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# ON 3-DIMENSIONAL $f$-KENMOTSU MANIFOLDS AND RICCI SOLITONS

# ПРО 3-ВИМІРНІ $f$-МНОГОВИДИ КЕНМОЦУ ТА СОЛІТОНИ РІЧЧІ

The object of the present paper is to study 3-dimensional $f$-Kenmotsu manifolds and Ricci solitons. Firstly we give an example of a 3 -dimensional $f$-Kenmotsu manifold. Then we consider Ricci-semisymmetric 3-dimensional $f$-Kenmotsu manifold and prove that a 3-dimensional $f$-Kenmotsu manifold is Ricci semisymmetric if and only if it is an Einstein manifold. Also $\eta$-parallel Ricci tensor in a 3-dimensional $f$-Kenmotsu manifold have been studied. Finally we study Ricci solitons in a 3-dimensional $f$-Kenmotsu manifold.

Метою даної статті є вивчення 3-вимірних $f$-многовидів Кенмоцу та солітонів Річчі. Спочатку наведено приклад 3-вимірного $f$-многовиду Кенмоцу. Потім розглянуто напівсиметричний за Річчі 3-вимірний $f$-многовид Кенмоцу і доведено, що 3-вимірний $f$-многовид Кенмоцу є напівсиметричним за Річчі тоді і тільки тоді, коли він є многовидом Ейнштейна. Також досліджено $\eta$-паралельний тензор Річчі у 3-вимірному $f$-многовиді Кенмоцу. Насамкінець, досліджено солітони Річчі у 3-вимірному $f$-многовиді Кенмоцу.

**1. Introduction.** Let $M$ be an almost contact manifold, i.e., $M$ is a connected $(2n+1)$-dimensional differentiable manifold endowed with an almost contact metric structure $(\phi, \xi, \eta, g)$ [1]. As usually, denote by $\Phi$ the fundamental 2-form of $M$, $\Phi(X,Y) = g(X, \phi Y)$, $X, Y \in \chi(M)$, $\chi(M)$ being the Lie algebra of differentiable vector fields on $M$.

For further use, we recall the following definitions [1, 5, 19]. The manifold $M$ and its structure $(\phi, \xi, \eta, g)$ is said to be:

(i) normal if the almost complex structure defined on the product manifold $M \times \mathbb{R}$ is integrable (equivalently $[\phi, \phi] + 2d\eta \otimes \xi = 0$),

(ii) almost cosymplectic if $d\eta = 0$ and $d\Phi = 0$,

(iii) cosymplectic if it is normal and almost cosymplectic (equivalently, $\nabla\phi = 0$, $\nabla$ being covariant differentiation with respect to the Levi–Civita connection).

The manifold $M$ is called locally conformal cosymplectic (respectively, almost cosymplectic) if $M$ has an open covering $\{U_t\}$ endowed with differentiable functions $\sigma_t : U_t \to \mathbb{R}$ such that over each $U_t$ the almost contact metric structure $(\phi_t, \xi_t, \eta_t, g_t)$ defined by

$$\phi_t = \phi, \qquad \xi_t = e^{\sigma_t}\xi, \qquad \eta_t = e^{-\sigma_t}\eta, \qquad g_t = e^{-2\sigma_t}g$$

is cosymplectic (respectively, almost cosymplectic).

Olszak and Rosca [14] studied normal locally conformal almost cosymplectic manifold. They gave a geometric interpretation of $f$-Kenmotsu manifolds and studied some curvature properties. Among others they proved that a Ricci symmetric $f$-Kenmotsu manifold is an Einstein manifold.

By an $f$-Kenmotsu manifold we mean an almost contact metric manifold which is normal and locally conformal almost cosymplectic.

