\mathrm{dr}(X)}{2}[-Y + \eta(Y)\xi] - \left(\frac{r}{2} - 3k\right)\eta(Y)\phi^2(\nabla_X\xi) - 2\phi^2((\nabla_X h')Y). \tag{7.2}$$

Making use of (2.10), the above equation implies

$$\phi^2((\nabla_X Q)Y) = \frac{\mathrm{dr}(X)}{2}[-Y + \eta(Y)\xi] - \left(\frac{r}{2} - 3k\right)\eta(Y)\phi^2(\nabla_X\xi) + 2\eta(Y)\phi^2(h'X + h'^2X). \tag{7.3}$$

In view of (2.11) and (7.3), we have

$$\frac{\mathrm{dr}(X)}{2}Y = 0,$$

that is, $r =$ constant.

Conversely, if r is constant, then the manifold is locally ϕ-Ricci symmetric. Thus we have the following theorem.

Theorem 7.1. *An almost Kenmotsu manifold M of dimension 3 with ξ belonging to the $(k,\mu)'$-nullity distribution and $h' \neq 0$ is locally ϕ-Ricci symmetric if and only if the scalar curvature r is a constant, provided the scalar curvature r is invariant under ξ.*

Hence from Theorem 6.1 and Theorem 7.1, we have the following corollary.

Corollary 7.1. *In an almost Kenmotsu manifold M of dimension 3 with ξ belonging to the $(k,\mu)'$-nullity distribution and $h' \neq 0$, the following statements are equivalent:*

1. *Ricci tensor is η-parallel;*
2. *manifold is locally ϕ-Ricci symmetric;*
3. *scalar curvature r is a constant, provided the scalar curvature r is invariant under ξ.*

8. Example of a 3-dimensional almost Kenmotsu manifold

We consider 3-dimensional manifold $M = \{(x,y,z) \in \mathbb{R}^3\}$, where (x,y,z) are the standard coordinates in $\mathbb{R}^3$. Let ξ, e_2, e_3 are three vector fields in $\mathbb{R}^3$ which satisfy [**9**]

$$[e_2, e_3] = 0, \qquad [\xi, e_2] = -e_2 - e_3, \qquad [\xi, e_3] = -e_2 - e_3.$$

Let g be the Riemannian metric defined by

$$g(\xi,\xi) = g(e_2,e_2) = g(e_3,e_3) = 1,$$
$$g(\xi,e_2) = g(\xi,e_3) = g(e_2,e_3) = 0.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, \xi)$ for any $Z \in T(M)$. Let ϕ be the $(1, 1)$ tensor field defined by

$$\phi(\xi) = 0, \qquad \phi(e_2) = e_3, \qquad \phi(e_3) = -e_2.$$

Using the linearity of ϕ and g, we have $\eta(\xi) = 1$, $\phi^2 X = -X + \eta(X)\xi$, and $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$ for any $X, Y \in \chi(M)$. Thus the structure (ϕ, ξ, η, g) is an almost contact structure. Also we have

$$h'\xi = 0, \qquad h'(e_2) = e_3, \qquad h'(e_3) = e_2.$$

The Riemannian connection ∇ of the metric g is given by the Koszul's formula

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned}$$

Using the Koszul's formula, we obtain

$$\begin{aligned} &\nabla_\xi \xi = 0, && \nabla_\xi e_2 = 0, && \nabla_\xi e_3 = 0, \\ &\nabla_{e_2} \xi = e_2 + e_3, && \nabla_{e_2} e_2 = -\xi, && \nabla_{e_2} e_3 = -\xi, \\ &\nabla_{e_3} \xi = e_2 + e_3, && \nabla_{e_3} e_2 = -\xi, && \nabla_{e_3} e_3 = -\xi. \end{aligned}$$

In view of the above relations, we get

$$\nabla_X \xi = -\phi^2 X + h'X$$

for any $X \in \chi(M)$. Therefore, the structure (ϕ, ξ, η, g) is an almost contact metric structure such that $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$, so that M is an almost Kenmotsu manifold.

By the above results, we can easily obtain the components of the curvature tensor R as follows:

$$\begin{aligned} &R(\xi, e_2)\xi = 2(e_2 + e_3), \qquad R(\xi, e_2)e_2 = -2\xi, \qquad R(\xi, e_2)e_3 = -2\xi, \\ &R(e_2, e_3)\xi = R(e_2, e_3)e_2 = R(e_2, e_3)e_3 = 0, \\ &R(\xi, e_3)\xi = 2(e_2 + e_3), \qquad R(\xi, e_3)e_2 = -2\xi, \qquad R(\xi, e_3)e_3 = -2\xi. \end{aligned}$$

With the help of the expressions of the curvature tensor, we conclude that the characteristic vector field ξ belongs to the $(k, \mu)'$-nullity distribution with $k = -2$ and $\mu = -2$.

Using the expressions of the curvature tensor, we find the values of the Ricci tensor S as follows:

$$S(\xi, \xi) = -4, \qquad S(e_2, e_2) = -2, \qquad S(e_3, e_3) = -2.$$

Therefore, the scalar curvature $r = S(\xi, \xi) + S(e_2, e_2) + S(e_3, e_3) = -8$, a constant. Hence, Theorem 6.1 and Theorem 7.1 are verified.

Acknowledgment. The authors are thankful to the referee for his/her valuable suggestions towards the improvement of the paper.

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