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On 4d rank-one $\mathcal{N} = 3$ superconformal field theories

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Abstract: We study the properties of 4d $\mathcal{N} = 3$ superconformal field theories whose rank is one, i.e. those that reduce to a single vector multiplet on their moduli space of vacua. We find that the moduli space can only be of the form $\mathbb{C}^3/\mathbb{Z}_\ell$ for ℓ=1, 2, 3, 4, 6, and that the supersymmetry automatically enhances to $\mathcal{N} = 4$ for ℓ=1, 2. In addition, we determine the central charges a and c in terms of ℓ, and construct the associated 2d chiral algebras, which turn out to be exotic $\mathcal{N} = 2$ supersymmetric W-algebras.

Keywords: Conformal and W Symmetry, Extended Supersymmetry, Supersymmetric gauge theory

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1 Introduction and summary

Four-dimensional non-gravitational theories with $\mathcal{N}=3$ supersymmetry have been mostly neglected in the literature, due to the well-known fact that any $\mathcal{N}=3$ supersymmetric Lagrangian automatically possesses $\mathcal{N}=4$ supersymmetry. The developments in the last several years on the supersymmetric dynamics tell us, however, that there are many 'non-Lagrangian' theories, i.e. strongly-coupled field theories which do not have obvious Lagrangian descriptions.

Therefore there can be non-Lagrangian $\mathcal{N}=3$ theories, some of whose general properties were first discussed in a paper by Aharony and Evtikhiev [1] from early December

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2015. Later in the same month, García-Etxebarria and Regalado made a striking discovery [2] that indeed such $\mathcal{N}=3$ theories appear on D3-branes probing a generalized form of orientifolds in F-theory.[1]

The aim of this note is to initiate the analysis of such concrete $\mathcal{N}=3$ theories in a purely field-theoretical manner. We mainly restrict attention to rank-1 theories, where the rank is defined as the dimension of the Coulomb branch of the theory considered as an $\mathcal{N}=2$ theory. We will find

- that the moduli space of supersymmetric vacua can only be of the form $\mathbb{C}^3/\mathbb{Z}_\ell$ for $\ell = 1, 2, 3, 4, 6$,
- that the supersymmetry is guaranteed to enhance to $\mathcal{N}=4$ for $\ell = 1, 2$, and therefore only $\ell = 3, 4, 6$ are allowed in the case of the genuine $\mathcal{N}=3$ theories,
- and that the central charges are given by $a = c = (2\ell - 1)/4$.

In addition we construct the 2d chiral algebras associated in the sense of [5] to these rank-1 $\mathcal{N}=3$ theories. We will find the following:

- The 2d chiral algebra contains the $\mathcal{N}=2$ super Virasoro subalgebra and a pair of bosonic chiral primary and antichiral primary with dimension $\ell/2$, as a consequence of the unitarity bounds and the operator product expansions of the 4d $\mathcal{N}=3$ superconformal algebra.
- The Jacobi identities of these operators close only for a finite number of central charges, including $c_{2d} = -3(2\ell - 1)$ as predicted from the construction of [5]. Furthermore, the null relation correctly encodes the structure of the moduli space of supersymmetric vacua at this value of the central charge.

Further studies of these chiral algebras will uncover the spectrum of BPS local operators in rank-1 $\mathcal{N}=3$ superconformal field theories (SCFTs), along the lines of [5–12].

All the findings in this note are consistent with, but do not prove, the existence of genuine $\mathcal{N}=3$ theories with $\ell = 3, 4, 6$. We also note that the findings do not preclude the existence of multiple distinct $\mathcal{N}=3$ theories with the same value of ℓ, although the data we compute in this note do not distinguish them.

The rest of the note is organized as follows: in section 2, we study basic properties of $\mathcal{N}=3$ rank-1 theories. We see that the moduli space of supersymmetric vacua is necessarily of the form $\mathbb{C}^3/\mathbb{Z}_\ell$ for $\ell = 1, 2, 3, 4, 6$, with an automatic enhancement to $\mathcal{N}=4$ when $\ell = 1, 2$. In section 3, we analyze the shortening conditions and the unitarity bounds of the $\mathcal{N}=3$ superconformal algebras to the extent necessary for us, and a few general properties of the associated 2d chiral algebra. In section 4, we use the results obtained so far to construct the 2d chiral algebra associated to $\mathcal{N}=3$ rank-1 theories for $\ell = 3, 4, 6$.

[1] Related holographic constructions of $\mathcal{N}=3$ systems were already discussed in a paper [3] from 1998, although no concrete models were identified there. The authors thank T. Nishioka for bringing this reference [3] to their attention. Also see a recent paper [4] discussing $\mathcal{N}=3$ holographic duals in (massive) type IIA and type IIB setups.

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	I_0	II	III	IV	I_0^*	IV^*	III^*	II^*
$\Delta(u)$	1	6/5	4/3	3/2	2	3	4	6
τ	arb.	ω	i	ω	arb.	ω	i	ω
g	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$

Table 1. The list of scale invariant rank-1 Seiberg-Witten geometries. The first line shows the name given by Kodaira; $\Delta(u)$ is the scaling dimension of the Coulomb branch operator u; τ is the complexified coupling at the generic points on the Coulomb branch; and g is the $\mathrm{SL}(2,\mathbb{Z})$ monodromy around the origin. On the row for τ, ω is a third root of unity, and arb. means that τ is arbitrary.

Note added: when this paper is completed, the authors learned from P. Argyres, M. Lotito, Y. Lü and M. Martone that they have an upcoming paper [13] which has a small overlap with but is largely complementary to this paper. The authors thank them for sharing the draft in advance.

2 Basic properties

2.1 Allowed forms of the moduli space

Let us start by analyzing the allowed form of the moduli space of vacua of an $\mathcal{N}=3$ rank-1 superconformal field theory. Regarding it as an $\mathcal{N}=2$ theory, its Coulomb branch should be a one-dimensional scale-invariant Seiberg-Witten geometry. Its classification is well-known: one just needs to go through Kodaira's list of singularities of elliptic fibrations and keep only the ones where the modulus of the elliptic fiber is constant. The resulting list is reproduced in table 1. In particular, the scaling dimension of the Coulomb branch operator u is fixed to be one of the eight possible values listed there.

The $\mathcal{N}=3$ supersymmetry relates the Higgs branch and the Coulomb branch of the theory regarded as an $\mathcal{N}=2$ theory. The Higgs branch at the origin $u=0$ of the Coulomb branch is then a hyperkähler cone of quaternionic dimension one. Such a one-dimensional cone is necessarily an asymptotically locally Euclidean space of the form $\mathbb{C}^2/\Gamma$ where Γ is a discrete subgroup of SU(2). As an $\mathcal{N}=3$ supersymmetric theory necessarily has a U(1) flavor symmetry as seen as an $\mathcal{N}=2$ theory, the space $\mathbb{C}^2/\Gamma$ should have a U(1) hyperkähler isometry. This restricts Γ to be of the form $\mathbb{Z}_\ell$. Let (z_+, z_-) be the coordinates of $\mathbb{C}^2$ before the quotient. Then, as an $\mathcal{N}=1$ theory, the Higgs branch are parameterized by three chiral operators $W_+ = z_+^\ell$, $W_- = z_-^\ell$ and $J = z_+ z_-$ satisfying

$$W^+ W^- \propto J^\ell. \tag{2.1}$$

Here, $W^\pm$ has dimension ℓ and U(1) charge $\pm\ell$, and J is the moment map of the U(1) symmetry.

The $\mathcal{N}=3$ symmetry rotates the $\mathcal{N}=2$ Coulomb branch to the $\mathcal{N}=2$ Higgs branch, and therefore relates the operator $W^\pm$ and u, as we will see this in more detail in section 3.1.

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This means that the integer ℓ should also be a number allowed as $\Delta(u)$. We conclude that $\ell = 1, 2, 3, 4, 6$. Combining the information on the Coulomb branch and the Higgs branch, we see that the full moduli space of supersymmetric vacua should be of the form $\mathbb{C}^3/\mathbb{Z}_\ell$, where ℓ is one from the list above.

That the moduli space of $\mathcal{N}=3$ theory is locally flat is known, see e.g. [14]. Let us check that the quotient by $\mathbb{Z}_\ell$ preserves $\mathcal{N}=3$ supersymmetry, at least away from the origin.[2] Note that away from the origin, the moduli space is smooth. As such, the theory is locally that of a single $\mathcal{N}=4$ U(1) vector multiplet. Let us denote the three chiral scalars of the vector multiplet by (z_0, z_+, z_-), such that $u = z_0^\ell$. The $\mathbb{Z}_\ell$ action acts as

$$(z_0, z_+, z_-) \mapsto (\gamma z_0, \gamma z_+, \gamma^{-1} z_-) \tag{2.2}$$

where $\gamma = e^{2\pi i/\ell}$. This is accompanied by an $\mathrm{SL}(2,\mathbb{Z})$ duality action g given in table 1.

The geometric action (2.2) is a part of the SU(4) R-symmetry of the free $\mathcal{N}=4$ multiplet, which determines its action of the four supercharges as

$$(Q_1, Q_2, Q_3, Q_4) \mapsto (\gamma^{1/2} Q_1, \gamma^{1/2} Q_2, \gamma^{1/2} Q_3, \gamma^{-3/2} Q_4). \tag{2.3}$$

The action of the duality transformation by g on the supercharges can be found e.g. in section 2.2 of [15]:

$$(Q_1, Q_2, Q_3, Q_4) \mapsto \gamma^{-1/2} (Q_1, Q_2, Q_3, Q_4). \tag{2.4}$$

Combined, we see that the action on the four supercharges is given by

$$(Q_1, Q_2, Q_3, Q_4) \mapsto (Q_1, Q_2, Q_3, \gamma^{-2} Q_4), \tag{2.5}$$

from which we conclude that all four supercharges are preserved for $\ell = 1, 2$ whereas only the first three supercharges are preserved for $\ell = 3, 4, 6$.