Apart from conformal curvature tensor, the projective curvature tensor is another important tensor from the differential geometric point of view. Let $M$ be a $(2n+1)$-dimensional Riemannian manifold. If there exist an one-to-one correspondence between each coordinate neighborhood of $M$ and a domain in Euclidean space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then $M$ is said to be locally projectively flat. For $n \geq 1$, $M$ is locally projectively flat if and only if the well known projective curvature tensor $P$ vanishes. Here $P$ is defined by [12]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}\{S(Y,Z)X - S(X,Z)Y\},$$

for $X, Y, Z \in \chi(M)$, where $R$ is the curvature tensor and $S$ is the Ricci tensor of $M$. In fact, $M$ is projectively flat (that is, $P = 0$) if and only if the manifold is of constant curvature [21, p. 84, 85]. Thus, the projective curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature. $\xi$-conformally flat $K$-contact manifolds have been studied by Zhen, Cabrerizo and Fernandez [22]. Likewise we define $\xi$-projectively flat $f$-Kenmotsu manifold. An $f$-Kenmotsu manifold is called $\xi$-projectively flat if the condition $P(X,Y)\xi = 0$ holds on $M$.

A Ricci soliton is a generalization of an Einstein metric. In a Riemannian manifold $(M, g)$, $g$ is called a Ricci soliton if [7]

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{1.1}$$

where $\pounds$ is the Lie derivative, $S$ is the Ricci tensor, $V$ is a complete vector field on $M$ and $\lambda$ is a constant. Compact Ricci solitons are the fixed points of the Ricci flow $\frac{\partial}{\partial t}g = -2S$ projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise as blow-up limits for the Ricci flow on compact manifolds.

The Ricci soliton is said to be shrinking, steady and expanding according as $\lambda$ is negative, zero and positive respectively. If the vector field $V$ is the gradient of a potential function $-f$, then $g$ is called a gradient Ricci soliton and equation (1.1) assumes the form

$$\nabla\nabla f = S + \lambda g.$$

A Ricci soliton on a compact manifold has constant curvature in dimension 2 [7] and also in dimension 3 [8]. For details we refer to Chow and Knopf [3] and Derdzinski [4]. We also recall the following significant result of Perelman [15]: A Ricci soliton on a compact manifold is a gradient Ricci soliton.

In [18], Sharma has started the study of Ricci solitons in $K$-contact manifolds. In a $K$-contact manifold the structure vector field $\xi$ is Killing, that is, $\pounds_\xi g = 0$, which is not in general, in an $f$-Kenmotsu manifold.

In the present paper we have studied some curvature properties of a 3-dimensional $f$-Kenmotsu manifold and Ricci solitons. The paper is organized as follows: After preliminaries we give an example of a regular 3-dimensional $f$-Kenmotsu manifold which is $\xi$-projectively flat. In Section 4, we consider Ricci semisymmetric 3-dimensional $f$-Kenmotsu manifolds and prove that a 3-dimensional $f$-Kenmotsu manifold is Ricci semisymmetric if and only if it is an Einstein manifold. Section 5 deals with $\eta$-parallel Ricci tensor in a 3-dimensional $f$-Kenmotsu manifold. Finally we study Ricci solitons in a 3-dimensional $f$-Kenmotsu manifold. We prove that if in a 3-dimensional $f$-Kenmotsu manifold the metric $g$ is a Ricci soliton and $V$ is pointwise collinear with $\xi$, then $g$ is $\eta$-Einstein and the Ricci soliton is expanding, provided $\lambda = 2(f^2 + f')$. We also prove that if the manifold is $\eta$-Einstein, then an $f$-Kenmotsu manifold admits a Ricci soliton, provided $f$ is constant.

**2. $f$-Kenmotsu manifolds.** Let $M$ be a real $(2n+1)$-dimensional differentiable manifold endowed with an almost contact structure $(\phi, \xi, \eta, g)$ satisfying

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1,$$

$$\phi\xi = 0, \qquad \eta \circ \phi = 0, \qquad \eta(X) = g(X, \xi), \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any vector fields $X, Y \in \chi(M)$, where $I$ is the identity of the tangent bundle $TM$, $\phi$ is a tensor field of $(1,1)$-type, $\eta$ is a 1-form, $\xi$ is a vector field and $g$ is a metric tensor field. We say that $(M, \phi, \xi, \eta, g)$ is an $f$-Kenmotsu manifold if the covariant differentiation of $\phi$ satisfies [13]:

$$(\nabla_X \phi)(Y) = f\big(g(\phi X, Y)\xi - \eta(Y)\phi X\big), \tag{2.2}$$

where $f \in C^\infty(M)$ such that $df \wedge \eta = 0$. If $f = \alpha = \text{constant} \neq 0$, then the manifold is a $\alpha$-Kenmotsu manifold [9]. 1-Kenmotsu manifold is a Kenmotsu manifold [10, 16]. If $f = 0$, then the manifold is cosymplectic [9]. An $f$-Kenmotsu manifold is said to be regular if $f^2 + f' \neq 0$, where $f' = \xi f$.

