On 3-dimensional Riemannian manifolds satisfying a certain condition on the curvature tensor

By

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(Received July 20, 1968)

1. Introduction

If a Riemannian manifold M is locally symmetric, then its curvature tensor R satisfies

$$(*) \qquad R(X, Y)\cdot R=0 \qquad \text{for all tangent vectors } X \text{ and } Y,$$

where the endomorphism $R(X, Y)$ operates on R as a derivation of tensor algebra at each point of M.

Conversely, does this algebraic condition (*) on the curvature tensor field R imply that M is locally symmetric (i. e. $\nabla R=0$) ?

One must exclude the 2-dimentional case, as was already observed by E. Cartan, 1. K. Nomizu has conjectured that the answer is affirmative in the case where M is irreducible and complete and dim. $M\geqq 3$. There are some partial or related results in this direction.

The main purpose of the present paper is to deal with the same problem about 3-dimensional Riemannian manifolds.

2. Reduction of condition (*) and some results

Let M be a 3-dimensional connected Riemannian manifold, then it is well known that the curvature tensor R of M is written in the form

$$(2.1) \qquad R(X, Y)=AX\wedge Y+X\wedge AY-\frac{1}{2}(\text{trace } A)X\wedge Y$$

where A is a field of symmetric endomorphism which corresponds to the Ricci tensor field S, that is, $g(AX, Y)=S(X, Y)$, g being the Riemannian metric and $X\wedge Y$ denotes the endomorphism which maps Z upon $g(Z, Y)X-g(Z, X)Y$.

At a point $x\in M$, let $\{e_1, e_2, e_3\}$ be an orthogonal basis of the tangent space $T_x(M)$ such that $Ae_i=\lambda_i e_i$, $i=1, 2, 3$.

Then, the equation (2. 1) implies

(2. 2) $$R(e_i, e_j)=(\lambda_i+\lambda_j-\frac{1}{2}\sum_{k=1}^{3}\lambda_k)e_i\wedge e_j.$$

By computing

$$(R(e_i, e_j)\cdot R)(e_k, e_l)=[R(e_i, e_j), R(e_k,e_l)]-R(R(e_i, e_j)e_k, e_l)$$
$$-R(e_k, R(e_i, e_j)e_l),$$

we find that it is zero except possibly in the case where $k=i$ and $l\neq i, j$ $(i\neq j)$. For this case we have

(2. 3) $$(R(e_i, e_j)\cdot R)(e_i, e_l)=(\lambda_j-\lambda_i)(\lambda_j+\lambda_i-\frac{1}{2}\sum_{k=1}^{3}\lambda_k)e_j\wedge e_l.$$

Thus, we see that the condition (*) is equivalent to

(2. 4) $$(\lambda_j-\lambda_i)(2(\lambda_j+\lambda_i)-\sum_{k=1}^{3}\lambda_k)=0, \qquad \text{for} \quad i\neq j.$$

Then, we have the following

THEOREM. *Let M be a 3-dimensional connected Riemannian manifold whose curvature tensor R satisfies the condition (*). If the rank of the Ricci form is* 3 *at some point of M, then M is a space of constant curvature.*

PROOF. We assume that the rank of the Ricci form is 3 at a point $x_0\in M$. Then, if $\lambda_1=\lambda_2$, $\lambda_2\neq\lambda_3$, then from (2. 4), we get

$$2(\lambda_1+\lambda_3)-(2\lambda_1+\lambda_3)=0.$$

Thus, we get $\lambda_3=0$. This is a contradiction.

Similarly, if $\lambda_1\neq\lambda_2$, $\lambda_2\neq\lambda_3$, $\lambda_3\neq\lambda_1$, then from (2. 4), we get

$$2(\lambda_1+\lambda_2)-\sum_{k=1}^{3}\lambda_k=0$$

and

$$2(\lambda_2+\lambda_3)-\sum_{k=1}^{3}\lambda_k=0.$$

Thus, we get $\lambda_1=\lambda_3$. This is a contradiction. Therefore, we can conclude that $\lambda_1=\lambda_2=\lambda_3$ at x_0.

