On C_α-Regularity of the Gradient of Solutions of Degenerate Parabolic Systems (*).

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Summary. – *We consider weak solutions* $u \in L_p([0, T], W_p^1(\Omega)) \cap L_\infty([0, T], L_2(\Omega))$ *of the degenerate parabolic (model)-system*

$$\frac{\partial u^i}{\partial t} - \operatorname{div}\left(|\nabla u|^{p-2}\, \nabla u^i\right) = 0 \quad \text{on } \Omega \subset \mathbf{R}^N, \quad 1 \leqq i \leqq m \text{ with } p > 2\,.$$

By local techniques it is proved, using sequences of time-space cylinders, which are adjusted to the alternative whether one is at a point of degeneracy or not, that the spatial gradient of u *is* α*-Höldercontinuous on compact subsets of* $\Omega \times [0, T]$ *with some* α *which depends only on* N *and* p.

1. – Introduction.

Let $\Omega \subset \mathbf{R}^N$ be an open set, $p > 2$, and consider the degenerate parabolic system

$$\frac{\partial u^i}{\partial t} - \operatorname{div}\left(|\nabla u|^{p-2} \nabla u^i\right) = 0\,, \quad 1 \leqq i \leqq m\,. \tag{1.1}$$

A vectorfunction $u = (u^1, \ldots, u^m)$ defined on $\Omega_T := (0, T] \times \Omega$ with

$$u^i \in L^\infty([0, T], L_2(\Omega)) \cap L_p([0, T], W_p^1(\Omega)) \tag{1.2}$$

is called a weak solution of (1.1), if

$$\iint_{\Omega_T} [u^i \varphi_t^i - |\nabla u|^{p-2} \nabla u^i \nabla \varphi^i]\, dx\, dt = 0 \quad \text{for all } \varphi \in C_0^1(\Omega_T)\,.$$

The aim of this paper is to give a proof, that for such solutions the gradient of u is locally α-Hölder continuous, with an exponent α, depending only on N and p (and not on u, m, or Ω_T).

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The corresponding degenerate elliptic system was first considered by UHLENBECK [10], where the similar conclusion was proved. Extensions, also for $p \in (1, 2)$, were given by TOLKSDORF [9].

The parabolic case appears to be considerably more complicated to treat. In the case of one equation, ALIKAKOS and EVANS [1] proved just continuity of ∇u; no modulus of continuity was obtained. Now in a recent article DIBENEDETTO and FRIEDMAN [5] gave a modulus $w(\varrho)$ of continuity for the gradient of solutions of (1.1) of the complicated form $w(\varrho) = (\log\log A/\varrho)^{-\sigma}$ with A and σ depending on Ω_T and some norm of u.

Inspired by their method we present the following new idea for proving $\nabla u \in C^{\alpha,\gamma\alpha}_{\mathrm{loc}}$:

One is used to consider balls of volume $c\varrho^N$ in the elliptic case and cylinders of volume $c\varrho^{N+2}$ in the parabolic case for proving (Hölder)-continuity. In the degenerate case we consider instead sequences of cylinders, whose volumes turn out to behave like $\varrho^{N+\gamma}$, and it might be, that the ratio of height over radius squared approaches infinity (at points at which the system degenerates), though the sequence of cylinders keeps nested.

It seems clear to us that this observation and the technique presented below also allows to prove Hölder-continuity of solutions of other degenerate problems, as there is e.g. the temperature in the two-phase Stefan problem [3], or of general porous medium equations [4]. We will come back to this problems elsewhere.

For more information on degenerate problems consult e.g. the references given in [2].

We start with the assumption, that for $K \subset \Omega_T$ we know already a uniform bound $\|\nabla u\|_{L_\infty(K)} \leqq M(K)$; see [5], Theorem 2.1. Furthermore by [5], Theorem 2.4, we know that

$$\iint_K |\nabla u|^{p-2} |D^2 u|^2 \, dx\, dt \leqq M_1(K)$$

the constants $M(K)$, $M_1(K)$ depending additionally on the norm of u in the spaces of (1.2). Then we prove the following

THEOREM (1). – *There is an $\alpha \in \big(0, 1/(p-1)\big)$, depending only on p and N, such that for all compact subsets $K \subset \Omega_T$ and all solutions u of* (1.1) *with $|\nabla u| \leqq M$ on K we have*

$$|\nabla u(x,t) - \nabla u(x',t')| \leqq C(K, M, p, N) \cdot \varrho^\alpha$$

for $\varrho = |x - x'| + |t - t'|^\gamma$, with $\gamma = 1/\big(2 + (2-p)\alpha\big) > \frac{1}{2}$.

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(1) After finishing this manuscript we were informed that DIBENEDETTO and FRIEDMAN have achieved a similar result (with an $\tilde{\alpha}$ depending additionally on K).

The proof is organized as follows. We will consider the differentiated system

$$(1.3)\qquad \frac{\partial}{\partial t}(u^i_{x_\beta}) - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u^i_{x_\beta} + \frac{\partial}{\partial x_\beta}(|\nabla u|^{p-2})\nabla u^i\right) = 0\,, \qquad 1 \leqq i \leqq m\,,$$

and will apply various testfunctions to it.

The typical cylinders used are defined in (2.0) below. In § 2 we present two alternatives A and B, and we will derive the theorem from them.

§ 3 contains the proof of alternative A—in fact, it is similar to a lemma, already used by the author in proving regularity results for elliptic systems (see [11], Lemma 4, p. 371 or [12]). It roughly states, that if $|\nabla u| \leqq (1-\varrho)\sup|\nabla u|$ on a fixed portion of a cylinder, then $\sup|\nabla u|$ is smaller by a fixed amount on some smaller cylinder.

§ 4 is devoted to the proof of alternative B, which is in the spirit of the italian school (see e.g. [6]), and we conclude with some remarks in § 5.

We restrict to the case $p > 2$, because there are (some more) technical difficulties in the case $p \in (2N/(N+2), 2)$. Furthermore only the model-problem is considered—it is easily possible to generalize to systems of the form

$$\frac{\partial u^i}{\partial t} - \operatorname{div}\left(A(t, x, |\nabla u|)\nabla u^i\right) = f_i$$

with certain conditions concerning A and f.

Last, the letter c will denote various constants, which may differ from line to line, but will depend only on N and p; and we use the summation convention throughout.

2. – The two alternatives and the proof of the theorem.

Note first that we may assume that $|\nabla u| < \frac{1}{2}$ on Ω_T—otherwise consider $\tilde{u}(x,t) = \alpha^{-1}u(x, \alpha^{2-p}t)$ for α large which solves the same equation on $\Omega_{\alpha^{p-2}T}$.

Let us define for $(x_0, t_0) \in \Omega_T$

$$(2.0)\qquad Q^\mu_R = Q^\mu_R(x_0, t_0) = B_R(x_0)\times[t_0 - \mu^{2-p}R^2, t_0] = B_R \times I^\mu_R$$

and consider only those $R, \mu \leqq \frac{1}{2}$, such that $Q^\mu_{2R}(x_0, t_0) \subset \Omega_T$.

We will show, that there are constants $\beta_0, \delta, \sigma, \theta, \varrho \in (0, \frac{1}{2})$, depending only on N and p, such that the following alternatives hold:

Suppose $|\nabla u| \leqq \mu$ on $Q^\mu_{2R}(x_0, t_0)$.

ALTERNATIVE A. – If meas $\{(x,t) \in Q^\mu_{R(1+\sigma)}(x_0, t_{00}) \,|\, |\nabla u|^2 < (1-\varrho)\mu^2\} \geqq \varrho|Q^\mu_{R(1+\sigma)}|$ then $|\nabla u| \leqq (1-\beta)\mu$ on $Q^{\mu(1-\beta)}_{2\delta R}(x_0, t_0)$ for all $\beta \in (0, \beta_0)$.