For an $f$-Kenmotsu manifold from (2.2) it follows that

$$\nabla_X \xi = f\{X - \eta(X)\xi\}. \tag{2.3}$$

The condition $df \wedge \eta = 0$ holds if $\dim M \geq 5$. In general this does not hold if $\dim M = 3$ [14].

In a 3-dimensional Riemannian manifold, we always have

$$R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y -$$

$$-\frac{\tau}{2}\big\{g(Y,Z)X - g(X,Z)Y\big\}. \tag{2.4}$$

In a 3-dimensional $f$-Kenmotsu manifold we have [14]

$$R(X,Y)Z = \left(\frac{\tau}{2} + 2f^2 + 2f'\right)(X \wedge Y)Z -$$

$$-\left(\frac{\tau}{2} + 3f^2 + 3f'\right)\big\{\eta(X)(\xi \wedge Y)Z + \eta(Y)(X \wedge \xi)Z\big\}, \tag{2.5}$$

$$S(X,Y) = \left(\frac{\tau}{2} + f^2 + f'\right)g(X,Y) - \left(\frac{\tau}{2} + 3f^2 + 3f'\right)\eta(X)\eta(Y), \tag{2.6}$$

where $\tau$ is the scalar curvature of $M$ and $f' = \xi(f)$.

From (2.5), we obtain

$$R(X,Y)\xi = -(f^2 + f')[\eta(Y)X - \eta(X)Y], \tag{2.7}$$

and (2.6) yields

$$S(X,\xi) = -2(f^2 + f')\eta(X). \tag{2.8}$$

From equations (2.7) and (2.8) it can be easily verified that $P(X,Y)\xi = 0$.

***Remark* 2.1.** A 3-dimensional $f$-Kenmotsu manifold is always $\xi$-projectively flat.

In the next section we verify the remark by an example.

**3. Example of a 3-dimensional $f$-Kenmotsu manifold.** We consider the three-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3,\ z \neq 0\}$, where $(x, y, z)$ are the standard coordinates in $\mathbb{R}^3$. The vector fields

$$e_1 = z^2 \frac{\partial}{\partial x}, \qquad e_2 = z^2 \frac{\partial}{\partial y}, \qquad e_3 = \frac{\partial}{\partial z}$$

are linearly independent at each point of $M$. Let $g$ be the Riemannian metric defined by

$$g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0,$$

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

Let $\eta$ be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$. Let $\phi$ be the $(1, 1)$ tensor field defined by $\phi(e_1) = -e_2$, $\phi(e_2) = e_1$, $\phi(e_3) = 0$.

Then using linearity of $\phi$ and $g$ we have

$$\eta(e_3) = 1, \qquad \phi^2 Z = -Z + \eta(Z) e_3,$$

$$g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M)$. Now, by direct computations we obtain

$$[e_1, e_2] = 0, \qquad [e_2, e_3] = -\frac{2}{z} e_2, \qquad [e_1, e_3] = -\frac{2}{z} e_1.$$

The Riemannian connection $\nabla$ of the metric tensor $g$ is given by the Koszul's formula which is

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) -$$

$$-g\big(X, [Y, Z]\big) - g\big(Y, [X, Z]\big) + g\big(Z, [X, Y]\big). \tag{3.1}$$

Using (3.1) we have

$$2g(\nabla_{e_1} e_3, e_1) = 2g\left(-\frac{2}{z} e_1, e_1\right),$$

$$2g(\nabla_{e_1} e_3, e_2) = 0 \qquad \text{and} \qquad 2g(\nabla_{e_1} e_3, e_3) = 0.$$