The enhancement to $\mathcal{N}=4$ when $\ell = 1, 2$ can be understood also as follows. When $\ell = 1, 2$, the hyperkähler cone $\mathbb{C}^2/\mathbb{Z}_\ell$ has a larger hyperkähler isometry SU(2) with corresponding moment map operators of dimension two. This in turn implies that the flavor symmetry as an $\mathcal{N}=2$ theory is larger than U(1). In [1] it was shown that genuine $\mathcal{N}=3$ theories cannot have any flavor symmetry current bigger than U(1) as $\mathcal{N}=2$ theory, meaning that the supersymmetry automatically enhances to $\mathcal{N}=4$ for $\ell = 1, 2$.

Finally, let us determine the central charges a and c of these theories labeled by ℓ. Very generally, any $\mathcal{N}=2$ superconformal field theory is believed to satisfy the relation

$$2a - c = \frac{1}{4} \sum_i (2\Delta(u_i) - 1) \tag{2.6}$$

where the sum runs over the independent generators of the Coulomb branch operators.

This relation was originally conjectured in [16] and a derivation that applies to a large subclass of $\mathcal{N}=2$ theories was given in [17]. It is not perfectly clear that the assumptions

[2] The analysis of the supercharges here is completely the same as the one given in García-Etxebarria and Regalado [2] done in F-theory. The point here is that it can be phrased in a completely field-theoretical manner.

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used in [17] is satisfied by strongly-coupled theories we are discussing here, but the authors think it is quite plausible.[3] Assuming the validity of the general formula, we then have

$$2a - c = \frac{2\ell - 1}{4}. \tag{2.7}$$

Now, in any $\mathcal{N}=3$ superconformal field theory, we have $a = c$, as originally shown in [1]. One way to re-derive it in our case is to go to the Higgs branch as an $\mathcal{N}=2$ theory. This process does not break $\mathrm{U}(1)_R$ symmetry in the $\mathcal{N}=2$ subalgebra, and hence the $\mathrm{U}(1)_R$-gravity-gravity anomaly, which is proportional to $a - c$, is conserved. On the Higgs branch the theory is just $\mathcal{N}=4$, and therefore $a - c = 0$.

From the known value of $2a - c$ above, we conclude that

$$a = c = \frac{2\ell - 1}{4}. \tag{2.8}$$

As mentioned above, the derivation here is not completely watertight, but we give a rather non-trivial consistency check in the rest of the paper.

2.2 Realizations

So far we concluded that the moduli space of a rank-1 $\mathcal{N}=3$ superconformal field theory is necessarily of the form $\mathbb{C}^3/\mathbb{Z}_\ell$ for $\ell = 1, 2, 3, 4, 6$. Here we give a brief survey of the known realizations of these theories.

When $\ell = 1, 2$, the theory automatically has $\mathcal{N}=4$ supersymmetry. For $\ell = 1$, the vacuum moduli space is simply $\mathbb{C}^3$ without any singularity, and therefore we can safely conclude that the only such theory is a theory of a single free U(1) vector multiplet. For $\ell = 2$, a realization is of course given by the $\mathcal{N}=4$ super Yang-Mills theory with gauge algebra $\mathfrak{su}(2)$. The gauge group can either be SU(2) or SO(3), depending on which we have two subtly different theories.[4]

Genuine $\mathcal{N}=3$ theories were first constructed in [2] using F-theory. Namely, they started from the F-theory setup of the form $\mathbb{R}^{1,3} \times \mathbb{C}^3 \times T^2$ where the last T^2 describes the axiodilaton of the Type IIB theory, took the quotient $(\mathbb{C}^3 \times T^2)/\mathbb{Z}_k$, and probed this background by r D3-branes. In particular, we have rank one theories when $r = 1$, and the moduli space of vacua is parameterized by the position of the D3-brane, that is $\mathbb{C}^3/\mathbb{Z}_k$. As the torus T^2 can have $\mathbb{Z}_k$ isometry only for $k = 1, 2, 3, 4, 6$, we get the same classification as we saw above.

There is a caveat however: we cannot directly identify the integer k governing the F-theory background and the integer ℓ governing the moduli space of the superconformal theory. When $k = 2$, there are two types of such $\mathbb{Z}_2$ quotient, up to the action of the $\mathrm{SL}(2,\mathbb{Z})$ duality of the type IIB. One is the $\mathrm{O3}^-$ plane and the other is the $\mathrm{SL}(2,\mathbb{Z})$

[3] It is known that this relation fails in gauge theories where part of the gauge symmetry is disconnected from the identity. For example, take $\mathcal{N}=4$ super Yang-Mills theory with gauge group U(1) and O(2). They both have $2a - c = 1/4$, but the Coulomb branch operator has dimension 1 for the former and 2 for the latter. In this note, when we speak about the moduli space of vacua, we declare that we do *not* impose the invariance under the disconnected part of the gauge group, or whatever that concept corresponds to in non-Lagrangian theories. The author expects that this relation holds under this condition.

[4] We declare that the $\mathcal{N}=4$ super Yang-Mills theory with gauge group O(2) belongs to the case $\ell = 1$, as discussed in Footnote 3.

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orbit containing $\mathrm{O3}^+$, $\widetilde{\mathrm{O3}}^-$, $\widetilde{\mathrm{O3}}^+$. Probing by one D3-brane, the former gives the $\mathcal{N}=4$ super Yang-Mills theory with gauge algebra $\mathfrak{so}(2)$, whereas the latter gives that with gauge algebra $\mathfrak{su}(2)$. As discussed in Footnote 3, we declare that when we discuss the moduli space we do not gauge by the disconnected part of the gauge group, and then the former has the moduli space $\mathbb{C}^3/\mathbb{Z}_{\ell=1}$ whereas the latter has $\mathbb{C}^3/\mathbb{Z}_{\ell=2}$. In both cases, ℓ divides k.

As discussed in [2], there are various versions of the $\mathbb{Z}_k$ quotients also for $k \neq 2$ in F-theory. Depending on the version, we will have a different discrete quotient

$$\mathbb{C}^3/\mathbb{Z}_\ell \to \mathbb{C}^3/\mathbb{Z}_k \tag{2.9}$$

where the left hand side is the moduli space of the superconformal theory and the right hand side is the F-theory background. We do not yet know which version of the $\mathbb{Z}_k$ quotient gives which divisor ℓ of k.

If there were a version such that $k = \ell$ for each $\ell = 3, 4, 6$, we would have an F-theoretic realization of an $\mathcal{N}=3$ rank-1 theory for each $\ell = 3, 4, 6$. This point is however not well understood and requires further study, and the details will be reported elsewhere [18]. We would like to point out that, even assuming this, the F-theory construction gives *a* realization; we do not yet know whether there are multiple subtly different versions of the theory for each $\ell = 3, 4, 6$ either.

The rank-1 $\mathcal{N}=3$ theories are already quite interesting even when considered as $\mathcal{N}=2$ theories, since they give rise to rank-1 $\mathcal{N}=2$ theories in addition to the known list consisting of the old ones [19–22] and the new ones [16, 23].[5] As already discussed, the $\mathcal{N}=3$ theory would have $\mathfrak{u}(1)$ flavor symmetry as an $\mathcal{N}=2$ theory.

A systematic study of all possible rank-1 $\mathcal{N}=2$ superconformal field theories and their mass deformations through the construction of the Seiberg-Witten curves and differentials is currently under investigation by Argyres, Lotito, Lü and Martone [25, 26]. The properties of the $\ell = 3$ theory we determined above match exactly with the entry in table 1 of [25] describing the IV^* singularity with $\mathfrak{u}(1)$ symmetry. The $\ell = 4$ and the $\ell = 6$ theories might similarly correspond to some of the entries in the same table. We immediately notice, however, that there are no entries corresponding to the III^* and II^* singularities marked there as having $\mathfrak{u}(1)$ flavor symmetry. This does not yet preclude the existence of the $\ell = 4$ and $\ell = 6$ theories, since in [25, 26] it was assumed that all the discrete symmetries acting on the mass parameters were considered as coming from the Weyl symmetry. In particular, in their construction, those marked as having $\mathfrak{su}(2)$ flavor symmetry can be interpreted as having $\mathbb{Z}_2 \ltimes \mathrm{U}(1)$ symmetry. This point clearly needs further study.[6]

3 4d $\mathcal{N}$=3 theories and the associated 2d chiral algebras

In this section, we work out some consequences of the 4d $\mathcal{N}=3$ superconformal algebras and state them in the $\mathcal{N}=2$ language. We also derive general properties of the 2d chiral algebra $\chi[\mathcal{T}]$ associated in the sense of [5] to an $\mathcal{N}=3$ superconformal theory $\mathcal{T}$. We mainly follow the conventions of [27].

[5] See [24] for an even newer rank-1 $\mathcal{N}=2$ theory with SU(4) symmetry.

[6] The authors thank P. Argyres, M. Lotito, Y. Lü and M. Martone for instructive discussions on this point, and for sharing their upcoming paper [13].

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	j_1	j_2	R	r	F	δ_1	δ_2
$\mathcal{Q}^1{}_+$	$+\frac{1}{2}$	0	$+\frac{1}{2}$	$+\frac{1}{2}$	0	-2	0
$\mathcal{Q}^1{}_-$	$-\frac{1}{2}$	0	$+\frac{1}{2}$	$+\frac{1}{2}$	0	0	0
$\mathcal{Q}^2{}_+$	$+\frac{1}{2}$	0	$-\frac{1}{2}$	$+\frac{1}{2}$	0	0	$+2$
$\mathcal{Q}^2{}_-$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$+\frac{1}{2}$	0	$+2$	$+2$
$\tilde{\mathcal{Q}}_{1\dot{+}}$	0	$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$+2$	0
$\tilde{\mathcal{Q}}_{1\dot{-}}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$+2$	$+2$
$\tilde{\mathcal{Q}}_{2\dot{+}}$	0	$+\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	0	0	-2
$\tilde{\mathcal{Q}}_{2\dot{-}}$	0	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	0	0	0
$\mathcal{Q}^3{}_+$	$+\frac{1}{2}$	0	0	$-\frac{1}{2}$	$+1$	0	0
$\mathcal{Q}^3{}_-$	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	$+1$	$+2$	0
$\tilde{\mathcal{Q}}_{3\dot{+}}$	0	$+\frac{1}{2}$	0	$+\frac{1}{2}$	-1	0	0
$\tilde{\mathcal{Q}}_{3\dot{-}}$	0	$-\frac{1}{2}$	0	$+\frac{1}{2}$	-1	0	$+2$

Table 2. The quantum numbers of supercharges.