ALTERNATIVE B. – If meas $\{(x,t)\in Q^{\mu}_{R(1+\sigma)}(x_0,t_0)\,|\,|\nabla u|^2<(1-\varrho)\mu^2\}<\varrho|Q^{\mu}_{R(1+\sigma)}|$ then

$$\mathop{\int\!\!\!\!-\!\!\!\!\int}_{Q_{i+1}}|\nabla u(x,s)-(\nabla u)_{i+1}|^2\,dx\,ds\leqq\theta\mathop{\int\!\!\!\!-\!\!\!\!\int}_{Q_i}|\nabla u(x,s)-(\nabla u)_i|^2\,dx\,ds$$

for $i\in\boldsymbol{N}_0$ with $Q_i=Q^{\mu}_{R\delta^i}$ and

$$(\nabla u)_i=\mathop{\int\!\!\!\!-\!\!\!\!\int}_{Q_i}\nabla u(x,s)\,dx\,ds=\frac{1}{|Q_i|}\iint_{Q_i}\nabla u(x,s)\,dx\,ds\,.$$

Concerning the constants, θ can be choosen arbitrarily, while σ and ϱ are determined in Lemma 4.5 and β_0 and δ at the end of § 3.

Let us now prove the theorem, assuming that A and B are correct.

Define

$$\mu_0=\tfrac{1}{2}\,,\qquad \mu_n=\tfrac{1}{2}(1-\beta)^n\,,\qquad R_n=\delta^nR_0\,.$$

Consider first two points (x_0,t_0) and (x_1,t_0) with the same time-coordinate, $|x_1-x_0|==\varrho\leqq R_0/2$, and we will assume, that

$$Q^{\mu_0}_{2R_0}(x_i,t_0)\subset\Omega_T\qquad\text{for } i=0,1\,.$$

Then for both points there are numbers $n_i=n_i(x_i,t_0)\in\boldsymbol{N}_0\cup\{\infty\}$, $i=0,1$ such that (by alternative A)

$$|\nabla u|\leqq\mu_j\qquad\text{on } Q^{\mu_j}_{2R_j}(x_i,t_0)\qquad\text{for } j=0,\dots,n_i \tag{2.1}$$

and (by alternative B)

$$\mathop{\int\!\!\!\!-\!\!\!\!\int}_{Q_{j,i}}|\nabla u(x,s)-(\nabla u)_{j,i}|^2\,dx\,ds\leqq4\theta^j\mu^2_{n_i}\qquad\text{for } j\in\boldsymbol{N}_0 \tag{2.2}$$

with

$$Q_{j,i}=Q^{\mu_{n_i}}_{R_{n_i}\delta^j}(x_i,t_0)\,,\qquad(\nabla u)_{i,j}=\mathop{\int\!\!\!\!-\!\!\!\!\int}_{Q_{j,i}}\nabla u(x,s)\,dx\,ds\,.$$

Here $n_i=\infty$ means, that A holds for all R_k, hence (2.1) for all $j\in\boldsymbol{N}_0$.

First of all, (2.2) implies, that $(\nabla u)_{j,i}$ is a Cauchy-sequence for $i=0,1$, because

$$|(\nabla u)_{j+1,i}-(\nabla u)_{j,i}|^2\leqq2|\nabla u(x,s)-(\nabla u)_{j+1,i}|^2+2|\nabla u(x,s)-(\nabla u)_{j,i}|^2$$

and integrating over $Q_{j+1,i}$ gives

$$|(\nabla u)_{j+1,i}-(\nabla u)_{j,i}|^2\leqq4\delta^{-N-2}\theta^j\mu^2_{n_i}\,.$$

Therefore $(\nabla u)_{j,i}$ converges to $\nabla u(x_i, t_0)$ (for almost all points) and

$$|\nabla u(x_i, t_0) - (\nabla u)_{j,i}|^2 \leq c(\theta)\,\delta^{-N-2}\theta^j \mu_{n_i}^2 \quad \text{for } j \in \boldsymbol{N}_0 . \tag{2.3}$$

We consider three different cases.

Case I. – $n_1 \leq n_0 < \infty$ and $2|x_1 - x_0| = 2\varrho \leq R_{n_0} \leq R_{n_1}$.
Then there is a number $j_0 \in \boldsymbol{N}_0$ such that

$$R_{n_0+j_0+1} < 2\varrho \leq R_{n_0+j_0} \tag{2.4}$$

and we estimate

$$\begin{aligned}|\nabla u(x_0, t_0) - \nabla u(x_1, t_0)|^2 &\leq 3|\nabla u(x_0, t_0) - (\nabla u)_{j_0,0}|^2 + 3|\nabla u(x_1, t_0) - (\nabla u)_{j_1,1}|^2 + \\ &+ 3|(\nabla u)_{j_0,0} - (\nabla u)_{j_1,1}|^2 \leq c(\theta^{j_0}\mu_{n_0}^2 + \theta^{j_1}\mu_{n_1}^2) + 3|(\nabla u)_{j_0,0} - (\nabla u)_{j_1,1}|^2\end{aligned} \tag{2.5}$$

with the number $j_1 \in \boldsymbol{N}$ still to choose. The last term in (2.5) can be estimated by the usual trick:

$$|(\nabla u)_{j_0,0} - (\nabla u)_{j_1,1}|^2 \leq 2|\nabla u(x, s) - (\nabla u)_{j_0,0}|^2 + 2|\nabla u(x, s) - (\nabla u)_{j_1,1}|^2$$

and integrating over $Q_{j_0,0} \cap Q_{j_1,1}$ gives

$$|(\nabla u)_{j_0,0} - (\nabla u)_{j_1,1}|^2 \leq 8\,\frac{|Q_{j_0,0}|}{|Q_{j_0,0} \cap Q_{j_1,1}|}\,\theta^{j_0}\mu_{n_0}^2 + 8\,\frac{|Q_{j_1,1}|}{|Q_{j_0,0} \cap Q_{j_1,1}|}\,\theta^{j_1}\mu_{n_1}^2 . \tag{2.6}$$

Now we choose $j_1 \in \boldsymbol{N}$, such that the two terms on the right hand side are equal in size. We note that

$$|Q_{j_0,0}|\theta^{j_0}\mu_{n_0}^2 = c\mu_{n_0}^{4-p}(\delta^{N+2})^{j_0+n_0}R_0^{N+2}\,\theta^{j_0} = c[(1-\beta)^{4-p}\delta^{N+2}]^{n_0}[\delta^{N+2}\theta]^{j_0}R_0^{N+2} .$$

Let

$$a := (1-\beta)^{4-p}\delta^{N+2} < 1 , \qquad b := \delta^{N+2}\theta < a . \tag{2.7}$$

Then we choose j_1 such that $a^{n_0}b^{j_0} = a^{n_1}b^{j_1}$, which means

$$n_0 - n_1 = K(j_1 - j_0) \quad \text{with} \quad K = \frac{\ln b}{\ln a} > 1 . \tag{2.8}$$

In order to estimate $|Q_{j_0,0} \cap Q_{j_1,1}|$, we note that

$$\big((1-\beta)^{2-p}\big)^{n_0}(\delta^2)^{n_0+j_0} \leq \big((1-\beta)^{2-p}\big)^{n_1}(\delta^2)^{n_1+j_1}$$

iff

$$(2.9) \qquad (1-\beta)^{2-p}(\delta^2)^{(K-1)/K} \leqq 1$$

and

$$R_{n_0+j_0} \leqq R_{n_1+j_1},$$

because

$$n_0 + j_0 \geqq n_1 + j_1$$

by (2.8).

Hence by (2.4)

$$|Q_{j_0,0} \cap Q_{j_1,1}| \geqq c(\delta^{N+2})^{n_0+j_0}\big((1-\beta)^{2-p}\big)^{n_0} R_0^{n+2}$$

Collecting all estimates gives

$$|\nabla u(x_0, t_0) - \nabla u(x_1, t_0)|^2 \leqq c\big((1-\beta)^{2n_0}\theta^{j_0} + (1-\beta)^{2n_1}\theta^{j_1}\big).$$

As $(1-\beta)^{2n_1}\theta^{j_1} \leqq (1-\beta)^{2n_0}\theta^{j_0}$, if

$$(2.10) \qquad (1-\beta)^2\theta^{-1/K} \geqq 1$$

we finally get

$$(2.11) \qquad |\nabla u(x_0, t_0) - \nabla u(x_1, t_0)| \leqq c(1-\beta)^{n_0}\theta^{j_0/2} \leqq c(1-\beta)^{n_0+j_0} \leqq c\left(\frac{\varrho}{R_0}\right)^\alpha$$

with

$$\alpha = \frac{\ln(1-\beta)}{\ln \delta} < 1 .$$

Case II. – $2\varrho > R_{n_0}$, $n_1 \leqq n_0 < \infty$.