Now, let $W=\{x\in M$; the rank of S is 3 at $x\}$, which is an open set. Let W_0 be the connected component of x_0 in W. Then, we can easily see that $\lambda_1=\lambda_2=\lambda_3\equiv\lambda(\neq 0)$ on W_0 and hence λ is constant on W_0. We now show that W_0 is actually equal to M. Let x be a point of $\overline{W}_0-W_0$. By the continuity argument for the characteristic polynomial of A, we see that the the rank of S is equal to 3 at x. Thus, W_0 is open and closed so that $W_0=M$. Therefore, M is an Einstein space with Ricci tensor $S=\lambda g$, and hence, by virtue of (2. 1), we see that M is a space of constant curvature $\frac{\lambda}{2}(\lambda\neq 0)$. q. e. d.

In the next place, we assume that the rank of A (or S) is 2 at some pont, say $x_0 \in M$. In this case, if $\lambda_3=0$ at x_0, then we see that $\lambda_1=\lambda_2 \neq 0$.

We shall now state a few examples of non-symmetric and irreducible Riemannian manifolds satisfying the condition (*).

Let M be a 2-dimensional Riemannian manifold with metric g, I an open interval of a real line R with natural metric dt^2 and $\bar{M}=M \times I$. The tangent space $T_{\bar{p}}(\bar{M})$ at a point $\bar{p} \in \bar{M} (\bar{p}=(p, t), p \in M$ and $t \in I)$ is considered as the direct sum $T_p(M)+T_t(I)$, where $T_p(M)$ and $T_t(I)$ are the tangent spaces at $p \in M$ and $t \in I$ respectively. That is, any $X \in T_p(M)$ is uniquely decomposed as

$$\bar{X}=X+X_I, \quad X \in T_p(M), \quad X_I \in T_t(I).$$

Now, we shall define the following Riemannian metric $\bar{g}$ on $\bar{M}$;

$$\bar{g}(\bar{X}, \bar{Y})=e^{-2\lambda} g(X, Y)+e^{-2\mu} dt(X_I) dt(Y_I),$$

where λ and μ are some functions of $t \in I$.

If we denote by $\{X, Y\}$ an orthonormal basis of vector fields on a neighborhood $U \subset M$, then $\left(\bar{X}=e^{\lambda} X,\ \bar{Y}=e^{\lambda} Y,\ \bar{Z}=e^{\mu} \frac{\partial}{\partial t}\right)$ is an orthonormal basis of vector fields on $U \times I \subset \bar{M}$.

Between the Riemannian connections ∇ and $\bar{\nabla}$ corresponding to g and $\bar{g}$, the following relations are valid;

$$
\begin{aligned}
\bar{\nabla}_{\bar{X}} \bar{X} &= e^{\lambda} g(Y, \nabla_X X) \bar{Y}+\lambda' e^{\mu} \bar{Z}=e^{\lambda} \nabla_X X+\lambda' e^{2\mu} \frac{\partial}{\partial t} \\
\bar{\nabla}_{\bar{Y}} \bar{Y} &= e^{\lambda} g(X, \nabla_Y Y) \bar{Y}+\lambda' e^{\mu} \bar{Z}=e^{\lambda} \nabla_Y Y+\lambda' e^{2\mu} \frac{\partial}{\partial t} \\
\bar{\nabla}_{\bar{X}} \bar{Y} &= e^{\lambda} g(X, \nabla_X Y) \bar{X}=e^{2\lambda} \nabla_X Y \\
\bar{\nabla}_{\bar{Y}} \bar{X} &= e^{\lambda} g(Y, \nabla_Y X) \bar{Y}=e^{2\lambda} \nabla_Y X \\
\bar{\nabla}_{\bar{X}} \bar{Z} &= -\lambda' e^{\mu} \bar{X}=-\lambda' e^{\lambda+\mu} X \\
\bar{\nabla}_{\bar{Y}} \bar{Z} &= -\lambda' e^{\mu} \bar{Y}=-\lambda' e^{\lambda+\mu} Y \\
\bar{\nabla}_{\bar{Z}} \bar{X} &= 0 \\
\bar{\nabla}_{\bar{Z}} \bar{Y} &= 0 \\
\bar{\nabla}_{\bar{Z}} \bar{Z} &= 0
\end{aligned}
\tag{2.5}
$$