3.1 $\mathcal{N}=3$ superconformal algebra and its $\mathcal{N}=2$ subalgebra

The 4d $\mathcal{N}=3$ superconformal algebra is $\mathfrak{su}(2,2|3)$, whose generators and (anti-)commutation relations in our notation are summarized in appendix A. In particular, the fermionic generators are $\mathcal{Q}^I{}_\alpha$, $\tilde{\mathcal{Q}}_{I\dot{\alpha}}$, $\mathcal{S}_I{}^\alpha$, $\tilde{\mathcal{S}}^{I\dot{\alpha}}$ for $\alpha=\pm$, $\dot{\alpha}=\dot{\pm}$ and $I=1,2,3$. This algebra has an $\mathcal{N}=2$ superconformal subalgebra containing $\mathcal{Q}^i_\alpha$, $\tilde{\mathcal{Q}}_{i\dot{\alpha}}$, $\mathcal{S}_i{}^\alpha$, $\tilde{\mathcal{S}}^{i\dot{\alpha}}$ for $i=1,2$, whose R-symmetry is $\mathfrak{u}(2)$ generated by $\mathcal{R}^i{}_j$ for $i,j=1,2$. The $\mathfrak{su}(2)_R$ and $\mathfrak{u}(1)_r$ charges are respectively given by

$$R \equiv \frac{1}{2}\left(\mathcal{R}^1{}_1 - \mathcal{R}^2{}_2\right), \qquad r \equiv \mathcal{R}^1{}_1 + \mathcal{R}^2{}_2 \,. \tag{3.1}$$

The $\mathcal{R}^i{}_j$ for $i,j=1,2$ and $\mathcal{R}^3{}_3$ generate an $\mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_r \oplus \mathfrak{u}(1)_F$ subalgebra of $\mathfrak{u}(3)$. Here we take $\mathfrak{u}(1)_F$ to be generated by

$$F \equiv 2\mathcal{R}^3{}_3 + r \,, \tag{3.2}$$

so that our $\mathcal{N}$=2 supercharges are neutral under $\mathfrak{u}(1)_F$. From the $\mathcal{N}$=2 viewpoint, F is a flavor charge.

The quantum numbers of the supercharges are listed in table 2 together with the eigenvalues of the following linear combinations of charges:

$$\begin{aligned} \delta_1 &\equiv \frac{1}{2}\{\mathcal{Q}^1{}_-, (\mathcal{Q}^1{}_-)^\dagger\} = E - 2j_1 - 2R - r \,, \\ \delta_2 &\equiv \frac{1}{2}\{\tilde{\mathcal{Q}}_{2\dot{-}}, (\tilde{\mathcal{Q}}_{2\dot{-}})^\dagger\} = E - 2j_2 - 2R + r \,, \end{aligned} \tag{3.3}$$

where E is the scaling dimension and j_1, j_2 are the $\mathfrak{so}(4)$ spins such that $\mathcal{M}_+{}^+ = j_1$, $\mathcal{M}^{\dot{+}}{}_{\dot{+}} = -j_2$. We will use the above two linear combinations of charges to discuss, in the next sub-section, the 2d chiral algebras associated in the sense of [5] to $\mathcal{N}=3$ SCFTs.

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The anti-commutation relations (A.3) imply various unitarity bounds on operators. In particular, the presence of the third set of supercharges implies the following unitarity bounds

$$\begin{aligned}\frac{1}{2}\{\mathcal{Q}^3{}_{\pm},\,(\mathcal{Q}^3{}_{\pm})^\dagger\} &= E \pm 2j_1 - F + r \geq 0\,,\\ \frac{1}{2}\{\tilde{\mathcal{Q}}_{3\dot{\pm}},\,(\tilde{\mathcal{Q}}_{3\dot{\pm}})^\dagger\} &= E \pm 2j_2 + F - r \geq 0\,.\end{aligned} \tag{3.4}$$

This particularly means that any scalar operator should have $E \geq |F - r|$.

3.1.1 Higgs branch operators

The $\mathcal{N}=3$ unitarity bounds (3.4) are further simplified for the special set of operators called Higgs branch operators. They are defined as local operators annihilated by all of $\mathcal{Q}^1{}_\alpha$, $(\mathcal{Q}^1{}_\alpha)^\dagger$ and $\tilde{\mathcal{Q}}_{2\dot{\alpha}}$, $(\tilde{\mathcal{Q}}_{2\dot{\alpha}})^\dagger$ for $\alpha = \pm$ and $\dot{\alpha} = \dot{\pm}$. Since they saturate the following bounds

$$\begin{aligned}\frac{1}{2}\{\mathcal{Q}^1{}_{\pm},\,(\mathcal{Q}^1{}_{\pm})^\dagger\} &= E \pm 2j_1 - 2R - r \geq 0\,,\\ \frac{1}{2}\{\tilde{\mathcal{Q}}_{2\dot{\pm}},\,(\tilde{\mathcal{Q}}_{2\dot{\pm}})^\dagger\} &= E \pm 2j_2 - 2R + r \geq 0\,,\end{aligned} \tag{3.5}$$

they are conformal primaries with $E = 2R$ and $j_1 = j_2 = r = 0$. For these operators, the $\mathcal{N}=3$ unitarity bounds (3.4) reduce to

$$E \geq |F|\,. \tag{3.6}$$

Moreover, Higgs branch operators are annihilated by all of $\mathcal{S}_I{}^\alpha = (\mathcal{Q}^I{}_\alpha)^\dagger$ and $\tilde{\mathcal{S}}^{I\dot{\alpha}} = (\tilde{\mathcal{Q}}_{I\dot{\alpha}})^\dagger$ for $I = 1, 2, 3$. Indeed $\mathcal{S}_1{}^\alpha$ and $\tilde{\mathcal{S}}^{2\dot{\alpha}}$ annihilate them by definition while the action of the other $\mathcal{S}_I{}^\alpha$, $\tilde{\mathcal{S}}^{I\dot{\alpha}}$ on Higgs branch operators breaks one of the unitarity bounds in (3.5). Therefore any Higgs branch operator is an $\mathcal{N}=3$ superconformal primary.

For rank-1 $\mathcal{N}=3$ SCFTs, we have seen in section 2 that there are three generators of the Higgs branch chiral ring, W^+, W^- and J. Since they respectively have $(E, F) = (\ell, \ell)$, $(\ell, -\ell)$ and $(2, 0)$, the $W^\pm$ saturate the $\mathcal{N}=3$ unitarity bound (3.6) but J does not. In particular, W^+ is annihilated by $\mathcal{Q}^1{}_\alpha$, $\tilde{\mathcal{Q}}_{2\dot{\alpha}}$, $\mathcal{Q}^3{}_\alpha$ (and their conjugates), while W^- is annihilated by $\mathcal{Q}^1{}_\alpha$, $\tilde{\mathcal{Q}}_{2\dot{\alpha}}$, $\tilde{\mathcal{Q}}_{3\dot{\alpha}}$ (and their conjugates).

3.1.2 Coulomb branch operators

Let us next consider Coulomb branch operators, which are defined as *scalar* local operators annihilated by all of $\tilde{\mathcal{Q}}_{1\dot{\alpha}}$, $(\tilde{\mathcal{Q}}_{1\dot{\alpha}})^\dagger$ and $\tilde{\mathcal{Q}}_{2\dot{\alpha}}$, $(\tilde{\mathcal{Q}}_{2\dot{\alpha}})^\dagger$ for $\dot{\alpha} = \dot{\pm}$. They saturate the following unitarity bounds

$$\begin{aligned}\frac{1}{2}\{\tilde{\mathcal{Q}}_{1\dot{\pm}},\,(\tilde{\mathcal{Q}}_{1\dot{\pm}})^\dagger\} &= E \pm 2j_2 + 2R + r \geq 0\,,\\ \frac{1}{2}\{\tilde{\mathcal{Q}}_{2\dot{\pm}},\,(\tilde{\mathcal{Q}}_{2\dot{\pm}})^\dagger\} &= E \pm 2j_2 - 2R + r \geq 0\,,\end{aligned} \tag{3.7}$$

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and therefore have $E = -r$ and $R = 0$ in addition to $j_1 = j_2 = 0$.[7] Moreover, they are neutral under any $\mathcal{N}=2$ flavor symmetry [28, 29], which implies they have $F = 0$. Then we see that Coulomb branch operators saturate the first unitarity bound in (3.4), and therefore are annihilated not only by $\tilde{\mathcal{Q}}_{1\dot{\alpha}}$, $\tilde{\mathcal{Q}}_{2\dot{\alpha}}$ (and their conjugates) but also by $\mathcal{Q}^3{}_\alpha$ (and its conjugate).[8] From the unitarity bounds (3.4) and (3.7), we also see that they are $\mathcal{N}=3$ superconformal primaries.