Then $R_{k+1} < 2\varrho \leq R_k$ for some $k \in \{0, \ldots, n_0 - 1\}$, $(x_1, t_0) \in Q^{\mu_k}_{2R_k}(x_0, t_0)$, and therefore

$$|\nabla u(x_0, t_0) - \nabla u(x_1, t_0)| \leqq 2 \sup_{\bar{Q}^{\mu_k}_{2R_k}(x_0, t_0)} |\nabla u| \leqq 2\mu_k = c(1-\beta)^k \leqq c\left(\frac{\varrho}{R_0}\right)^\alpha .$$

Case III. – If $n_0 = \infty$, then $|\nabla u| \leqq \mu_n$ on $Q^{\mu_n}_{R_n}(x_0, t_0)$, $n \in \boldsymbol{N}$. Choose $\tilde{n}_0 \in \boldsymbol{N}$ with $R_{\tilde{n}_0+1} < 2\varrho \leqq R_{\tilde{n}_0}$ and the same reasoning as in case II yields the result.

Next consider two points of the form (x_0, t_0), (x_0, t_1) and let $\varrho^2 = |t_0 - t_1|$. Then case I is given, if

$$2\varrho^2 \leqq \tfrac{1}{2}\big((1-\beta)^{2-p}\delta^2\big)^{n_0} R_0^2$$

and j_0 is determined by

$$\delta^2 < 4\varrho^2 R_0^{-2}\big((1-\beta)^{p-2}\delta^{-2}\big)^{n_0}\delta^{-2j_0} \leqq 1 .$$

The same reasoning as above leads to

$$|\nabla u(x_0, t_0) - \nabla u(x_0, t_1)| \leqq c(1-\beta)^{n_0}\theta^{j_0/2} .$$

Take $\alpha_1 < 1$ with $(1-\beta) = \big((1-\beta)^{2-p}\delta^2\big)^{\alpha_1}$, which is possible, if

$$(1-\beta)^{p-1} > \delta^2 . \tag{2.12}$$

We may also assume, that $\theta^{\frac{1}{2}} \leqq \delta^{2\alpha_1}$, from which we get

$$|\nabla u(x_0, t_0) - \nabla u(x_0, t_1)| \leqq c\left(\frac{|t_0 - t_1|}{R_0^2}\right)^{\alpha_1} \tag{2.13}$$

and the same estimate holds in the corresponding cases II and III. Combining (2.11) and (2.13), we see that

$$|\nabla u(x, t) - \nabla u(x', t')| \leqq c(K)\varrho^\alpha$$

for $\varrho = |x - x'| + |t - t'|^\gamma$ with

$$\gamma = \frac{1}{2 + (2-p)\alpha} \qquad \text{and} \qquad \alpha = \frac{|\ln(1-\beta)|}{|\ln \delta|}$$

depending only on N, p, on compact subsets $K \subset (0, T) \times \Omega$. $c(K)$ depends also on $\sup_K |\nabla u|$.

It remains to check the conditions (2.7), (2.9), (2.10), (2.12), which are easily seen to be fulfilled for β small enough.

3. - Proof of alternative A.

A similar statement in the context of elliptic systems was given by us in [11], Lemma 4, p. 371—in fact, the method relies on an old device by Moser. Let $\varepsilon < \varrho$ and

$$V = -\ln\big(1 - |\nabla u|^2\mu^{-2} + \varepsilon\big) + \ln \varrho . \tag{3.1}$$

With $R_0 = (1+\sigma)R \leqq \frac{3}{2}R$ we have by assumption

$$|\{V_+ = 0\} \cap Q^\mu_{R_0}| \geqq \varrho |Q^\mu_{R_0}| \tag{3.2}$$

and the trivial estimate $V_+ \leqq \ln(\varrho/\varepsilon)$ on Q^μ_{2R}. The aim is to give a better estimate on some smaller cylinder $Q^\mu_{R_1}$, which will be achieved in three steps.

STEP 1. – We have

$$\iint\limits_{Q^\mu_{\frac{3}{2}R}} |\nabla V_+|^2\, dx\, ds \leqq c\mu^{2-p} R^N \left(1 + \ln\frac{\varrho}{\varepsilon}\right). \tag{3.3}$$

PROOF OF (3.3). – Take $2\mu^{-2}u^i_{x_\beta}(1 - |\nabla u|^2\mu^{-2} + \varepsilon)^{-1}\xi$ with $\xi \geqq 0$ as a test-function in (1.3). As $V_{x_\alpha} = \mu^{-2}(|\nabla u|^2)_{x_\alpha}(1 - |\nabla u|^2\mu^{-2} + \varepsilon)^{-1}$, we get

$$\iint |\nabla u|^{p-2}|\nabla V|^2 \xi\, dx\, ds + \iint V_t \xi\, dx\, ds + \iint |\nabla u|^{p-2}\, V_{x_\alpha}\xi_{x_\alpha}\, dx\, ds + \\ + (p-2)\iint |\nabla u|^{p-4} u^i_{x_\alpha} u^i_{x_\beta} V_{x_\beta}\xi_{x_\alpha}\, dx\, ds \leqq 0\,. \tag{3.4}$$

Let $\eta(x, t)$ denote the usual cutoff function on Q^μ_{2R} with respect to $Q^\mu_{\frac{3}{2}R}$ and insert $\xi = \eta^2$ in (3.4). Then by standard estimations

$$\iint\limits_{Q^\mu_{\frac{3}{2}R}} |\nabla u|^{p-2}|\nabla V|^2\, dx\, ds \leqq c \iint\limits_{Q^\mu_{2R}} |\nabla u|^{p-2}|\nabla\eta|^2\, dx\, ds + 2\iint\limits_{Q^\mu_{2R}} |V||\eta_t|\, dx\, ds \\ + \int\limits_{B_{2R}} |V(x, t_0)|\, dx \leqq cR^N\left(1 + \ln\left(\frac{\varrho}{\varepsilon}\right)\right).$$

As $|\nabla u|^2 \geqq (1-\varrho)\mu^2$ on $\{V_+ > 0\}$, we get (3.3).

STEP 2. – There are $\gamma < 1$, $\delta > 0$ (with $(1+\delta)(1+\sigma) < \frac{3}{2}$), such that either

$$\iint\limits_{Q^{\tilde\mu}_{R_0(1+\delta)}} (V - \gamma M)^2_+\, dx\, ds \leqq cR_0^2 \iint\limits_{Q^{\tilde\mu}_{R_0(1+\delta)}} |\nabla V_+|^2\, dx\, ds \tag{3.5}$$

for some $\tilde\mu = c(\varrho)\mu > \mu$ or

(3.7) below is not fulfilled.

PROOF OF (3.5). – Let $A_{k,\varrho}(t) := \{x \in B_\varrho | V(x, t) > k\}$. First of all, there is a $t_1 \in$ $\in I := [t_0 - R_0^2\mu^{2-p}, t_0 - (\varrho/2)R_0^2\mu^{2-p}]$, such that

$$|A_{0,R_0}(t_1)| \leqq \left(1 - \frac{\varrho}{2}\right)|B_{R_0}| \tag{3.6}$$

because otherwise $|\{V_+ > 0\} \cap Q^\mu_{R_0}| \geqq \int\limits_I |A_{0,R_0}(s)|\, ds > (1-\varrho)|Q^\mu_{R_0}|$ contradicting (3.2).

