

ON A BOUND FOR THE RATE OF CONVERGENCE IN THE MULTIDIMENSIONAL CENTRAL LIMIT THEOREM

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1. Introduction

In recent years many papers concerned with estimation of the rate of convergence in the central limit theorem in R^k have appeared (see [1], [2], [6]–[8], [10], [13]–[16]). They have significantly extended our knowledge in this area. We shall mention here two recent results which are most closely related to the estimate obtained in the present paper.

V. Rotar [14], applying the method of characteristic functions, obtained a “nonuniform” estimate. It was a generalization of the corresponding one dimensional result of S. Nagaev [11] which had been extended to the case of differently distributed summands by A. Bikyalis [9]. Under the assumption that the summands are identically distributed, Rotar’s result can be formulated in the following manner. If P_n is the distribution of the normalized sum $n^{-1/2} \sum_{i=1}^n \xi_i$ of nondegenerate, independent, identically distributed random variables with values in R^k such that $\mathcal{E}\xi_1 = 0$, $\mathcal{E}|\xi_1|^3 < \infty$, and Q is the normal distribution with the same first and second moments as ξ_1 , then for any absolutely measurable convex set $E \subset R^k$

$$(1.1) \quad |P_n(E) - Q(E)| \leq c(k) \frac{\mathcal{E}(\Delta^{-1}\xi_1, \xi_1)^{3/2}}{1 + \nu_\Delta^3(E)} n^{-1/2},$$

where $c(k)$ depends only on k , Δ is the covariance matrix of ξ_1 and $\nu_\Delta(E)$ is defined in formula (3.2) below.

On the other hand, V. Paulauskas [13], applying the method of composition of H. Bergström [3]–[6] and using the results of the author [16], derived a bound in terms of “pseudo moments” which, in the notation introduced above, takes the form

$$(1.2) \quad |P_n(E) - Q(E)| \leq c(k) \nu_3' n^{-1/2},$$

where

$$(1.3) \quad \nu_3' = \max(\nu_3, \nu_3^{1/4}), \quad \nu_3 = \int_{R^k} (\Delta^{-1}x, x)^{3/2} |P - Q|(dx).$$

(Here $|P - Q|$ denotes the variation of the measure $P - Q$.)

From the theorem to be proved in this paper by using the method of composition, both results (1.1) and (1.2) follow (see remarks to the theorem). Furthermore, the resultant bound differs from (1.2) in that v'_3 becomes $\tilde{v}_3 = \max(v_3, v_3^{k/(k+3)})$ which not only improves (1.2) but also is the best possible result in the sense that it is impossible to replace the exponent $k/(k+3)$ by $m > k/(k+3)$. It is appropriate also to mention that our theorem is new even in the one dimensional case where it is an improvement on the classical bound of Berry-Esseen. Finally we note that although this theorem is formulated and proved for convex sets E , it can be extended, in the spirit of R. Bhattacharya [7], to sets of a more general type.

Hereafter the following notation will be used: $c, c(k)$, with or without indices will denote, respectively, absolute constants and constants depending only on the dimension k (the same symbol may be used for different constants); \mathcal{C} will denote the class of all measurable convex subsets of R^k (by measurability we always mean absolute measurability); \mathcal{P}_k denotes the set of all nondegenerate probability measures on R^k with mean zero and finite third moments; $|\mu|$ for any signed measure μ on R^k denotes its variation; finally, for $T > 0$, N_T (and φ_T) will denote the normal distribution (and its density) with mean zero and covariance matrix $T^{-2}I$, where I is the $(k \times k)$ identity matrix. In addition, the partial derivatives $(\partial/\partial x_u)f, (\partial^2/\partial x_u \partial x_v)f, \dots$ for any differentiable function f on R^k will be denoted, respectively, by $\partial_u^1 f, \partial_{u,v}^2 f, \dots$, for the sake of brevity.

2. Some lemmas

In proving the theorem, we shall use a series of lemmas to which this section is devoted.

LEMMA 2.1 *For any probability measure P on R^k*

$$(2.1) \quad \sup_{E \in \mathcal{C}} |P_n(E) - N_1(E)| \leq 2 \sup_{E \in \mathcal{C}} |(P - N_1) * N_T|(E) + 24 \left[\frac{\Gamma[(k+1)/2]}{\Gamma(k/2)} \right]^2 T^{-1}.$$

Lemma 2.1 can be proved in exactly the same way as Lemma 2 in [16]. The only difference is that instead of the bound used there for $N_1[\delta(E, \pm h)]$, where $\delta(E, h) = E^h - E, \delta(E, -h) = (E^c)^h - E^c$, and $A^h, h > 0$, for any $A \subset R^k$ is an h -neighborhood of A , it is necessary to use a more precise estimate which follows from the results of B. von Bahr [2], namely, for any measurable convex set E for all $h > 0$

$$(2.2) \quad N_1(\delta(E, \pm h)) \leq \sqrt{2} \frac{\Gamma[(k+1)/2]}{\Gamma(k/2)} h.$$

REMARK 2.1. It may be that the dependence of the right side of 2.2 on k is unnecessary, that is, that a bound of the form

$$(2.3) \quad N_1(\delta(E, \pm h)) \leq ch,$$

is valid, where c is an absolute constant. In any case, this is true for spheres.