For rank-1 $\mathcal{N}=3$ SCFTs, there is only one Coulomb branch operator u. Its $E = -r$ is determined by the fact that u can be regarded as a Higgs branch operator with respect to another set of $\mathcal{N}=2$ supercharges, say $\mathcal{Q}^3{}_\alpha$, $\tilde{\mathcal{Q}}_{3\dot{\alpha}}$, $\mathcal{Q}^2{}_\alpha$, $\tilde{\mathcal{Q}}_{2\dot{\alpha}}$. With this new choice of $\mathcal{N}=2$ symmetry, $\mathcal{Q}^1{}_\alpha$ and $\tilde{\mathcal{Q}}_{1\dot{\alpha}}$ are regarded as the "third" set of supercharges. Since u is annihilated by $\mathcal{Q}^3{}_\alpha$, $\tilde{\mathcal{Q}}_{2\dot{\alpha}}$ and their conjugates, it is indeed regarded as a Higgs branch operator with respect to the new $\mathcal{N}=2$ supersymmetry. Moreover, u is annihilated by the anti-chiral part of the "third" set of supercharges, $\tilde{\mathcal{Q}}_{1\dot{\alpha}}$. This implies that u is mapped to W^- (and vice versa) by exchanging $(\mathcal{Q}^1{}_\alpha, \tilde{\mathcal{Q}}_{1\dot{\alpha}})$ and $(\mathcal{Q}^3{}_\alpha, \tilde{\mathcal{Q}}_{3\dot{\alpha}})$. Since this exchanging is a part of the $\mathrm{U}(3)_R$ symmetry of the theory, we see that the scaling dimension of u is given by $\Delta(u) = \Delta(W^-) = \ell$.[9]

More generally, for any 4d $\mathcal{N}=3$ SCFT, exchanging $(\mathcal{Q}^1{}_\alpha, \tilde{\mathcal{Q}}_{1\dot{\alpha}})$ and $(\mathcal{Q}^3{}_\alpha, \tilde{\mathcal{Q}}_{3\dot{\alpha}})$ maps a Coulomb branch operator to a Higgs branch operator saturating the second unitarity bound in (3.4). Since $E = 2R$ for Higgs branch operators is an integer, we see that $E = -r$ for Coulomb branch operators is always an integer for any 4d $\mathcal{N}=3$ SCFT.[10]

3.2 Identifying the 2d $\mathcal{N}$=2 super Virasoro multiplet

In this sub-section, we show that the 2d chiral algebra $\chi[\mathcal{T}]$ corresponding in the sense of [5] to any 4d $\mathcal{N}=3$ SCFT, $\mathcal{T}$, contains an $\mathcal{N}=2$ super Virasoro algebra.[11]

First of all, let us recall that Schur operators are defined as local operators with $\delta_1 = \delta_2 = 0$, where δ_1, δ_2 are defined in (3.3). Their quantum numbers satisfy

$$j_1 + j_2 = E - 2R\,, \qquad j_1 - j_2 = -r\ . \tag{3.8}$$

The unitarity implies that they are operators annihilated by $\mathcal{Q}^1{}_-$, $(\mathcal{Q}^1{}_-)^\dagger$ and $\tilde{\mathcal{Q}}_{2\dot{-}}$, $(\tilde{\mathcal{Q}}_{2\dot{-}})^\dagger$. Any local operator which is not a Schur operator has $\delta_1 > 0$ or $\delta_2 > 0$. It was shown in [5] that the space of Schur operators in any 4d $\mathcal{N}=2$ SCFT has a structure of a 2d chiral algebra. In particular, every 4d Schur operator $\mathcal{O}$ maps to a 2d local operator $\chi[\mathcal{O}]$ with 2d chiral operator product expansions (OPEs) determined by 4d OPEs. The 2d chiral algebra always contains a Virasoro subalgebra with the identification

$$L_0 = E - R\ . \tag{3.9}$$

[7] Here $j_1 = 0$ follows from the fact that Coulomb branch operators are, by definition, scalars. The absence of local operators saturating these bounds with $j_1 \neq 0$ in a large class of 4d $\mathcal{N}=2$ SCFTs were discussed in [28].

[8] The conjugates of Coulomb branch operators have $E = r$ and saturate the second bound in (3.4).

[9] Exchanging $(\mathcal{Q}^2{}_\alpha, \tilde{\mathcal{Q}}_{2\dot{\alpha}})$ and $(\mathcal{Q}^3{}_\alpha, \tilde{\mathcal{Q}}_{3\dot{\alpha}})$ maps the conjugate of u to W^+ and vice versa. In other words, $\overline{u}$ and W^+ are in the same irreducible $\mathcal{N}=3$ super multiplet.

[10] This also follows from the fact that $\mathcal{R}^2{}_2 - \mathcal{R}^3{}_3 = r - R - \frac{F}{2}$ has only integer eigenvalues as $\mathcal{R}^1{}_1 - \mathcal{R}^2{}_2 = 2R$.

[11] This was also noticed by O. Aharony, M. Evtikhiev and R. Yacoby (unpublished).

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The general discussion for $\mathcal{N}=2$ SCFTs in [5] tells us that our theory $\mathcal{T}$ has at least the following bosonic Schur operators:

- The highest weight component of the $\mathrm{SU}(2)_R$ current, $\mathcal{J}^{11}_{++}$, with $E=3$, $R=1$ and $F=0$.[12] The corresponding 2d operator

$$T \equiv \chi[\mathcal{J}^{11}_{++}] \tag{3.10}$$

  is the 2d stress tensor.

- The highest weight component of the $\mathrm{U}(1)_F$ moment map operator, J^{11}, with $E=2$, $R=1$ and $F=0$. This is a Higgs branch operator in the sense of section 3.1, and was denoted by J in section 2. The corresponding 2d operator

$$J \equiv \chi[J^{11}] \tag{3.11}$$

  is an affine U(1) current.

Other bosonic Schur operators will be discussed in the next sub-section.

Since our theory $\mathcal{T}$ has $\mathcal{N}=3$ symmetry, there are extra supercharges $\mathcal{Q}^3{}_\alpha$, $\tilde{\mathcal{Q}}_{3\dot\alpha}$. From table 2, we see that $\mathcal{Q}^3{}_+$ and $\tilde{\mathcal{Q}}_{3\dot+}$ have $\delta_1=\delta_2=0$ and therefore act on the space of Schur operators.[13] This means that fermionic Schur operators are created by acting $\mathcal{Q}^3{}_+$ and $\tilde{\mathcal{Q}}_{3\dot+}$ on the above bosonic ones. For example, $\mathcal{Q}^3{}_+J^{11}$ and $\tilde{\mathcal{Q}}_{3\dot+}J^{11}$ are two fermionic Schur operators, which are non-vanishing due to the unitarity bounds (3.4).[14] Moreover, they are conformal primaries because J^{11} is an $\mathcal{N}=3$ superconformal primary as shown in section 3.1. Then, as shown in appendix B, the corresponding 2d operators

$$G \equiv \chi[\mathcal{Q}^3{}_+J^{11}]\,, \qquad \overline{G} \equiv \chi[\tilde{\mathcal{Q}}_{3\dot+}J^{11}] \tag{3.12}$$

are Virasoro primaries. From (3.9), we see that their holomorphic dimension is $\frac{3}{2}$. Moreover, G and $\overline{G}$ respectively have charge $+1$ and -1 under J since $\mathcal{Q}^3{}_+$ and $\tilde{\mathcal{Q}}_{3\dot+}$ have $\mathrm{U}(1)_F$ charge ±1.

Let us next consider $\{\mathcal{Q}^3{}_+,\,\tilde{\mathcal{Q}}_{3\dot+}\}J^{11}$ and $[\mathcal{Q}^3{}_+,\,\tilde{\mathcal{Q}}_{3\dot+}]J^{11}$. While the former is a conformal descendant of J^{11}, the latter is a bosonic Schur operator with $E=3$, $R=1$ and moreover is a conformal primary. According to [5, 27, 30], the only such Schur operator is the highest weight component of the $\mathrm{SU}(2)_R$ current, $\mathcal{J}^{11}_{++}$, in the stress tensor multiplet. Assuming that the theory $\mathcal{T}$ contains only one stress tensor multiplet,[15] we conclude that

$$\mathcal{J}^{11}_{++} = \frac{1}{2}[\mathcal{Q}^3{}_+,\,\tilde{\mathcal{Q}}_{3\dot+}]J^{11}. \tag{3.13}$$

More generally we identify $\mathcal{J}^{ij}_{\alpha\dot\alpha} = \frac{1}{2}[\mathcal{Q}^3{}_\alpha,\,\tilde{\mathcal{Q}}_{3\dot\alpha}]J^{ij}$.

[12] Here, we follow the convention of [5]. Namely, $\mathcal{J}^{11}_{++}$ is the highest weight $\mathfrak{su}(2)_R\oplus\mathfrak{so}(4)$ component of the $\mathrm{SU}(2)_R$ current $\mathcal{J}^{ij}_{\alpha\dot\alpha}$.

[13] On the other hand, $\mathcal{Q}^3{}_-$ and $\tilde{\mathcal{Q}}_{3\dot-}$ have either $\delta_1>0$ or $\delta_2>0$, and therefore their actions cannot create any Schur operator. They map any local operator to a non-Schur operator or zero.

[14] In the language of [27], these operators are respectively in the $\mathcal{D}_{\frac{1}{2}(0,0)}$ and the $\overline{\mathcal{D}}_{\frac{1}{2}(0,0)}$ multiplets.

[15] If $\mathcal{T}$ is the union of two or more mutually non-interacting SCFTs, it contains more than one stress tensor multiplet. In this paper, we only consider irreducible rank-one $\mathcal{N}=3$ SCFTs which cannot be divided into two or more separate $\mathcal{N}=2$ SCFTs.

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Hence, the four Schur operators J^{11}, $\mathcal{Q}^3{}_+J^{11}$, $\tilde{\mathcal{Q}}_{3\dot{+}}J^{11}$ and $\mathcal{J}^{11}_{+\dot{+}}$ are in the same $\mathcal{N}=3$ superconformal multiplet as the stress tensor.[16] This means that the corresponding 2d chiral operators J, G, $\overline{G}$ and T are also in a 2d super multiplet. It is a standard fact that in 2d, the energy momentum tensor T, a U(1) current J, and two fermionic dimension 3/2 currents G, $\overline{G}$ of U(1) charge ± 1 necessarily form the $\mathcal{N}=2$ super Virasoro algebra. Therefore, we see that the 2d chiral algebra $\chi[\mathcal{T}]$ associated in the sense of [5] to a 4d $\mathcal{N}=3$ SCFT contains the $\mathcal{N}=2$ super Virasoro algebra. The 2d central charge is

$$c_{2d} = -12c_{4d} \tag{3.14}$$

as in [5].