Now let $\varphi(x)$ denote the cutoff function on $B_{R_0(1+\delta)}$ with respect to B_{R_0} and insert $\xi = 2V_+(x,t)\varphi^2(x)\chi_{[t_1,t_2]}(t)$ in (3.4) for any $t_2 \in [t_1, t_0]$. Then

$$\int_{B_{R_0}} V_+(x,t_2)^2\,dx \leqq \int_{B_{R_0(1+\delta)}} V_+(x,t_1)^2\,dx + c \iint_{B_{R_0(1+\delta)}\times[t_1,t_2]} V_+(x,s)|\nabla u|^{p-2}|\nabla\varphi|^2\,dx\,ds \leqq$$
$$\leqq \left(1-\frac{\varrho}{4}\right)|B_{R_0}|\ln^2\left(\frac{\varrho}{\varepsilon}\right) + c(t_0-t_1)\mu^{p-2}(\delta R_0)^{-2}|B_{R_0(1+\delta)}|\ln\left(\frac{\varrho}{\varepsilon}\right).$$

Hence, if $\gamma < 1$ and $2^{N-1}\delta < \varrho/8$

$$(\gamma M)^2|A_{\gamma M,R_0}(t_2)| \leqq |B_{R_0(1+\delta)}|\left(\left(1-\frac{\varrho}{4}\right)\cdot\frac{\ln^2 \varrho/\varepsilon}{(1+\delta)^N} + c\delta^{-2}\ln\left(\frac{\varrho}{\varepsilon}\right)\right).$$

We will assume

$$\ln^2\left(\frac{\varrho}{\varepsilon}\right) \leqq (1+\delta)^N M^2 \quad \text{and} \quad c\delta^{-2}\ln\left(\frac{\varrho}{\varepsilon}\right) \leqq \frac{\varrho}{8}M^2. \tag{3.7}$$

Then

$$|A_{\gamma M,R_0(1+\delta)}(t_2)| \leqq |B_{R_0(1+\delta)}\setminus B_{R_0}| + |A_{\gamma M,R_0}t_2)| \leqq$$
$$\leqq |B_{R_0(1+\delta)}|\left(N\delta + \gamma^{-2}\left(1-\frac{\varrho}{8}\right)\right) \leqq \left(1-\frac{\varrho}{9}\right)|B_{R_0(1+\delta)}|$$

if δ small and γ close to 1 enough, depending on ϱ. By [7], p. 56, we infer

$$\int_{B_{R_0(1+\delta)}} (V(x,t_2)-\gamma M)_+^2\,dx \leqq cR_0^2\int_{B_{R_0(1+\delta)}} |\nabla V_+|^2\,dx.$$

Integrating over $t_2 \in [t_0 - ((\varrho/2)R_0^2)\mu^{2-p}, t_0] = [t_0 - (R_0(1+\delta))^2\tilde\mu^{2-p}, t_0]$ we get

$$\iint_{Q^{\tilde\mu}_{R_0(1+\delta)}} (V-\gamma M)_+^2\,dx\,ds \leqq cR_0^2\iint_{Q_{R_0(1+\delta)}} |\nabla V_+|^2\,dx\,ds$$

with $\tilde\mu^{2-p}(1+\delta)^2 = \mu^{2-p}(\varrho/2)$, hence (3.5).

STEP 3. – Consists of the following lemma, which is a variant of [8], Theorem 6.2, p. 105. As we must be careful about the dependence on μ, we present a proof below.

LEMMA. – Let $\varrho_0, \sigma_0 < \frac{1}{2}$ and $M := \sup_{Q^\mu_{(1-\sigma_0)\varrho_0}} V_+ > 0$. Suppose there is a $\gamma < 1$, such that for all $\varrho \leqq \varrho_0$, $\sigma \leqq \sigma_0$, $k \geqq \gamma M$

$$\mu^{2-p}\sup_{I^\mu_{(1-\sigma)\varrho}}\left\{\int_{B_{(1-\sigma)\varrho}} (V-k)_+^2\,dx\right\} + \iint_{Q^\mu_{(1-\sigma)\varrho}} |\nabla(V-k)_+|^2\,dx\,dt \leqq c_1(\sigma\varrho)^{-2}\iint_{Q^\mu_\varrho} (V-k)_+^2\,dx\,dt. \tag{3.8}$$

Then

$$M^2 \leqq c(c_1, \sigma_0, \gamma) \cdot \iint_{Q^\mu_{\varrho_0}} (V - \gamma M)^2_+ \, dx\, dt \,. \tag{3.9}$$

PROOF. – Let $\tau := \frac{1}{2}(1 + 1/\gamma) > 1$, $k_0 \geqq \gamma M$ and define for $s \in \mathbf{N}_0$

$$k_s := \tau k_0 - k_0 \tau^{-s}(\tau - 1) \nearrow \tau k_0$$
$$\varrho_s := \left((1-\sigma_0) + \sigma_0 \cdot 2^{-s}\right)\varrho_0 \searrow (1-\sigma_0)\varrho_0\,, \qquad \varrho_s^* := \tfrac{1}{2}(\varrho_s + \varrho_{s+1}) > \varrho_{s+1}$$

and let $\varphi_s(x)$ denote a cutoff-function, which equals 1 on $B_{\varrho_{s+1}}$ and 0 outside $B_{\varrho_s^*}$.

Set $I_s := \iint_{Q^\mu_{\varrho_s}} (V - k_s)^2_+ \, dx\, dt$.

Using Sobolev's inequality and (3.8), we get:

$$\begin{aligned} I_{s+1} &\leqq \iint_{Q^\mu_{\varrho_s^*}} (V - k_{s+1})^2_+ \varphi_s^2(x)\, dx\, dt \\ &\leqq c \int_{I^\mu_{\varrho_s^*}} |A_{k_{s+1},\varrho_s^*}(t)|^{2/N} \cdot \left(\int_{B_{\varrho_s^*}} |\nabla (V - k_{s+1})_+|^2 \varphi_s^2 + (V - k_{s+1})^2_+ |\nabla \varphi_s|^2 \, dx \right) dt \\ &\leqq c \cdot c_1 \cdot \sup_{t \in I^\mu_{\varrho_s^*}} |A_{k_{s+1},\varrho_s}(t)|^{2/N} (\varrho_s - \varrho_{s+1})^{-2} I_s \,. \end{aligned}$$

On the other hand

$$c_1 I_s (\varrho_s - \varrho_{s+1})^{-2} \geqq c\mu^{2-p} \sup_{I^\mu_{\varrho_s^*}} \left\{ \int_{B_{\varrho_s^*}} (V - k_s)^2_+ \, dx \right\} \geqq c\mu^{2-p}(k_{s+1} - k_s)^2 \sup_{I^\mu_{\varrho_s^*}} |A_{k_{s+1},\varrho_s^*}(t)|$$

which combines to the recursion relation

$$I_{s+1} \leqq c\left((\varrho_s - \varrho_{s+1})^{-2} I_s\right)^{1+2/N} [\mu^{(p-2)}(k_{s+1} - k_s)^{-2}]^{2/N} \leqq c[\mu^{p-2} k_0^{-2} \varrho_0^{-(N+2)}]^{2/N} b^s \cdot I_s^{1+2/N} \tag{3.10}$$

with c depending only on c_1, τ, σ_0, N and $b = 4^{1+2/N}\tau^{4/N} > 1$.

Now if $0 \leqq I_{s+1} \leqq A b^s I_s^{1+2/N}$ with $b > 1$, then $I_s \to 0$, if $I_0 \leqq A^{-N/2} b^{-N^2/4}$ (see [8] Lemma 5.6, p. 95).

As $I_0 \leqq \iint_{Q^\mu_{\varrho_0}} (V - \gamma M)^2_+ \, dx\, dt$, it suffices, that

$$\iint_{Q^\mu_{\varrho_0}} (V - \gamma M)^2_+ \, dx\, dt \leqq c\mu^{2-p} k_0^2 \varrho_0^{N+2} \,.$$

Hence let $k_0 = \max \{\gamma M, c(\iint_{Q^\mu_{\varrho_0}} (V - \gamma M)^2_+ \, dx\, dt)^{\frac{1}{2}}\}$ and we conclude from $I_s \to 0$, that

$$M \leqq \tau k_0 \,.$$

As $\tau\gamma < 1$, the lemma is proved.

Now inserting $\xi = 2(V-k)_+\eta^2$ in (3.4) with appropriate cutoff-functions, we see that (3.8) is valid indeed, where we use again, that $|\nabla u|^2 > (1-\varrho)\mu^2$ on $\{V>0\}$.