Hence $\nabla_{e_1} e_3 = -\frac{2}{z} e_1$. Similarly, $\nabla_{e_2} e_3 = -\frac{2}{z} e_2$ and $\nabla_{e_3} e_3 = 0$. (3.1) further yields

$$\nabla_{e_1} e_2 = 0, \qquad \nabla_{e_1} e_1 = \frac{2}{z} e_3,$$

$$\nabla_{e_2} e_2 = \frac{2}{z} e_3, \qquad \nabla_{e_2} e_1 = 0,$$

$$\nabla_{e_3} e_2 = 0, \qquad \nabla_{e_3} e_1 = 0.$$

From the above it follows that the manifold satisfies $\nabla_X \xi = f\{X - \eta(X)\xi\}$ for $\xi = e_3$, where $f = -\frac{2}{z}$. Hence we conclude that $M$ is an $f$-Kenmotsu manifold. Also $f^2 + f' \neq 0$. Hence $M$ is a regular $f$-Kenmotsu manifold.

It is known that

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{3.2}$$

With the help of the above formula and using (3.2) it can be easily verified that

$$R(e_1, e_2)e_3 = 0, \qquad R(e_2, e_3)e_3 = -\frac{6}{z^2}e_2, \qquad R(e_1, e_3)e_3 = -\frac{6}{z^2}e_1,$$

$$R(e_1, e_2)e_2 = -\frac{4}{z^2}e_1, \qquad R(e_3, e_2)e_2 = -\frac{6}{z^2}e_3, \qquad R(e_1, e_3)e_2 = 0,$$

$$R(e_2, e_1)e_1 = -\frac{4}{z^2}e_2, \qquad R(e_2, e_3)e_1 = 0, \qquad R(e_3, e_1)e_1 = -\frac{6}{z^2}e_3.$$

From the above expressions of the curvature tensor we obtain

$$S(e_1, e_1) = g(R(e_1, e_2)e_2, e_1) + g(R(e_1, e_3)e_3, e_1) = -\frac{10}{z^2}.$$

Similarly, we have

$$S(e_2, e_2) = -\frac{10}{z^2} \qquad \text{and} \qquad S(e_3, e_3) = -\frac{12}{z^2}.$$

Now from the expressions of curvature tensors and Ricci tensor we have

$$P(e_1, e_2)e_3 = P(e_1, e_3)e_3 = P(e_2, e_3)e_3 = 0,$$

that is, $M$ is $\xi$-projectively flat. Hence the Remark 2.1 is verified.

**4. Ricci-semisymmetric 3-dimensional *f*-Kenmotsu manifolds.** An $f$-Kenmotsu manifold is called Ricci-semisymmetric if $R(X,Y) \cdot S = 0$ [20], where $R(X,Y)$ is treated as a derivation of the tensor algebra for any tangent vectors $X$, $Y$. Then

$$S(R(X,Y)U, V) + S(U, R(X,Y)V) = 0. \tag{4.1}$$

Putting $X = U = \xi$ in (4.1) we have

$$S(R(\xi, Y)\xi, V) + S(\xi, R(\xi, Y)V) = 0. \tag{4.2}$$

Then using (2.7) in (4.2) we obtain

$$(f^2 + f')\big[\eta(Y)S(\xi, V) - S(Y, V)\big] + (f^2 + f')\big[g(Y, V)S(\xi, \xi) - \eta(V)S(\xi, Y)\big] = 0.$$

Now using (2.8) in the above equation yields

$$S(Y, V) = -2\big(f^2 + f'\big)g(Y, V),$$

on the set of $M$ on which $f \neq 0$. Thus we obtain the following:

**Proposition 4.1.** *Let $M$ be a Ricci-semisymmetric 3-dimensional regular $f$-Kenmotsu manifold. Then $M$ is an Einstein manifold.*

Since Ricci symmetric manifold implies Ricci-semisymmetric, therefore Proposition 4.1 generalizes the result of Olszak and Rosca [14].

Now we prove the following:

**Theorem 4.1.** *Let $M$ be a 3-dimensional non-cosymplectic $f$-Kenmotsu manifold. The following conditions are equivalent*:

(i) *$M$ is an Einstein manifold,*

(ii) *the Ricci tensor $S$ of $M$ is parallel* ($\nabla S = 0$),

(iii) *$M$ is Ricci-semisymmetric.*

***Proof.*** (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are clear. (iii) $\Rightarrow$ (i) by Proposition 4.1.