The $\mathfrak{sl}(2|1)$ subalgebra of the $\mathcal{N}=2$ super Virasoro algebra can be explicitly seen in the $\mathcal{N}=3$ superconformal algebra. Indeed, L_0, $L_{\pm 1}$ was identified as $L_{-1} = \frac{1}{2}\mathcal{P}_{+\dot{+}}$, $L_1 = \frac{1}{2}\mathcal{K}^{\dot{+}+}$ and $L_0 = E - R$ in [5],[17] and our identification (3.12) means

$$G_{-\frac{1}{2}} = \frac{1}{2}\mathcal{Q}^3{}_+, \quad \overline{G}_{-\frac{1}{2}} = \frac{1}{2}\tilde{\mathcal{Q}}_{3\dot{+}}, \quad G_{\frac{1}{2}} = \frac{1}{2}\tilde{\mathcal{S}}^{3\dot{+}}, \quad \overline{G}_{\frac{1}{2}} = \frac{1}{2}\mathcal{S}_3{}^+, \quad J_0 = F\ . \tag{3.15}$$

It is then straightforward to show that, under these identifications, L_0, $L_{\pm 1}$, J_0 and $G_{\pm\frac{1}{2}}$, $\overline{G}_{\pm\frac{1}{2}}$ generate a subalgebra of $\mathfrak{su}(2,2|3)$ which acts as $\mathfrak{sl}(2|1)$ on the space of Schur operators.

3.3 2d operators corresponding to Higgs branch operators

In addition to the above Schur operators, the Higgs branch operators are all Schur operators. We here show the following two statements:

1. For any Higgs branch operator $\mathcal{O}$, $\chi[\mathcal{O}]$ is a superprimary operator.

2. For any Higgs branch operator $\mathcal{O}$ with $E = \pm F$, $\chi[\mathcal{O}]$ is a(n) (anti-)chiral superprimary.

In the next section, we will use the second statement to identify the 2d chiral algebras corresponding in the sense of [5] to rank-1 $\mathcal{N}=3$ SCFTs.

Let us first show the first statement. Suppose that $\mathcal{O}$ is a Higgs branch operator. Since $\mathcal{O}$ is a Hall-Littlewood operator in the language of [5, 30], $\chi[\mathcal{O}]$ is a Virasoro primary in two dimensions (as shown in section 3.2.4 of [5] and reviewed in appendix B). Therefore we only need to show that $\chi[\mathcal{O}]$ is annihilated by $G_{n+\frac{1}{2}}$ for $n \geq 0$. Since $\mathcal{O}$ is an $\mathcal{N}=3$ superconformal primary as shown in section 3.1, it is annihilated by $(\mathcal{Q}^3{}_+)^\dagger$, $(\tilde{\mathcal{Q}}_{3\dot{+}})^\dagger$. This means that $\chi[\mathcal{O}]$ is annihilated by $G_{\frac{1}{2}}$ and $\overline{G}_{\frac{1}{2}}$. Therefore, for all $n \geq 2$,

$$G_{n+\frac{1}{2}} = \frac{2}{n-1}[L_n, G_{\frac{1}{2}}]\,, \qquad \overline{G}_{n+\frac{1}{2}} = \frac{2}{n-1}[L_n, \overline{G}_{\frac{1}{2}}] \tag{3.16}$$

[16] Further actions of $\mathcal{Q}^3{}_+$ or $\tilde{\mathcal{Q}}_{3\dot{+}}$ on these operators do not create any new Schur operators up to their conformal descendants.

[17] The extra factor of $\frac{1}{2}$ comes from our different normalization of $\mathcal{P}_{\alpha\dot{\alpha}}$ and $\mathcal{K}^{\dot{\alpha}\alpha}$.

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also annihilate $\chi[\mathcal{O}]$. Finally,

$$G_{\frac{3}{2}}\,\chi[\mathcal{O}] = \frac{2}{3}L_2 G_{-\frac{1}{2}}\,\chi[\mathcal{O}]\,, \qquad \overline{G}_{\frac{3}{2}}\,\chi[\mathcal{O}] = \frac{2}{3}L_2 \overline{G}_{-\frac{1}{2}}\,\chi[\mathcal{O}]\,, \tag{3.17}$$

are vanishing because $G_{-\frac{1}{2}}\chi[\mathcal{O}]$ and $\overline{G}_{-\frac{1}{2}}\chi[\mathcal{O}]$ are Virasoro primaries (see appendix B). Hence, $\chi[\mathcal{O}]$ is a superprimary in two dimensions.

Let us next consider the second statement. Note that the requirement $E = \pm F$ is precisely the condition that one of the unitarity bounds in (3.4) is saturated since $j_{1,2} = r = 0$ here. Therefore, if a Higgs branch operator, $\mathcal{O}$, has $E = +F$ (or $E = -F$), then $\mathcal{O}$ is annihilated by $\mathcal{Q}^3{}_\alpha$ (or $\tilde{\mathcal{Q}}_{3\dot{\alpha}}$). This particularly means that the corresponding 2d operator, $\chi[\mathcal{O}]$, is annihilated by $G_{-\frac{1}{2}}$ (or $\overline{G}_{-\frac{1}{2}}$). Thus, we see that any Higgs branch operator with $E = F$ (or $E = -F$) maps to an anti-chiral (or chiral) superprimary in two dimensions.

4 Construction of the associated 2d chiral algebras

Based on the properties we uncovered in the previous section, here we proceed to the construction of the 2d chiral algebras associated in the sense of [5] to the 4d $\mathcal{N}=3$ rank-1 superconformal field theories, whose moduli space is of the form $\mathbb{C}^3/\mathbb{Z}_\ell$ where $\ell = 1, 2, 3, 4, 6$.

As shown in section 3.2, the 2d chiral algebra has an $\mathcal{N}=2$ super Virasoro algebra as a subalgebra. In addition, the Higgs branch operators as 4d $\mathcal{N}=2$ theory give rise to generators of the 2d chiral algebra, as was shown in [5]. In our setup, the Higgs branch operators in 4d are generated by W_+, W_- and J, whose dimensions are ℓ, ℓ, 2 and the U(1) charges are ℓ, $-\ell$, 0 respectively, with one relation

$$W_+ W_- \propto J^\ell. \tag{4.1}$$

As shown in section 3.2, $\chi[J]$ is the bottom component of the super energy momentum tensor, and $\chi[W_+]$ ($\chi[W_-]$) is a(n) (anti)chiral primary of dimension $\ell/2$. Below, we use the following shorthand notations for them:

$$J := \chi[J], \qquad W := \chi[W_+], \qquad \overline{W} := \chi[W_-]. \tag{4.2}$$

In the cases studied previously in the literature e.g. [5–7], it was often the case that the entire 2d chiral algebras were generated by taking repeated operator product expansions of the Higgs branch operators. We use this empirical feature as a working hypothesis and will find out that it leads to a consistent answer. As it is important, let us record here our Assumption:

> The 2d chiral algebra is generated by the $\mathcal{N}=2$ super Virasoro multiplet J, a bosonic chiral primary W and a bosonic antichiral primary $\overline{W}$, both of dimension $\ell/2$.

We will see below that for $\ell = 3$, this assumption uniquely fixes $c_{2\mathrm{d}}$ to be -15, consistent with the 4d central charge $c_{4\mathrm{d}} = (2\ell - 1)/4$ derived in (2.8) with the standard

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mapping $c_{2\mathrm{d}} = -12c_{4\mathrm{d}}$. Furthermore, we see that the construction automatically leads to a null relation of the form

$$W\overline{W} \propto J^3 + (\text{composite operators constructed from } J \text{ and (super)derivatives}), \tag{4.3}$$

reproducing the Higgs branch relation.

Similarly, for $\ell = 4$, the allowed $c_{2\mathrm{d}}$ are -21, -9 and 12, with the Higgs branch relation reproduced for $c_{2\mathrm{d}} = -21$, and for $\ell = 6$, the allowed $c_{2\mathrm{d}}$ are -33, -15 and 18, with the Higgs branch relation reproduced for $c_{2\mathrm{d}} = -33$.

Before proceeding, we note that the 2d chiral algebras satisfying the assumption above were constructed in [31, 32] for $\ell = 3$ but with W and $\overline{W}$ implicitly taken to be fermionic. This choice was more natural for a 2d unitary algebra, since the spin of W and $\overline{W}$ is half-integral. In this case the allowed central charge was $c_{2\mathrm{d}} = +9$. The 2d chiral algebras for $\ell = 4$ and $\ell = 6$ with bosonic $W_\pm$ were constructed in [33], with the allowed central charges as listed above. The null relation leading to the Higgs branch relation was not studied there.