and

$$
\begin{aligned}
[\bar{Z}, \bar{X}] &= \lambda' e^{\mu} \bar{X}=\lambda' e^{\lambda+\mu} X \\
[\bar{Z}, \bar{Y}] &= \lambda' e^{\mu} \bar{Y}=\lambda' e^{\lambda+\mu} Y \\
[\bar{X}, \bar{Y}] &= e^{\lambda}\{g(X, \nabla_X Y) \bar{X}-g(Y, \nabla_Y X) \bar{Y}\}=e^{2\lambda}[X, Y].
\end{aligned}
$$

Using these equations, we get the following relations between the curvature tensors $\bar{R}$ and R corresponding to $\bar{\nabla}$ and ∇;

$$\bar{R}(\bar{Y},\bar{X})\bar{X}=e^{3\lambda}R(Y,X)X-\lambda'^2e^{\lambda+2\mu}Y$$

$$\bar{R}(\bar{X},\bar{Z})\bar{X}=-e^{3\mu}(\lambda''+\lambda'\mu'-\lambda'^2)\frac{\partial}{\partial t}$$

$$\bar{R}(\bar{Y},\bar{Z})\bar{X}=0$$

$$\bar{R}(\bar{X},\bar{Y})\bar{Z}=0$$

$$\bar{R}(\bar{X},\bar{Z})\bar{Z}=e^{\lambda+2\mu}(\lambda''+\lambda'\mu'-\lambda'^2)X.$$

Now, let us assume the condition

(2.6) $$\lambda''+\lambda'\mu'-\lambda^2=0,$$

then the rank of the Ricci form $\bar{S}$ of $\bar{M}$ is 2 or 0 at every point of $\bar{M}$. In fact, we can see that

$$\bar{S}(\bar{X},\bar{X})=\bar{S}(\bar{Y},\bar{Y})=\bar{g}(\bar{R}(\bar{Y},\bar{X})\bar{X},\bar{Y})$$

$$=e^{2\lambda}(e^{\lambda}g(R(Y,X)X,Y)-\lambda'^2e^{2\mu})$$

$$\bar{S}(\bar{X},\bar{Y})=\bar{S}(\bar{X},\bar{Z})=\bar{S}(\bar{Y},\bar{Z})=\bar{S}(\bar{Z},\bar{Z})=0,$$

that is,

(2.7) $$\bar{S}=\begin{pmatrix} e^{2\lambda}(Ke^{2\lambda}-\lambda'^2e^{2\mu}) & 0 & 0 \\ 0 & e^{2\lambda}(Ke^{2\lambda}-\lambda'^2e^{2\mu}) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where K is the Gaussian curvature of M.

To find out the pairs of functions λ, μ which satisfy the differential equation (2.6), we assume that μ is given. Then by Bernoulli's formula, we get $\frac{1}{\lambda'}=-e^{\mu}\int e^{-\mu}dt$. Using the last equation, we can choose the pairs

(I) $\begin{cases}\lambda=t\\ \mu=t\end{cases}$ $(t\in I=R)$, (II) $\begin{cases}\lambda=-\log t\\ \mu=0\end{cases}$ $(t\in I=R_+)$

and etc. (R_+: a positive half line)

Therefore, we see that the Riemannian manifolds $\bar{M}$ with the metric $\bar{g}$ corresponding to the pairs of functions λ, μ like (I), (II) are irreducible by (2.5) and these curvature tensors satisfy the condition (*). And moreover, they are not symmetric, because any 3-dimensional symmetric Riemannian manifold whose Ricci tensor has the rank equal to 2 or 0 is reducible.

But, as is easily seen, they are not complete. Therefore, with respect to Nomizu's conjecture, the assumption of completeness is essential.

REMARK 1. In the case (I), we assumed that $K\not\equiv 1$.

REMARK 2. For 3-dimentional Riemannian manifolds, the condition $R(X,Y)\cdot R=0$ is equivalent to the condition $R(X, Y)\cdot S=0$.

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