Theorem 4.1 is proved.

***Remark* 4.1.** It is obvious that by the formula (2.4) the conditions (i) – (iii) in Theorem 4.1 can be replaced by the following conditions:

(i) $M$ is of constant curvature,

(ii) $M$ is locally symmetric ($\nabla R = 0$),

(iii) $M$ is semisymmetric.

## 5. $\eta$-parallel Ricci tensor.

**Definition 5.1.** *The Ricci tensor $S$ of a 3-dimensional $f$-Kenmotsu manifold is called $\eta$-parallel if it satisfies*

$$(\nabla_W S)(\phi X, \phi Y) = 0$$

*for all vector fields $W$, $X$ and $Y$.*

The notion of $\eta$-parallel Ricci tensor for Sasakian manifolds was introduced by Kon [11].

Let us consider a 3-dimensional $f$-Kenmotsu manifold. Replacing $X$ by $\phi X$ and $Y$ by $\phi Y$ in (2.6), we have

$$S(\phi X, \phi Y) = \left(\frac{\tau}{2} + f^2 + f'\right) g(\phi X, \phi Y). \tag{5.1}$$

Using (2.1) in (5.1) we get

$$S(\phi X, \phi Y) = \left(\frac{\tau}{2} + f^2 + f'\right) \{g(X, Y) - \eta(X)\eta(Y)\}. \tag{5.2}$$

Differentiating the equation (5.2) covariantly with respect to $W$, we obtain

$$(\nabla_W S)(\phi X, \phi Y) = \left(\frac{d\tau(W)}{2} + 2f(Wf) + (Wf')\right) \{g(X, Y) - \eta(X)\eta(Y)\} +$$

$$+ \left(\frac{\tau}{2} + f^2 + f'\right) \{-fg(W, X)\eta(Y) + fg(W, Y)\eta(X)\}.$$

Suppose the Ricci tensor is $\eta$-parallel. Then, we get from above

$$\left(\frac{d\tau(W)}{2} + 2f(Wf) + (Wf')\right) \{g(X, Y) - \eta(X)\eta(Y)\} +$$

$$+ \left(\frac{\tau}{2} + f^2 + f'\right) \{-fg(W, X)\eta(Y) + fg(W, Y)\eta(X)\} = 0. \tag{5.3}$$

Then putting $X = Y = e_i$ in (5.3), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold, and taking summation over $i$, $1 \leq i \leq 3$, we obtain

$$\frac{d\tau(W)}{2} + 2f(Wf) + (Wf') = 0.$$

Thus we have the following:

**Theorem 5.1.** *If in a* $3$*-dimensional* $f$*-Kenmotsu manifold the Ricci tensor* $S$ *is* $\eta$*-parallel, then the scalar curvature is constant, provided* $f$ *is a constant.*

**6. Ricci solitons.** Suppose that a 3-dimensional $f$-Kenmotsu manifold admits a Ricci soliton. It is well known that $\nabla g = 0$. Since $\lambda$ in the Ricci soliton equation (1.1) is a constant, so $\nabla \lambda g = 0$. Thus $\pounds_V g + 2S$ is parallel.

In [2] Calin and Crasmareanu proved that a symmetric parallel second order tensor in an $f$-Kenmotsu manifold is a constant multiple of the metric tensor on the set on which $f \neq 0$. Hence we conclude that $\pounds_V g + 2S = \beta$ (say) is a constant multiple of the metric tensor $g$, that is,

$$(\pounds_V g + 2S))(X, Y) = \beta(X, Y) = \beta(\xi, \xi) g(X, Y),$$

where $\beta(\xi, \xi)$ is given by

$$\beta(\xi, \xi) = (\pounds_V g + 2S))(\xi, \xi) = -4(f^2 + f').$$

So $\pounds_V g + 2S + 2\lambda g$ reduces to $(-4(f^2 + f') + 2\lambda)g$. Using (1.1) we get $\lambda = 2(f^2 + f')$. Thus we have the following:

**Proposition 6.1.** *In a* $3$*-dimensional regular* $f$*-Kenmotsu manifold the Ricci soliton* $(g, \lambda, V)$ *is shrinking or expanding according as* $(f^2 + f')$ *is negative or positive.*