4.1 Conventions

In the computations below, we use the 2d $\mathcal{N}=2$ holomorphic superspace, where the coordinate Z consists of the bosonic coordinate z and the fermionic coordinates θ and $\overline{\theta}$. We mostly follow the convention of Krivonos and Thielemans [34], where the Mathematica package `SOPEN2defs` we will use to compute the $\mathcal{N}=2$ superconformal operator product expansion was developed and described.[18]

We define the superderivatives to be

$$\mathcal{D} = \partial_\theta - \frac{1}{2}\overline{\theta}\partial_z, \qquad \overline{\mathcal{D}} = \partial_{\overline{\theta}} - \frac{1}{2}\theta\partial_z, \tag{4.4}$$

Then a chiral superfield W and an antichiral $\overline{W}$ satisfy

$$\overline{\mathcal{D}}W = 0, \qquad \mathcal{D}\overline{W} = 0 \tag{4.5}$$

respectively. The operator product expansions can be usefully done using covariant combinations

$$Z_{12} = z_1 - z_2 + \frac{1}{2}(\theta_1\overline{\theta}_2 - \theta_2\overline{\theta}_1), \qquad \theta_{12} = \theta_1 - \theta_2, \qquad \overline{\theta}_{12} = \overline{\theta}_1 - \overline{\theta}_2. \tag{4.6}$$

Then the energy momentum superfield $J(Z)$ has the operator product expansion

$$J(Z_1)J(Z_2) \sim \frac{c/3 + \theta_{12}\overline{\theta}_{12}J}{Z_{12}^2} + \frac{-\theta_{12}\mathcal{D}J + \overline{\theta}_{12}\overline{\mathcal{D}}J + \theta_{12}\overline{\theta}_{12}\partial J}{Z_{12}} \tag{4.7}$$

and a superprimary $\mathcal{O}$ with dimension Δ and U(1) charge F has the operator product expansion with J given by

$$J(Z_1)\mathcal{O}(Z_2) \sim \Delta\frac{\theta_{12}\overline{\theta}_{12}\mathcal{O}}{Z_{12}^2} + \frac{-F\mathcal{O} - \theta_{12}\mathcal{D}\mathcal{O} + \overline{\theta}_{12}\overline{\mathcal{D}}\mathcal{O} + \theta_{12}\overline{\theta}_{12}\partial\mathcal{O}}{Z_{12}}. \tag{4.8}$$

[18] Note that they called the operators satisfying $\overline{\mathcal{D}}W$ antichiral primary, but we call such operators chiral primary and vice versa. They also had a typo in their super OPE of the superconformal algebra in their (7), where $c/4$ should be $c/3$.

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Here, in the equations (4.7)and (4.8) and below, the operators on the right hand sides of the operator product expansions are always taken to be at $Z = Z_2$.

In our convention the (anti)chiral primaries are those with $\Delta = F/2$ ($\Delta = -F/2$). Note that our 2d algebra is not unitary, and therefore, $\Delta = F/2$ does not immediately imply that the antichiral derivative to vanish. Rather, we use the fact that W and $\overline{W}$ come from 4d Higgs branch operators W_+ and W_- of an $\mathcal{N}=3$ theory to conclude that they are (anti)chiral primaries.

The normal ordered product of two operators $\mathcal{O}_1$ and $\mathcal{O}_2$ is defined as the constant term, i.e. the term without any power of θ_{12}, $\overline{\theta}_{12}$ or Z_{12} in the operator product expansion of $\mathcal{O}_1$ and $\mathcal{O}_2$. Note that this does not always agree with the normal ordered product of two operators defined as the constant part of the operator product expansion of the bottom components on the non-superspace parametrized only by z. The normal ordered product of more than two operators are defined by recursively taking the operator product expansions from the right, i.e. $\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\cdots = (\mathcal{O}_1(\mathcal{O}_2(\mathcal{O}_3\cdots)))$.

4.2 Strategy

Our computational strategy is quite simple. We first require the operator product expansions of J with itself (4.7), and that W, $\overline{W}$ have the operator product expansions with respect to J given by (4.8) where $\Delta = \ell/2$ and $F = \pm\ell$, and that

$$W(Z_1)W(Z_2) \sim \text{regular}, \qquad \overline{W}(Z_1)\overline{W}(Z_2) \sim \text{regular}. \tag{4.9}$$

The only operator product expansion that needs to be worked out is that of W and $\overline{W}$.

Our assumption implies that only J and composite operators constructed out of it appear in the singular part of this operator product expansion. Demanding that $W(Z_1)\overline{W}(Z_2)$ to be annihilated by $\overline{\mathcal{D}}_1$ and $\mathcal{D}_2$, we find that it has the form

$$W(Z_1)\overline{W}(Z_2) \sim \sum_{d=1}^{\ell} \frac{1}{{Z_{12}}^d}\left(\frac{d}{2}\frac{\theta_{12}\overline{\theta}_{12}}{Z_{12}} + 1 + \theta_{12}\mathcal{D}\right)\mathcal{O}_{\ell-d} \tag{4.10}$$

where $\mathcal{O}_d$ is an operator of dimension d constructed out of J and its (super)derivatives. In particular, $\mathcal{O}_0$ is a constant and $\mathcal{O}_1 \propto J$. We arbitrarily choose $\mathcal{O}_1 = J$ to fix the normalization of W and $\overline{W}$. Demanding the closure of the Jacobi identity among J, W and $\overline{W}$ then fixes all other $\mathcal{O}_d$. Note that this is just the standard fact that when we fix the normalization of a primary (this time, the identity operator) in an operator product expansion, the contribution of all the descendants are automatically fixed. The explicit expressions for $\mathcal{O}_d$ are given in [33].

The only nontrivial procedure is to check the closure of the Jacobi identity among W, W and $\overline{W}$; the analysis of the Jacobi identity for the triple $\overline{W}$, $\overline{W}$ and W is similar, thanks to the discrete symmetry exchanging W and $\overline{W}$.

The computations can be performed easily and quickly using `SOPEN2defs`, the Mathematica package written by Krivonos and Thielemans [34]. On a 2012 notebook computer, the computation time was dominated by the time needed to type expressions into a notebook. The entire computation of Jacobi identities etc. took at most a few minutes. The

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Mathematica notebook detailing the computations below is available as ancillary files on the arXiv page for this paper.

4.3 Results

4.3.1 $\ell = 1$

When $\ell = 1$, the operator product expansion (4.10) just means that W and $\overline{W}$ are free, consisting of two bosons q, $\overline{q}$ of dimension 1/2 with $q\overline{q} \sim 1/z$ and two neutral fermions λ, $\overline{\lambda}$ of dimension 1 with $\lambda\overline{\lambda} \sim 1/z^2$. These are as they should be, since the 4d theory itself is free. We can define $J = W\overline{W}$ to reproduce the (rather trivial) Higgs branch relation. This J automatically has the correct operator product expansion (4.7) with $c_{2\mathrm{d}} = -3$, which agrees with the expected formula $c_{2\mathrm{d}} = -3(2\ell - 1)$. In fact this case was already essentially discussed in [5].

4.3.2 $\ell = 2$

When $\ell = 2$, the operator product expansion (4.10) together with the other operator product expansions mean that W, J, $\overline{W}$ generate a small $\mathcal{N}=4$ super Virasoro algebra. As such, the operator product expansions close for arbitrary value of $c_{2\mathrm{d}}$. Explicitly, we need to choose $\mathcal{O}_0 = -c/3$ and $\mathcal{O}_1 = J$.

It is still instructive to see when there can be null relations representing the Higgs branch relation $W\overline{W} \propto J^2$. In the language of the 2d chiral algebra, this should correspond to a null relation of the form

$$W\overline{W} - (a_1 J^2 + a_2 J' + a_3[\mathcal{D}, \overline{\mathcal{D}}]J) = 0. \tag{4.11}$$

Demanding that the left hand side to be an $\mathcal{N}=2$ primary, we find that only two choices are possible:

$$(c_{2\mathrm{d}}, a_1, a_2, a_3) = (-9, 1/4, 1/2, 1/4) \quad \text{or} \quad (6, -1/2, 1/2, 0). \tag{4.12}$$

It turns out, however, that only the first choice makes the left hand side of (4.11) to be an $\mathcal{N}=4$ primary. For example, with the second choice, repeated operator product expansions of (4.11) with W leads to an additional null relation $W^2 = 0$, which we do not like. We see that the Higgs branch relation is only compatible when $c_{2\mathrm{d}} = -9 = 3(2\ell - 1)$. Moreover, for $c_{2\mathrm{d}} = -9$, the null relation (4.11) is precisely the condition that the 2d stress tensor $\frac{1}{2}[\mathcal{D}, \overline{\mathcal{D}}]J$ is given by the Sugawara stress tensor associated with the SU(2) current formed by W, J and $\overline{W}$. This is consistent with the discussion in section 5.3.2 of [5]. Before proceeding, we note that the null relation above for $c_{2\mathrm{d}} = -9$ leads to new null operators given by

$$\begin{aligned} \mathcal{X} &= \mathcal{D}\partial W - J\mathcal{D}W + 2(\mathcal{D}J)W, \\ \overline{\mathcal{X}} &= \overline{\mathcal{D}}\partial\overline{W} + J\mathcal{D}\overline{W} - 2(\overline{\mathcal{D}}J)\overline{W}. \end{aligned} \tag{4.13}$$

4.3.3 $\ell = 3$

The fun starts at $\ell = 3$. We find that the Jacobi identity for W, W, $\overline{W}$ does not close for general values of $c_{2\mathrm{d}}$, since the failure of the Jacobi identity contains terms proportional to

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the identity operator. These terms all vanish when and only when $c_{2\mathrm{d}} = -15$. Note that this is exactly what the 4d $\mathcal{N}=3$ analysis dictates: $c_{2\mathrm{d}} = -12c_{4\mathrm{d}} = -3(2\ell - 1)$.

With this value of the central charge, the $W\overline{W}$ operator product expansion (4.10) is given by

$$\mathcal{O}_0 = \frac{5}{3}, \quad \mathcal{O}_1 = J, \quad \mathcal{O}_2 = \frac{1}{4}J^2 + \frac{1}{2}\partial J + \frac{1}{4}[\mathcal{D}, \overline{\mathcal{D}}]J. \tag{4.14}$$

The failure of the Jacobi identity for W, W, $\overline{W}$ now contains only terms proportional to

$$\mathcal{X} = \mathcal{D}\partial W - J\mathcal{D}W + 3(\mathcal{D}J)W \tag{4.15}$$

and $\mathcal{D}\mathcal{X} = -4\mathcal{D}J\mathcal{D}W$. One finds that $\mathcal{X}$ happens to be an $\mathcal{N}=2$ superprimary, so it is possible to impose the null relation $\mathcal{X} = 0$ as far as the operator product expansion with J is concerned. After imposing this null relation and its $\mathcal{N}=2$ descendants, we find that the Jacobi identity for W, W and $\overline{W}$ closes.