But combining (3.3) and (3.5), we see that either

$$\iint\limits_{Q^{\tilde\mu}_{R_0(1+\delta)}} (V-\gamma M)_+^2\,dx\,ds \leq c\ln\left(\frac{\varrho}{\varepsilon}\right) \qquad \text{for } M = \sup_{Q^{\tilde\mu}_{R_0}} V_+\,,$$

hence

$$M^2 \leq c\ln\left(\frac{\varrho}{\varepsilon}\right) \tag{3.11a}$$

by the lemma, or (3.7) is not correct which amounts to

$$M \leq (1+\delta)^{-N/2}\ln\left(\frac{\varrho}{\varepsilon}\right). \tag{3.11b}$$

Choose ε small enough, such that $c \leq (1+\delta)^{-N}\ln(\varrho/\varepsilon)$, which reduces (3.11$a$) to (3.11$b$) and by elementary calculations we get

$$|\nabla u| \leq (1-\beta)\mu \qquad \text{on } Q^{\tilde\mu}_R$$

with β just depending on ϱ. Remembering, that $\tilde\mu = c(\varrho)\mu$, we see that also

$$|\nabla u| \leq (1-\beta)\mu \qquad \text{on } Q^{\mu(1-\beta)}_{2\delta R}$$

where $\delta \leq \delta_0(\varrho)$, $\beta \leq \beta_0(\varrho)$.

4. – Proof of alternative B.

This proof is essentially done along the lines of DIBENEDETTO and FRIEDMAN [5], § 4, § 5. We just have to be careful about the dependence on μ and the different cylinders used. We suppose again $|\nabla u| \leq \mu < \frac{1}{2}$ on Q^{μ}_{2R} and start with

LEMMA 4.1. – Let $W = (W^i_\alpha)$ be any vector in $\boldsymbol{R}^{N\times m}$. Then

$$\sup_{I^{\mu}_{R/2}} \int\limits_{B_{R/2}} |\nabla u - W|^2\,dx + \iint\limits_{Q^{\mu}_{R/2}} |\nabla u|^{p-2}|D^2 u|^2\,dx\,ds \leq \\ \leq cR^{-2}\iint\limits_{Q^{\mu}_{R}} (|\nabla u|^{p-2} + \mu^{p-2})|\nabla u - W|^2\,dx\,ds\,. \tag{4.1}$$

PROOF. – Use the test-function $(u^i_{x_\beta} - W^i_\beta)\eta^2$ in (1.3), with η an appropriate cutoff-function. Noting that $|\eta_t| \leq 2\mu^{p-2}R^{-2}$, the lemma follows by standard estimations.

The idea of the following is to use the locally linearized system with constant coefficients with the hope, that the good estimates for its solutions can be used to derive at least not so bad estimates for the function u in question. This technique was employed with great success by the italian school—we refer here to the book of GIAQUINTA [6].

For the following lemma in the elliptic version compare e.g. p. 78.

LEMMA 4.2. – Let $V = (V^i_\alpha) \in \mathbf{R}^{m\times N}$ with $\mu/2 \leqq |V| \leqq 2\mu$ and let v be the unique solution of the linear parabolic system (cf. [8], p. 573)

$$v^i_t - \operatorname{div}\left(|V|^{p-2}\nabla v^i + (p-2)|V|^{p-4}V^j_\beta v^j_{x_\beta} V^i\right) = 0 \quad \text{in } Q^\mu_{R/2}, \tag{4.2}$$

$v - u = 0$ on the parabolic boundary of $Q^\mu_{R/2}$.

Then we have for all $\varrho < R/4$, $W \in \mathbf{R}^{m\times N}$

$$\iint\limits_{Q^\mu_\varrho} |\nabla v - (\nabla v)_\varrho|^2\, dx\, ds \leqq c\left(\frac{\varrho}{R}\right)^{N+4} \iint\limits_{Q^\mu_{R/2}} |\nabla v - W|^2\, dx\, ds \tag{4.3}$$

with some c, depending only on p and N.

PROOF. – Let $C_\varrho = B_\varrho(0)\times[-\varrho^2, 0]$ and consider $w(x, s) = v(x_0 + (R/2)x, t_0 + \frac{1}{4}\mu^{2-p}R^2 s)$ solving

$$w^i_s - (A^{ij}_{\alpha\beta} w^j_{x_\beta})_{x_\alpha} = 0 \quad \text{on } C_1 \tag{4.4}$$

with the constant matrix

$$A^{ji}_{\alpha\beta} = |\hat{V}|^{p-2}\delta_{ij}\delta_{\alpha\beta} + (p-2)|\hat{V}|^{p-4}\hat{V}^i_\beta \hat{V}^j_\alpha$$

where we set $\hat{V} = V\mu^{-1}$, $\frac{1}{2} \leqq |\hat{V}| \leqq 2$.

Note that

$$c_-(p)|\eta|^2 \leqq A^{ij}_{\alpha\beta}\eta^i_\alpha \eta^j_\beta \leqq c_+(p)|\eta|^2 \quad \text{for } \eta \in \mathbf{R}^{m\times N}$$

with $0 < c_-(p) \leqq c_+(p)$ independent of μ and V. It is clear, that w is C^∞ inside. Now differentiate (4.4) and test with

$$(\nabla w - \tilde{W})\eta^2$$

which gives

$$\iint\limits_{C_{\frac{1}{2}}} |D^2 w|^2\, dx\, ds \leqq c(p) \iint\limits_{C_1} |\nabla w - \tilde{W}|^2\, dx\, ds\,.$$

Repeating this procedure (now with $\tilde{W} = 0$) for the higher order derivatives gives for all $k \in \boldsymbol{N}$

$$\iint_{C_{\frac{1}{2}}} \sum_{j=0}^{k} \left| \frac{\partial^j}{\partial s^j} D^k w \right|^2 dx\, ds \leq c(p, k) \iint_{C_1} |\nabla w - \tilde{W}|^2\, dx\, ds$$

where $D^k w$ denotes the vector of all spatial derivatives of order k. Hence by Sobolev

$$\max_{C_{\frac{1}{2}}} |D^2 w|^2 + |D^3 w|^2 \leq c(p, N) \iint_{C_1} |\nabla w - \tilde{W}|^2\, dx\, ds\,.$$

Now let $f(s) = \fint_{B_r} \nabla w(x, s)\, dx$. Then $\fint_{[-r^2, 0]} f(s)\, ds = (\nabla w)_r = \fint\!\!\fint_{C_r} \nabla w(x, s)\, dx\, ds$ and

$$\int_{-r^2}^{0} |f(s) - (\nabla w)_r|^2\, ds \leq c r^4 \int_{-r^2}^{0} |f'(s)|^2\, ds \leq c r^6 \sup_s |f'(s)|^2 \leq c r^6 \sup_{C_r} |\nabla w_t|^2 \leq c r^6 \sup_{C_r} |D^3 w|^2$$

with the last inequality following directly from (4.4). Hence

$$\begin{aligned}\iint_{C_r} |\nabla w(x, s) - (\nabla w)_r|^2\, dx\, ds &\leq 2 \int_{-r^2}^{0} \int_{B_r} |\nabla w(x, s) - f(s)|^2\, dx\, ds + 2 C r^N \int_{-r^2}^{0} |f(s) - (\nabla w)_r|^2\, ds \leq \\ &\leq c \int_{-r^2}^{0} r^2 \int_{B_r} |D^2 w|^2\, dx\, ds + C r^{N+6} \sup_{C_r} |D^3 w|^2 \leq C r^{N+4} \sup_{C_{\frac{1}{2}}} \{|D^2 w|^2 + |D^3 w|^2\} \leq \\ &\leq C r^{N+4} \iint_{C_1} |\nabla w(x, s) - \tilde{W}|^2\, dx\, ds \qquad \text{for } r < \frac{1}{2}\,.\end{aligned}$$

Let $\tilde{W} = (R/2) W$ and transform back to get (4.3)—note, that $\fint\!\!\fint_{C_r} \nabla w(x. s)\, dx\, ds = R/2 \fint\!\!\fint_{Q^\mu_{rR/2}} \nabla v(x, s)\, dx\, ds$.