In particular, let $V$ be pointwise collinear with $\xi$, i.e., $V = b\xi$, where $b$ is a function on the 3-dimensional $f$-Kenmotsu manifold. Then

$$(\pounds_V g + 2S + 2\lambda g)(X, Y) = 0,$$

which implies that

$$g(\nabla_X b\xi, Y) + g(\nabla_Y b\xi, X) + 2S(X, Y) + 2\lambda g(X, Y) = 0,$$

or,

$$bg(\nabla_X \xi, Y) + (Xb)\eta(Y) + bg(\nabla_Y \xi, X) + (Yb)\eta(X) + 2S(X, Y) + 2\lambda g(X, Y) = 0. \tag{6.1}$$

Using (2.3) in (6.1), we obtain

$$2fbg(X, Y) - 2fb\eta(X)\eta(Y) + (Xb)\eta(Y) + (Yb)\eta(X) + 2S(X, Y) + 2\lambda g(X, Y) = 0. \tag{6.2}$$

In (6.2) replacing $Y$ by $\xi$ it follows that

$$Xb + (\xi b)\eta(X) - 4(f^2 + f')\eta(X) + 2\lambda\eta(X) = 0. \tag{6.3}$$

Putting $X = \xi$ in (6.3), we get

$$\xi b = 2(f^2 + f') - \lambda.$$

Putting this value in (6.3), we obtain

$$Xb + (2(f^2 + f') - \lambda)\eta(X) - 4(f^2 + f')\eta(X) + 2\lambda\eta(X) = 0,$$

or,

$$db = \{2(f^2 + f') - \lambda\}\eta. \tag{6.4}$$

Applying $d$ on (6.4), we get

$$\{\lambda - 2(f^2 + f')\}d\eta = 0.$$

Since $d\eta \neq 0$, we have

$$2(f^2 + f') - \lambda = 0. \tag{6.5}$$

Using (6.5) in (6.4) yields $b$ is a constant. Therefore from (6.2) it follows

$$S(X,Y) = -(\lambda + fb)g(X,Y) + fb\eta(X)\eta(Y), \tag{6.6}$$

which implies that $M$ is an $\eta$-Einstein manifold. This leads to the following:

**Theorem 6.1.** *If in a* 3*-dimensional $f$-Kenmotsu manifold the metric $g$ is a Ricci soliton and $V$ is pointwise collinear with $\xi$, then $V$ is a constant multiple of $\xi$ and $g$ is an $\eta$-Einstein manifold of the form* (6.6) *and the Ricci soliton is expanding provided* $\lambda = 2(f^2 + f')$.

The converse of the above theorem is not true, in general. However if we take $f = \text{constant}$, i.e., if we consider a 3-dimensional $\eta$-Einstein $f$-Kenmotsu manifold, then it admits a Ricci soliton. This can be proved as follows:

Let $M$ be a 3-dimensional $\eta$-Einstein $f$-Kenmotsu manifold and $V = \xi$. Then

$$S(X,Y) = \gamma g(X,Y) + \delta\eta(X)\eta(Y), \tag{6.7}$$

where $\gamma$ and $\delta$ are certain scalars.

Now using (2.3)

$$(\pounds_\xi g)(X,Y) = g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) = 2f\big\{g(X,Y) - \eta(X)\eta(Y)\big\}.$$

Therefore

$$(\pounds_\xi g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 2(f + \gamma + \lambda)g(X,Y) - 2(f - \delta)\eta(X)\eta(Y). \tag{6.8}$$

From equation (6.8) it follows that $M$ admits a Ricci soliton $(g, \xi, \lambda)$ if $f + \gamma + \lambda = 0$ and $\delta = f = \text{constant}$. From (6.7) we have using (2.8), $-2f^2 = \gamma + \delta$. Hence $\gamma = -2f^2 - f = \text{constant}$. Therefore $\lambda = -(\gamma + \delta) = \text{constant}$. So we have the following:

**Theorem 6.2.** *If a* 3*-dimensional $f$-Kenmotsu manifold is $\eta$-Einstein of the form $S = \gamma g + \delta\eta \otimes \eta$, then the manifold admits a Ricci soliton $(g, \xi, -(\gamma + \delta))$.*

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