Similarly, we find that the Jacobi identity for W, $\overline{W}$ and $\overline{W}$ closes after demanding that the composite operator

$$\overline{\mathcal{X}} = \overline{\mathcal{D}}\partial\overline{W} + J\overline{\mathcal{D}W} - 3(\overline{\mathcal{D}}J)\overline{W} \tag{4.16}$$

is null.

One further finds that the operator product of $\mathcal{X}$ and W is regular, while that of $\mathcal{X}$ and $\overline{W}$ contains operators whose scaling dimensions are larger than that of $\mathcal{X}$. Similar statements hold $\overline{\mathcal{X}}$. This guarantees that $\mathcal{X}$ and $\overline{\mathcal{X}}$ are the operators with lowest dimension among the null states to be removed.

Another null state is obtained by taking the operator product expansion of $\overline{W}$ with $\mathcal{X}$, whose coefficient of $\overline{\theta}_{12}/Z_{12}^2$ is proportional to

$$\mathcal{Y} = 36W\overline{W} - (J^3 + 9(\partial J)J + 6J[\mathcal{D}, \overline{\mathcal{D}}]J + 6\overline{\mathcal{D}}J\mathcal{D}J + 6[\mathcal{D}, \overline{\mathcal{D}}]\partial J + 7\partial^2 J). \tag{4.17}$$

This operator is null, and correctly represents the 4d Higgs branch relation $W\overline{W} \propto J^3$.

4.3.4 $\ell = 4, 5, 6$

The analysis for ℓ larger than three can similarly be done. For $\ell = 4$, we find that the failure of the Jacobi identity for W, W, $\overline{W}$ contains terms proportional to the identity operator times $(c - 12)(c + 9)(c + 21)$. For each possible case $c = -21$, -9 and 12, we find that the Jacobi identity can be satisfied by imposing a null relation. But we find that the null relation is only consistent with the expected Higgs branch relation when $c_{2\mathrm{d}} = -21 = -3(2\ell - 1)$, again the value that follows from our $\mathcal{N}=3$ analysis.

For $\ell = 6$, we find that the values of $c_{2\mathrm{d}}$ allowed by the closure of the Jacobi identity is $c = -33$, -15 and 18. Again, the null relation is compatible with the Higgs branch relation only for $c_{2\mathrm{d}} = -33 = -3(2\ell - 1)$. In both cases $\ell = 4, 6$, we find that the basic null operators are

$$\begin{aligned} \mathcal{X} &= \mathcal{D}\partial W - J\mathcal{D}W + \ell(\mathcal{D}J)W, \\ \overline{\mathcal{X}} &= \overline{\mathcal{D}}\partial\overline{W} + J\overline{\mathcal{D}W} - \ell(\overline{\mathcal{D}}J)\overline{W}. \end{aligned} \tag{4.18}$$

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Note that the null operators for the cases $\ell = 2, 3$ are given by the same expressions, see (4.13), (4.15), (4.16).

We can also analyze the case $\ell = 5$. Here we find that the Jacobi identities are only consistent for $c_{2\mathrm{d}} = -27 = -3(2\ell - 1)$, and the null relation are generated by the same $\mathcal{X}$ and $\overline{\mathcal{X}}$ given in (4.18). A descendant by $\overline{W}$ of $\mathcal{X}$ generates a new null relation of the form $W\overline{W} \propto J^5 +$ (operators involving (super)derivatives). Note that the existence of the $\ell = 5$ super W-algebra does *not* contradict with the fact that there should not be the $\mathcal{N}=3$ theory with $\ell = 5$ in four dimensions.

The analysis so far suggests that there is a series of super W-algebras generated by the $\mathcal{N}=2$ super Virasoro algebra plus (anti)chiral primaries $W_\pm$ of dimension $\ell/2$ with $c_{2\mathrm{d}} = -3(2\ell - 1)$, with the basic null fields as given in (4.18). The operator product expansion of W and $\overline{W}$ has the form (4.10). A descendant of the null field seems to automatically give the relation of the form

$$\ell^2 W\overline{W} = \frac{(2(\ell-1))!}{\ell!} J^\ell + (\text{descendants}), \tag{4.19}$$

where the coefficients are guessed from the examples so far. Note that we normalized W and $\overline{W}$ by demanding $\mathcal{O}_1 = J$ in (4.10). It would be interesting, as a question purely in two dimensions, to see whether such a series of 2d chiral algebras indeed exists.

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A The 4d $\mathcal{N}=3$ superconformal algebras

The 4d $\mathcal{N}$=3 superconformal algebra is $\mathfrak{su}(2,2|3)$ whose bosonic part is $\mathfrak{su}(2,2)\oplus\mathfrak{u}(3)$. The $\mathfrak{u}(3)$ R-symmetry is generated by $\mathcal{R}_I{}^J$ for $I, J = 1, 2, 3$ subject to

$$\left[\mathcal{R}^I{}_J, \mathcal{R}^K{}_L\right] = \delta^K_J \mathcal{R}^I{}_L - \delta^I_L \mathcal{R}^K{}_J \ . \tag{A.1}$$

The fermionic generators of $\mathfrak{su}(2,2|3)$ are $\mathcal{Q}^I{}_\alpha$, $\tilde{\mathcal{Q}}_{I\dot\alpha}$ and $\mathcal{S}_I{}^\alpha$, $\tilde{\mathcal{S}}^{I\dot\alpha}$ for $I = 1, 2, 3$, $\alpha = \pm$ and $\dot\alpha = \dot\pm$. Their R-charges can be read off from

$$\begin{aligned}
[\mathcal{R}^I{}_J,\, \mathcal{Q}^K{}_\alpha] &= \delta^K_J \mathcal{Q}^I{}_\alpha - \frac{1}{4}\delta^I_J \mathcal{Q}^K{}_\alpha \ , & [\mathcal{R}^I{}_J,\, \tilde{\mathcal{Q}}_{K\dot\alpha}] &= -\delta^I_K \tilde{\mathcal{Q}}_{J\dot\alpha} + \frac{1}{4}\delta^I_J \tilde{\mathcal{Q}}_{K\dot\alpha} \ , \\
[\mathcal{R}^I{}_J,\, \mathcal{S}_K{}^\alpha] &= -\delta^I_K \mathcal{S}_J{}^\alpha + \frac{1}{4}\delta^I_J \mathcal{S}_K{}^\alpha \ , & [\mathcal{R}^I{}_J,\, \tilde{\mathcal{S}}^{K\dot\alpha}] &= \delta^K_J \tilde{\mathcal{S}}^{I\dot\alpha} - \frac{1}{4}\delta^I_J \tilde{\mathcal{S}}^{K\dot\alpha} \ .
\end{aligned} \tag{A.2}$$

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The anti-commutation relations among $\mathcal{Q}^I{}_\alpha$, $\tilde{\mathcal{Q}}_{I\dot{\alpha}}$, $\mathcal{S}_I{}^\alpha$, $\tilde{\mathcal{S}}^{I\dot{\alpha}}$ are given by

$$\begin{aligned}
\{\mathcal{Q}^I{}_\alpha,\, \mathcal{S}_J{}^\beta\} &= 2\delta^I_J\delta^\beta_\alpha\mathcal{H} + 4\delta^I_J\mathcal{M}_\alpha{}^\beta - 4\delta^\beta_\alpha\mathcal{R}^I{}_J\,, & \{\mathcal{Q}^I{}_\alpha,\, \tilde{\mathcal{Q}}_{J\dot{\alpha}}\} &= 2\delta^I_J\mathcal{P}_{\alpha\dot{\alpha}}\,,\\
\{\tilde{\mathcal{S}}^{I\dot{\alpha}},\, \tilde{\mathcal{Q}}_{J\dot{\beta}}\} &= 2\delta^I_J\delta^{\dot{\alpha}}_{\dot{\beta}}\mathcal{H} - 4\delta^I_J\tilde{\mathcal{M}}^{\dot{\alpha}}{}_{\dot{\beta}} + 4\delta^{\dot{\alpha}}_{\dot{\beta}}\mathcal{R}^I{}_J\,, & \{\tilde{\mathcal{S}}^{I\dot{\alpha}},\, \mathcal{S}_J{}^\alpha\} &= 2\delta^I_J\mathcal{K}^{\dot{\alpha}\alpha}\,.
\end{aligned} \tag{A.3}$$

Here $\mathcal{H}$ is the Hamiltonian whose eigenvalue is the scaling dimension, and $\mathcal{M}_\alpha{}^\beta$, $\tilde{\mathcal{M}}^{\dot{\alpha}}{}_{\dot{\beta}}$ are generators of $\mathfrak{so}(4)$ subalgebra of $\mathfrak{su}(2,2)$. They satisfy

$$\begin{aligned}
&[\mathcal{H},\, \mathcal{Q}^I{}_\alpha] = \frac{1}{2}\mathcal{Q}^I_\alpha\,, \quad [\mathcal{H},\, \tilde{\mathcal{Q}}_{I\dot{\alpha}}] = \frac{1}{2}\tilde{\mathcal{Q}}_{\dot{\alpha}}\,, && [\mathcal{H},\, \mathcal{S}_I{}^\alpha] = -\frac{1}{2}\mathcal{S}_I{}^\alpha\,, \quad [\mathcal{H},\, \tilde{\mathcal{S}}^{I\dot{\alpha}}] = -\frac{1}{2}\tilde{\mathcal{S}}^{I\dot{\alpha}}\,,\\
&[\mathcal{M}_\alpha{}^\beta,\, \mathcal{Q}^I{}_\gamma] = \delta^\beta_\gamma\mathcal{Q}^I{}_\alpha - \frac{1}{2}\delta^\beta_\alpha\mathcal{Q}^I{}_\gamma\,, && [\mathcal{M}_\alpha{}^\beta, \mathcal{S}_I{}^\gamma] = -\delta^\gamma_\alpha\mathcal{S}_I{}^\alpha + \frac{1}{2}\delta^\beta_\alpha\mathcal{S}_I{}^\gamma\,,\\
&[\tilde{\mathcal{M}}^{\dot{\alpha}}{}_{\dot{\beta}},\, \tilde{\mathcal{Q}}_{I\dot{\gamma}}] = -\delta^{\dot{\alpha}}_{\dot{\gamma}}\tilde{\mathcal{Q}}_{I\dot{\beta}} + \frac{1}{2}\delta^{\dot{\alpha}}_{\dot{\beta}}\tilde{\mathcal{Q}}_{I\dot{\gamma}}\,, && [\tilde{\mathcal{M}}^{\dot{\alpha}}{}_{\dot{\beta}},\, \tilde{\mathcal{S}}^{I\dot{\gamma}}] = \delta^{\dot{\gamma}}_{\dot{\beta}}\tilde{\mathcal{S}}^{I\dot{\alpha}} - \frac{1}{2}\delta^{\dot{\alpha}}_{\dot{\beta}}\tilde{\mathcal{S}}^{I\dot{\gamma}}\,.
\end{aligned} \tag{A.4}$$