LEMMA 4.3. – Let v be given by (4.2). Then for $\varrho < R/2$

$$\iint_{Q^\mu_\varrho} |\nabla u - \nabla v|^2\, dx\, dt \leq c \Big(\mu^{-2} \iint_{Q^\mu_R} |\nabla u - V|^2\, dx\, dt\Big)^{2/N^*} \iint_{Q^\mu_R} |\nabla u - V|^2\, dx\, dt$$

with $N^* = N$ if $N > 2$ and $N^* > 2$ if $N = 2$.

PROOF. – Subtracting (4.2) from (1.1) gives

$$(4.5) \qquad (u^i - v^i)_t - \operatorname{div} \big(|V|^{p-2} (\nabla u^i - \nabla v^i) + (p-2) |V|^{p-4} V^j_\beta (u^j_{x_\beta} - v^j_{x_\beta}) V^i\big) = \operatorname{div} \boldsymbol{H}^i$$

and by [5], (4.8), p. 103, we have

$$|\boldsymbol{H}| \leq c \mu^{-1} (|\nabla u| + |V|)^{p-2} |\nabla u - V|^2\,.$$

Multiplying (4.5) by $(u^i - v^i)$, summing over i and integrating over $Q^\mu_{R/2}$ gives

$$|V|^{p-2}\iint\limits_{Q^\mu_{R/2}} |\nabla u - \nabla v|^2\,dx\,dt \leq \int\limits_{Q^\mu_{R/2}} |H||\nabla u - \nabla v|\,dx\,dt$$

hence

$$\iint\limits_{Q^\mu_{R/2}} |\nabla u - \nabla v|^2\,dx\,dt \leq c\mu^{2-2p}\iint\limits_{Q^\mu_{R/2}} (|\nabla u| + |V|)^{2p-4}|\nabla u - V|^4\,dx\,dt\,. \tag{4.6}$$

Let $r = R/2$ and consider

$$I = \int\limits_{B_r} (|\nabla u| + |V|)^{2p-4}|\nabla u - V|^4\,dx\,.$$

Splitting the integrand in

$$\{(|\nabla u| + |V|)^{(p-2)(1-1/q)}|\nabla u - V|^{2(1-1/q)}\}\times\{(|\nabla u| + |V|)^{(p-2)/q+(p-2)}|\nabla u - V|^{2/q+2}\}$$

and using Hölder's inequality gives for $q > 1$:

$$I \leq c\mu^{(p-2)(1-1/q)}\Big(\int\limits_{B_r} |\nabla u - V|^2\Big)^{1-1/q}\cdot\Big(\int\limits_{B_r} (|\nabla u| + |V|)^{(p-2)(1+q)}|\nabla u - V|^{2+2q}\Big)^{1/q} \leq$$

$$\leq c\mu^{p-2(1-1/q)}\Big(\int\limits_{B_r} |\nabla u - V|^2\Big)^{(1-1)/q}\Big(\int\limits_{B_r} \{(|\nabla u| + |V|)^{(p-2)/2}|\nabla u - V|\}^{2^q}\Big)^{1/q}.$$

Now we need the elementary inequality

$$\big||U|^\alpha U - |V|^\alpha V\big| \geq c(|U| + |V|)^\alpha|U - V| \quad \text{for } \alpha > 0\,, \tag{4.7}$$

which is proved in [5], p. 103, and we get

$$I \leq c\mu^{p-2(1-1/q)}\Big(\int\limits_{B_r} |\nabla u - V|^2\Big)^{1-1/q}\Big(\int\limits_{B_r} \big||\nabla u|^{(p-2)/2}\nabla u - |V|^{(p-2)/2}V\big|^{2q}\Big)^{1/q}. \tag{4.8}$$

For abbreviation let $X = |\nabla u|^{(p-2)/2}\nabla u - |V|^{(p-2)/2}V$. Estimate the first factor of (4.8) with the help of (4.1) and take $q = s/(s-2)$ (hence $1 - 1/q = 2/s$). Recall (4.6), this gives

$$\iint\limits_{Q^\mu_r} |\nabla u - \nabla v|^2\,dx\,dt \leq cR^{2N/s}\mu^{2-p}\Big(\mu^{-2}\mathop{\int\!\!\!\!-\!\!\!\!\int\!\!\!\!-}\limits_{Q^\mu_R} |\nabla u - V|^2\,dx\,dt\Big)^{2/s}\cdot$$

$$\cdot\int\limits_{I^\mu_r}\Big(\int\limits_{B_r} |X|^{2s/(s-2)}\,dx\Big)^{(s-2)/s}dt\,. \tag{4.9}$$

Now the last factor is estimated by Sobolev's inequality

$$\|X\|_{2s/(s-2)} \leqq c\|X\|_2 R^{-N/s} + c\|\nabla X\|_2 R^{1-N/s}$$

for $s = N^*$ on balls $B_{R/2}$.

This implies, using $|X|^2 \leqq c|\nabla u - V|^2 \mu^{p-2}$

$$\int\limits_{I^\mu_r}\Big(\int\limits_{B_r}|X|^{2s/(s-2)}\,dx\Big)^{(s-2)/s} dt \leqq cR^{-2N/s}\mu^{p-2}\iint\limits_{Q^\mu_r}|\nabla u - V|^2\,dx\,dt + $$
$$+ cR^{2(1-N/s)}\iint\limits_{Q^\mu_r}|\nabla u|^{p-2}|D^2u|^2\,dx\,dt \leqq cR^{-2N/s}\mu^{p-2}\iint\limits_{Q^\mu_R}|\nabla u - V|^2\,dx\,dt$$

by (4.1), which combined with (4.9) proves the lemma.

Next we give a condition, which implies the conclusion of Alternative B (compare [5], Lemma 4.4, 4.5). Keep in mind, that $|\nabla u| \leqq \mu$ on Q^μ_{2R}.

LEMMA 4.4. – Let $V_0 \in \boldsymbol{R}^{N\times m}$, $\frac{3}{4}\mu \leqq |V_0| \leqq \frac{3}{2}\mu$ and $\Theta \in (0,1)$. Then there are ε, δ, positive, small and depending only on Θ, N, and p, such that

$$\text{(4.10)}\qquad ⨍\!\!⨍_{Q^\mu_R}|\nabla u - V_0|^2\,dx\,dt \leqq \varepsilon\mu^2$$

implies: For $i \in \boldsymbol{N}_0$ there holds for $Q_i = Q^\mu_{R\delta^i}$

(i) $⨍\!\!⨍_{Q_{i+1}}|\nabla u - (\nabla u)_{i+1}|^2\,dx\,dt \leqq \Theta ⨍\!\!⨍_{Q_i}|\nabla u - (\nabla u)_i|^2\,dx\,dt$;

(ii) $⨍\!\!⨍_{Q_i}|\nabla u - (\nabla u)_i|^2\,dx\,dt \leqq \varepsilon\mu^2$;

(iii) $|V_0 - (\nabla u)_i| \leqq \dfrac{\mu}{4}$

with $(\nabla u)_i = ⨍\!\!⨍_{Q_i}\nabla u\,dx\,dt$.

REMARK. – We can find a $\delta > 0$ of this kind, which is smaller than a given number, say $\delta_0(\varrho)$ from alternative A.

PROOF. – We note first the triviality that $\iint\limits_{Q_i}|\nabla u - (\nabla u)_i|^2\,dx\,dt \leqq \iint\limits_{Q_i}|\nabla u - W|^2\,dx\,dt$ for all constant vectors $W \in \boldsymbol{R}^{N\times m}$. Then the remark is easily seen to be true, because whenever we have found a δ_1 such that (i)-(iii) holds, choose $k \in \boldsymbol{N}$ with $\delta = \delta_1^k \leqq \delta_0(\varrho)$ and consider those Q_i with $i \equiv 0 \bmod k$. Then the lemma is true with this δ.