On the other hand, $P_{\alpha\dot{\alpha}}$ and $\mathcal{K}^{\dot{\alpha}\alpha}$ have the following commutation relations with the supercharges:

$$\begin{aligned}
&[\mathcal{K}^{\dot{\alpha}\alpha},\, \mathcal{Q}^I{}_\beta] = 2\delta^\alpha_\beta\,\tilde{\mathcal{S}}^{I\dot{\alpha}}\,, && [\mathcal{K}^{\dot{\alpha}\alpha}\tilde{\mathcal{Q}}_{I\dot{\beta}}] = 2\delta^{\dot{\alpha}}_{\dot{\beta}}\,\mathcal{S}_I{}^\alpha\,,\\
&[\mathcal{P}_{\alpha\dot{\alpha}},\, \mathcal{S}_I{}^\beta] = -2\delta^\beta_\alpha\tilde{\mathcal{Q}}_{I\dot{\alpha}}\,, && [\mathcal{P}_{\alpha\dot{\alpha}},\, \tilde{\mathcal{S}}^{I\dot{\beta}}] = -2\delta^{\dot{\beta}}_{\dot{\alpha}}\tilde{\mathcal{Q}}^I{}_\alpha\,.
\end{aligned} \tag{A.5}$$

The hermiticity is given by

$$\begin{aligned}
&(\mathcal{Q}^I{}_\alpha)^\dagger = \mathcal{S}_I{}^\alpha\,, && (\tilde{\mathcal{Q}}_{I\dot{\alpha}})^\dagger = \tilde{\mathcal{S}}^{I\dot{\alpha}}\,, && (\mathcal{R}^I{}_J)^\dagger = \mathcal{R}^J{}_I\,,\\
&(\mathcal{M}_\alpha{}^\beta)^\dagger = \mathcal{M}_\beta{}^\alpha\,, && (\tilde{\mathcal{M}}_\alpha{}^\beta)^\dagger = \tilde{\mathcal{M}}_\beta{}^\alpha\,, && (\mathcal{H})^\dagger = \mathcal{H}\,, \qquad (\mathcal{P}_{\alpha\dot{\alpha}})^\dagger = \mathcal{K}^{\dot{\alpha}\alpha}\,.
\end{aligned} \tag{A.6}$$

B Detailed computations

We here show that $G \equiv \chi[\mathcal{Q}^3{}_+ J^{11}]$ and $\overline{G} \equiv \chi[\tilde{\mathcal{Q}}_{3\dot{+}} J^{11}]$ are Virasoro primaries. To that end, we first recall the argument of sub-section 3.2.4 of [5], where the authors proved that any Hall-Littlewood (HL) operators map to Virasoro primaries in two dimensions. Here, HL operators are defined as local operators annihilated by $\mathcal{Q}^1{}_-$, $\tilde{\mathcal{Q}}_{2\pm}$ and their conjugates, and therefore are Schur operators. The unitarity bounds in (3.5) imply that HL operators have $E = 2R - r$, $j_1 = -r$ and $j_2 = 0$. Now, suppose that $\{\mathcal{O}_i\}$ is a basis of the space of Schur operators, and that $\mathcal{O}_1$ is a HL operator. We also use a short-hand notation $\hat{\mathcal{O}}_i \equiv \chi[\mathcal{O}_i]$ for the corresponding 2d operators. Then the OPE of $\hat{\mathcal{O}}_1$ and the 2d stress tensor $T(z)$ is of the form

$$T(z)\,\hat{\mathcal{O}}_1(0) = \sum_i \frac{\hat{\mathcal{O}}_i(0)}{z^{2+h_1-h_i}}\,, \tag{B.1}$$

where h_i is the eigenvalue of L_0 for $\hat{\mathcal{O}}_i$. From equation (3.6) of [5], h_i is given by

$$h_i = R^{(i)} + j_1^{(i)} + j_2^{(i)}, \tag{B.2}$$

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where $R^{(i)}$ and $(j_1^{(i)}, j_2^{(i)})$ are the $\mathrm{SU}(2)_R$ charge and the spins of $\mathcal{O}_i$. Since any Schur operator satisfy $r = j_2 - j_1$, this is equivalent to

$$h_i = R^{(i)} + |r^{(i)}| + 2\min\left(j_1^{(i)}, j_2^{(i)}\right), \tag{B.3}$$

where $r^{(i)}$ is the $\mathrm{U}(1)_r$ charge of O_i. Since HL operators have $\min(j_1, j_2) = j_2 = 0$, we have

$$h_1 = R^{(1)} + |r^{(1)}|. \tag{B.4}$$

Therefore (B.1) is rewritten as

$$T(z)\hat{\mathcal{O}}_1(0) = \sum_{\mathcal{O}_i:\ \text{Schur}} \frac{\hat{\mathcal{O}}_i(0)}{z^{2+\Delta R^{(i)} - 2\min\left(j_1^{(i)}, j_2^{(i)}\right)}}, \tag{B.5}$$

where $\Delta R^{(i)} \equiv R^{(1)} - R^{(i)}$. The $\mathrm{U}(1)_R$ charge dependence drops out because $T(z)$ is neutral under $\mathrm{U}(1)_R$.

Recall here that the 2d stress tensor $T(z)$ is given by a linear combination of the 4d $\mathrm{SU}(2)_R$ current $\mathcal{J}^{ij}_{+\dot{+}}$ [5]. Since the $\mathrm{SU}(2)_R$ current is an $\mathrm{SU}(2)_R$ triplet, $T(z)$ is a linear combination of operators with $R = 0, \pm 1$. Therefore an OPE with $T(z)$ changes the $\mathrm{SU}(2)_R$ charge by ± 1 or 0, namely

$$\Delta R^{(i)} = \pm 1,\ 0\,, \tag{B.6}$$

depending on i. Moreover, from (3.8), we see that Schur operators have

$$j_1 = \frac{E - 2R - r}{2}, \qquad j_2 = \frac{E - 2R + r}{2}, \tag{B.7}$$

whose right-hand sides are positive semi-definite because of the unitarity bounds (3.5). Therefore the worst possible singularity in (B.5) is of order three. On the other hand, since any Hall-Littlewood operator is a conformal primary, it is annihilated by L_1.[19] Therefore the singularity of order three in (B.5) vanishes. This means that $\hat{\mathcal{O}}_1(z)$ is a Virasoro primary.

Now we generalize the above discussion to the cases in which $\mathcal{O}_1 = \mathcal{Q}^3{}_+ J^{11}$ and $\mathcal{O}_1 = \tilde{\mathcal{Q}}_{3\dot{+}} J^{11}$. The only difference is that $\mathcal{O}_1$ is no longer a HL operator, and therefore (B.5) is replaced by

$$T(z)\hat{\mathcal{O}}_1(0) = \sum_{\mathcal{O}_i:\ \text{Schur}} \frac{\hat{\mathcal{O}}_i(0)}{z^{\frac{5}{2}+\Delta R^{(i)} - 2\min\left(j_1^{(i)}, j_2^{(i)}\right)}}. \tag{B.8}$$

However, the worst possible singularity is still of order three since, as discussed in [5], any 2d OPE corresponding to a 4d OPE should be single-valued. Moreover, since $\mathcal{Q}^3{}_+ J^{11}$ and $\tilde{\mathcal{Q}}_{3\dot{+}} J^{11}$ are conformal primaries, the corresponding 2d operator $\hat{\mathcal{O}}_1$ is annihilated by L_1. Therefore the worst singularity in the above OPE is of order two. Thus, we see that $G \equiv \chi[\mathcal{Q}^3{}_+ J_F^{11}]$ and $\overline{G} \equiv \chi[\tilde{\mathcal{Q}}_{3\dot{+}} J_F^{11}]$ are Virasoro primaries.

Note here that exactly the same argument tells us that, for any Higgs branch operator $\mathcal{O}$, it follows that $\chi[\mathcal{O}]$, $\chi[\mathcal{Q}^3{}_+\mathcal{O}]$ and $\chi[\tilde{\mathcal{Q}}_{3\dot{+}}\mathcal{O}]$ are Virasoro primaries.[20]

[19] Recall that L_1 is identified with $\mathcal{K}^{\dot{+}+}$. See equation (2.19) of [5].

[20] Note also that the same argument fails in the case of $\hat{\mathcal{O}}_1 = \chi[\frac{1}{2}[\mathcal{Q}^3{}_+,\ \tilde{\mathcal{Q}}_{3\dot{+}}]J_F^{11}]$. In this case, the worst possible singularity in (B.5) is of order four, which is consistent with the identification of $\chi[\frac{1}{2}[\mathcal{Q}^3{}_+,\ \tilde{\mathcal{Q}}_{3\dot{+}}]J^{11}]$ as the 2d stress tensor.

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