We start with $V = V_0$ and use only $\mu/2 \leqq |V| \leqq 2\mu$ so that in view of (iii), we can proceed inductively. By assumption and Lemma 4.3 we get

$$\iint\limits_{Q^\mu_{R/2}} |\nabla u - V_0|^2\, dx\, dt \leqq c \iint\limits_{Q^\mu_R} |\nabla u - V_0|^2\, dx\, dt$$

which implies by (4.3) (if $\delta < \frac{1}{4}$)

$$\iint\limits_{Q^\mu_{\delta R}} |\nabla v - (\nabla v)_{\delta R}|^2\, dx\, dt \leqq c\delta^{N+4} \iint\limits_{Q^\mu_R} |\nabla u - V_0|^2\, dx\, dt\,. \tag{4.11}$$

Now by lemma 4.3 and (4.11)

$$\iint\limits_{Q^\mu_{\delta R}} |\nabla u - (\nabla u)_{\delta R}|^2\, dx\, dt \leqq \iint\limits_{Q^\mu_{\delta R}} |\nabla u - (\nabla v)_{\delta R}|^2\, dx\, dt \leqq 2 \iint\limits_{Q^\mu_{\delta R}} |\nabla u - \nabla v|^2\, dx\, dt + \\ + 2 \iint\limits_{Q^\mu_{\delta R}} |\nabla v - (\nabla v)_{\delta R}|^2\, dx\, dt \leqq c(\varepsilon^{2/N^*} + \delta^{N+4}) \iint\limits_{Q^\mu_R} |\nabla u - V_0|^2\, dx\, dt$$

and we see that

$$\fint\!\!\fint\limits_{Q^\mu_{\delta R}} |\nabla u - (\nabla u)_{\delta R}|^2\, dx\, dt \leqq c(\delta^2 + \varepsilon^{2/N^*}\delta^{-(N+2)}) \fint\!\!\fint\limits_{Q^\mu_R} |\nabla u - V_0|^2\, dx\, dt \\ \leqq \Theta \fint\!\!\fint\limits_{Q^\mu_R} |\nabla u - V_0|^2\, dx\, dt$$

if we choose first δ and then ε, depending on Θ.

We assume additionally that

$$\varepsilon\delta^{-(N+2)} \leqq \frac{1}{16}(1 - \Theta^{\frac{1}{2}})^2\,. \tag{4.12}$$

It remains to check the third assertion

$$|(\nabla u)_{\delta R} - V_0|^2 = \left| \fint\!\!\fint\limits_{Q^\mu_{\delta R}} (\nabla u - V_0)\, dx\, dt \right|^2 \leqq \fint\!\!\fint\limits_{Q^\mu_{\delta R}} |\nabla u - V_0|^2\, dx\, dt \leqq \\ \leqq \delta^{-(N+2)} \fint\!\!\fint\limits_{Q^\mu_R} |\nabla u - V_0|^2\, dx\, dt \leqq \varepsilon\delta^{-(N+2)}\mu^2 \leqq \frac{1}{16}\mu^2\,.$$

We can continue this process, now with $V = (\nabla u)_{\delta R}$, if we show that (iii) remains true. But as above, we have

$$|(\nabla u)_{\delta^{i+1}R} - (\nabla u)_{\delta^i R}|^2 \leqq \Theta^i \delta^{-(N+2)} \varepsilon\mu^2 \leqq \frac{1}{16}\Theta^i \mu^2 (1 - \Theta^{\frac{1}{2}})^2$$

by (4.12), hence

$$|(\nabla u)_{\delta^{i+1}R}-V_0|\leqq\sum_{j=0}^{\infty}\Theta^{j/2}(1-\Theta^{\frac{1}{2}})\frac{\mu}{4}\leqq\frac{\mu}{4}.$$

So all what is left to prove alternative B is the following lemma, differing only from [5], Lemma 5.1 by the use of the cylinder Q_R^μ instead of Q_R^1. In order to keep this paper selfcontained, we give the full proof.

LEMMA 4.5. – There are σ, $\varrho\in(0,\frac{1}{2})$, such that

$$\operatorname{meas}\left\{(x,t)\subset Q_{R(1+\sigma)}(x_0,t_0)\big||\nabla u|^2<(1-\varrho)\mu^2\right\}<\varrho|Q^\mu_{R(1+\sigma)}|$$

implies: There is a $V_0\in\boldsymbol{R}^{N\times m}$, $\frac{3}{4}\mu\leqq|V_0|\leqq\frac{3}{2}\mu$, such that

$$\fint\!\!\fint_{Q_R^\mu}|\nabla u-V_0|^2\,dx\,dt\leqq\varepsilon\mu^2\quad\text{holds}\tag{4.10}$$

with ε determined in Lemma 4.4.

PROOF. – We mimicry the proof of [5], Lemma 5.1. Define $W:=\left(|\nabla u|^2-\right.$ $\left.-(1-2\varrho)\mu^2\right)$ and use the test-function $2u^i_{x_\beta}W_+\eta^2$ in (1.3) with η denoting a cutoff-function on Q^μ_{2R} with respect to $Q^\mu_{(1+\sigma)R}$. We get by standard estimations

$$\iint_{Q^\mu_{(1+\sigma)R}}|\nabla u|^{p-2}|D^2u|^2W_+\,dx\,dt\leqq c\iint_{Q^\mu_{2R}}W_+^2(|\nabla u|^{p-2}|\nabla\eta|^2+|\eta_t|)\,dx\,dt.$$

Observing that $W_+\leqq2\varrho\mu^2$ on Q^μ_{2R} and $W_+\geqq\varrho\mu^2$ on

$$A^\varrho_{(1+\sigma)R}:=\left\{(x,t)\in Q^\mu_{(1+\sigma)R}\big||\nabla u|^2>(1-\varrho)\mu^2\right\}$$

we get

$$\iint_{A^\varrho_{(1+\sigma)R}}|\nabla u|^{p-2}|D^2u|^2\,dx\,dt\leqq c(\varrho\mu^2)^{-1}4\varrho^2\mu^4\iint_{Q^\mu_{2R}}\left(|\nabla u|^{p-2}|\nabla\eta|^2+|\eta_t|\right)dx\,dt\leqq c\varrho\mu^2R^N.\tag{4.13}$$

Note that we have also from (4.1)

$$\iint_{Q^\mu_{(1+\sigma)R}}|\nabla u|^{p-2}|D^2u|^2\,dx\,dt\leqq c\mu^2R^N.\tag{4.14}$$

Now define for $t\in I^\mu_{(1+\sigma)R}$

$$V(t):=\fint_{B_{(1+\sigma)R}}|\nabla u|^{(p-2)/2}\,\nabla u\,dx.$$

We want to prove

$$\iint\limits_{Q^\mu_{(1+\sigma)R}} \left||\nabla u|^{(p-2)/2}\nabla u - V(t)\right|^2 dx\, dt \leq c\mu^2 \varrho^{1/(N+1)} R^{N+2}. \tag{4.15}$$

We need the inequality

$$\int\limits_{B_\varrho} |w|^2\, dx \leq c \int\limits_{B_\varrho} (|\nabla w|^2)^{N/(N+1)}\, dx \cdot \left(\int\limits_{B_\varrho} |w|\, dx\right)^{2/(N+1)}$$

valid for functions having average zero on B_ϱ; see e.g. [7], (2.10), p. 45 with $\alpha = N/(N+1)$, $r = 1$, $p = 2$, $m = 2N/(N+1)$.

Let $w = |\nabla u|^{(p-2)/2}\nabla u - V(t)$, hence $|w| \leq c\mu^{p/2}$, and we get

$$\iint\limits_{Q^\mu_{(1+\sigma)R}} \left||\nabla u|^{(p-2)/2} u - V(t)\right|^2 dx\, dt \leq cR^{2N/(N+1)} \mu^{p/(N+1)} \iint\limits_{Q^\mu_{(1+\sigma)R}} (|\nabla u|^{p-2}|D^2 u|^2)^{N/(N+1)}\, dx\, dt.$$

Split the region of integration into the two parts $A^\varrho_{(1+\sigma)R}$ and $Q^\mu_{(1+\sigma)R} \setminus A^\varrho_{(1+\sigma)R}$. By (4.13) we get

$$\iint\limits_{A^\varrho_{(1+\sigma)R}} (|\nabla u|^{p-2}|D^2 u|^2)^{N/(N+1)}\, dx\, dt \leq c\left(\iint\limits_{A^\varrho_{(1+\sigma)R}} |\nabla u|^{p-2}|D^2 u|^2\, dx\, dt\right)^{N/(N+1)} |Q^\mu_{(1+\sigma)R}|^{1/(N+1)} \leq$$
$$\leq c\varrho^{N/(N+1)} \mu^{2N/(N+1)} R^{N^2/(N+1)} \mu^{(2-p)/(N+1)} R^{(N+2)/(N+1)}$$

and using the assumption of the lemma and (4.14) we have

$$\iint\limits_{Q^\mu_{(1+\sigma)R} \setminus A_{(1+\sigma)R}} (|\nabla u|^{p-2}|D^2 u|^2)^{N/(N+1)}\, dx\, dt \leq$$
$$\leq c\left(\iint\limits_{Q^\mu_{(1+\sigma)R}} |\nabla u|^{p-2}|D^2 u|^2\, dx\, dt\right)^{N/(N+1)} |Q^\mu_{(1+\sigma)R} \setminus A^\varrho_{(1+\sigma)R}|^{1/(N+1)} \leq$$
$$\leq c\mu^{2N/(N+1)} R^{N^2/(N+1)} \cdot \varrho^{1/(N+1)} \mu^{(2-p)/(N+1)} R^{(N+2)/(N+1)}.$$

Collecting the estimates gives (4.15).

Now define $g(t)$ by $V(t) = g(t)|g(t)|^{(p-2))/2}$; note, that $|g(t)| \leq \mu$. By (4.7)

$$\left||\nabla u|^{(p-2)/2}\nabla u - V(t)\right|^2 \geq c|\nabla u|^{p-2}|\nabla u - g(t)|^2$$

which together with (4.15) implies

$$(1-\varrho)^{(p-2)/2}\mu^{p-2} \iint\limits_{A^\varrho_{(1+\sigma)R}} |\nabla u - g(t)|^2\, dx\, dt \leq c\mu^2 \varrho^{1/(N+1)} R^{N+2}.$$

Let $(\nabla u)_R(t) = \fint_{B_R} \nabla u(x,t)\,dx$; then obviously

$$\int_{B_R} |\nabla u(x,t) - (\nabla u)_R(t)|^2\,dx \le \int_{B_R} |\nabla u(x,t) - g(t)|^2\,dx$$

hence

$$\iint_{Q_R^\mu} |\nabla u - (\nabla u)_R(t)|^2\,dx\,dt \le \iint_{A^\varrho_{(1+\sigma)R}} |\nabla u - g(t)|^2\,dx\,dt + 2\mu^2 |Q^\mu_{(1+\sigma)R} \setminus A^\varrho_{(1+\sigma)R}| \le \le c\mu^{4-p}\varrho^{1/(N+1)} R^{N+2} \tag{4.16}$$

where the assumption of the lemma was used again.

Next we want to estimate

$$D = \iint_{Q_R^\mu} |(\nabla u)_R(t) - (\nabla u)_R|^2\,dx\,dt\,,$$

which can be bounded by

$$\begin{aligned} D &\le |Q_R^\mu| \sup_{t\in I_R^\mu} \left| \frac{1}{|B_R|}\int_{B_R} \nabla u(x,t)\,dx - \frac{1}{|I_R^\mu|}\int_{I_R^\mu}\left(\frac{1}{|B_R|}\int_{B_R}\nabla u(x,s)\,dx\right)ds\right|^2 \le \\ &\le |Q_R^\mu||B_R|^{-2} \sup_{t,s\in I_R^\mu} \left|\int_{B_R} \big(\nabla u(x,t) - \nabla u(x,s)\big)\,dx\right|^2 \le \\ &\le c\mu^{2-p} R^{2-N} \sup_{t,s\in I_R^\mu} \left|\int_{B_R} \big(\nabla u(x,t) - \nabla u(x,s)\big)\,dx\right|^2 . \end{aligned}$$

Let φ be a cutoff function on $B_{(1+\sigma)R}$ with respect to B_R. Then

$$E := \left|\int_{B_R} \big(\nabla u(x,t) - \nabla u(x,s)\big)\,dx\right| \le \left|\int_{B_{(1+\sigma)R}} \big(\nabla u(x,t) - \nabla u(x,s)\big)\varphi(x)\,dx\right| + 2\mu |B_{(1+\sigma)R}\setminus B_R|\,.$$

Integrate (1.3) from s to t, multiply by φ and integrate over $B_{(1+\sigma)R}$ to get

$$\begin{aligned} E &\le c\mu\sigma R^N + \left|\int_s^t \int_{B_{(1+\sigma)R}} \Big(|\nabla u|^{p-2}\,\nabla u_{x_\alpha} + \nabla\big(|\nabla u|^{p-2}\big)\,u_{x_\alpha}\Big)\varphi_{x_\alpha}\,dx\,d\tau\right| \le c\mu\sigma R^N + \\ &+ c(\sigma R)^{-1}\mu^{(p-2)/2} \iint_{Q^\mu_{(1+\sigma)R}} |\nabla u|^{(p-2)/2} |D^2 u|\,dx\,d\tau\,. \end{aligned}$$

Now the right hand side is treated the same way as in the proof of (4.15) above, giving

$$\iint_{Q^\mu_{(1+\sigma)R}} |\nabla u|^{(p-2)/2} |D^2 u|\,dx\,d\tau \le c\mu^{(4-p)/2}\varrho^{\frac{1}{2}} R^{N+1}$$

and we get

$$E \leqq c\mu(\sigma + \varrho^{\frac{1}{2}}\sigma^{-1})R^N .$$

Therefore

$$D \leqq c\mu^{4-p}R^{N+2}(\sigma + \varrho^{\frac{1}{2}}\sigma^{-1})^2 .$$

Recalling (4.16) gives the final estimate

$$\iint\limits_{Q_R^\mu} |\nabla u(x,t) - (\nabla u)_R|^2\, dx\, dt \leqq c\mu^{4-p}R^{N+2}(\varrho^{1/(N+1)} + \sigma^2 + \varrho\sigma^{-2}) .$$

So choose first σ and then ϱ suitably small, depending on ε, to get

$$\fint\!\!\fint\limits_{Q_R^\mu} |\nabla u(x,t) - V_0|^2\, dx\, dt \leqq \varepsilon\mu^2 \tag{4.17}$$

with $V_0 := (\nabla u)_R$.

It remains to check that $\frac{3}{4}\mu \leqq |V_0| \leqq \frac{3}{2}\mu$. But $|(\nabla u)_R| \leqq \mu$ trivially; and the other estimate follows this way: By (4.17) we have

$$\Bigl(\iint\limits_{Q_R^\mu} |\nabla u|^2\, dx\, dt\Bigr)^{\frac{1}{2}} \leqq (\varepsilon^{\frac{1}{2}}\mu + |V_0|)|Q_R^\mu|^{\frac{1}{2}} .$$

But

$$\iint\limits_{Q_R^\mu} |\nabla u|^2\, dx\, dt \geqq \iint\limits_{Q_{(1+\sigma)R}^\mu} |\nabla u|^2\, dx\, dt - \mu^2|Q_{(1+\sigma)R}^\mu \setminus Q_R^\mu| \geqq \iint\limits_{A_{(1+\sigma)R}^\varrho} |\nabla u|^2\, dx\, dt - c\sigma\mu^2|Q_R^\mu| \geqq$$

$$\geqq (1-\varrho)\mu^2(1+\sigma)^{N+2}|Q_R^\mu| - c\sigma\mu^2|Q_R^\mu| .$$

Hence

$$|V_0| \geqq \mu[(1-\varrho)(1+\sigma)^{N+2} - c\sigma]^{\frac{1}{2}} - \varepsilon^{\frac{1}{2}}\mu \geqq \tfrac{3}{4}\mu$$

if ϱ, σ, and ε are suitable small.

Thereby the proof of the lemma is completed.

5. – Remarks.

1. – One can get global estimates (on $[\varepsilon, T]\times\Omega$) for Neumann boundary-conditions.
2. – Further generalizations concerning lower order terms and a non-zero right hand side indicated in [5] and [9] are possible.
3. – Existence of weak solutions follows by standard Galerkin-approximation, as an *a-priori*-estimate in the spaces defined in (1.2) is easy available.

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