Indeed, by immediate calculation one is easily convinced of the validity of 2.3 for spheres centered at zero. From this and the formula representing the non-central χ^2 distribution in terms of central χ^2 distributions (see, for example, [17]), it follows that for a sphere S with center at (a_1, \dots, a_k) of arbitrary radius z

$$(2.4) \quad N_1(\delta(S, \pm h)) \leq e^{-a^2/2} \sum_{i=0}^{\infty} \frac{1}{i!} \left(\frac{a^2}{2}\right)^i N_1^{(i)}(\delta(S^{(i)}, \pm h)) \leq ch,$$

where $a^2 = \sum_{j=1}^k a_j^2$, $N_1^{(i)}$ is the $(k + 2i)$ dimensional normal distribution with mean zero and identity covariance matrix, and $S^{(i)}$ is a sphere in R^{k+2i} of radius z with center at zero.

LEMMA 2.2. *If $\{\xi_i\}$, for $i = 1, 2, \dots$, is a sequence of independent random variables with common distribution $P \in \mathcal{P}_k$, P_n the distribution of the normalized sum $n^{-1/2} \sum_{i=1}^n \xi_i$, and Q the normal distribution with the same first and second moments as P , then*

$$(2.5) \quad \sup_{E \in \mathcal{E}} |P_n(E) - Q(E)| \leq c_1(k) \bar{v}_3 n^{-1/2}, \quad n = 1, 2, \dots,$$

where

$$(2.6) \quad \bar{v}_3 = |P - Q|(x : (\Delta^{-1}x, x) \leq 1) + \int_{(\Delta^{-1}x, x) > 1} (\Delta^{-1}x, x)^{3/2} |P - Q|(dx)$$

and Δ is the covariance matrix of the distribution P . The constant $c_1(k) \leq c'k^{5/2}$.

PROOF. The proof of this lemma differs little from the proof of Theorem 1 in [16] which, as will be clear later, it makes more precise.

First, let us note that it is sufficient to prove the lemma in the case where $\Delta = I$. Indeed, let

$$(2.7) \quad \{t_i = (t_{i,1}, \dots, t_{i,k}), i = 1, 2, \dots, k\}$$

be elements of R^k such that the real random variables (t_i, ξ_1) , $i = 1, 2, \dots, k$, are uncorrelated. Denote by A the matrix $(t_{i,j}/\mathcal{E}^{1/2}(t_i, \xi_1)^2)$. Let \bar{P} be the distribution of the variable $A\xi_1$ and \bar{P}_n the distribution of the normalized sum $n^{-1/2} \sum_{i=1}^n A\xi_i$. Obviously, the covariance matrix of \bar{P} is equal to I .

Because $A\mathcal{E} = \mathcal{E}$ and for any measurable set E , $P_n(E) = \bar{P}_n(AE)$, $Q(E) = N_1(AE)$, it follows that

$$(2.8) \quad \sup_{E \in \mathcal{E}} |P_n(E) - Q(E)| = \sup_{E \in \mathcal{E}} |\bar{P}_n(E) - N_1(E)|.$$

On the other hand, since $\xi_1 = A^{-1}(A\xi_1)$ and $A\xi_1$ has the identity covariance matrix, $\Delta = (A^{-1})(A^{-1})^*$. Consequently,

$$(2.9) \quad \begin{aligned} (\Delta^{-1}x, x)^{3/2} &= |Ax|^3, & \{x : (\Delta^{-1}x, x) \leq 1\} &= A^{-1}S_1, \\ \{x : (\Delta^{-1}x, x) > 1\} &= A^{-1}S_1^c, \end{aligned}$$

where S is a sphere with unit radius and center at zero and, therefore,

$$(2.10) \quad \begin{aligned} \bar{v}_3 &= |P - Q|(A^{-1}S_1) + \int_{A^{-1}S_1^c} |Ax|^3 |P - Q|(dx) \\ &= |\bar{P} - N_1|(S_1) + \int_{S_1^c} |x|^3 |\bar{P} - N_1|(dx). \end{aligned}$$

Now, keeping the notation of [16], we shall indicate only those modifications in the proof of Theorem 1 of [16] which are necessary for the proof of this lemma.

In the first place, for any measurable set E

$$(2.11) \quad \begin{aligned} |(\bar{P} - N_1)(E)| &\leq \frac{1}{2} |\bar{P} - N_1|(R^k) \\ &\leq \frac{1}{2} \left(|\bar{P} - N_1|(S_1) + \int_{S_1^c} |x|^3 |\bar{P} - N_1|(dx) \right) = \frac{1}{2} \bar{v}_3, \end{aligned}$$

so that the lemma is true for $n = 1$.

We must now bound $n|[U_0 * (P_{(n)} - N_{n^{1/2}})](E)|$ in the following manner. (Compare (21) of [16]; below, for brevity, we shall put $H_1 = P_{(n)} - N_{n^{1/2}}$.)

$$(2.12) \quad \begin{aligned} n|(U_0 * H_1)(E)| &\leq \frac{c_1 \tau_0^3}{6n^{1/2}} \int_{R^k} \left(\sum_{j=1}^k |x_j| \right)^3 |\bar{P} - N_1|(dx) \\ &\leq \frac{\sqrt{2}}{3} \frac{c_1 k^{3/2} \bar{v}_3}{n^{1/2}}. \end{aligned}$$

(Here we have used the inequality

$$(2.13) \quad v_s \leq \bar{v}_3, \quad s \leq 3,$$

for $s = 3$, where

$$(2.14) \quad v_s = \int_{R^k} |x|^s |\bar{P} - N_1|(dx) = \int_{R^k} (\Delta^{-1}x, x)^{s/2} |P - Q|(dx),$$

is the s th pseudo moment of the distribution P .) Analogously, the estimates of the terms $|(U_i * H_1)(E)|$, for $i = 1, 2, \dots, n - 2$, (compare equation (22) of [16]) are changed to

$$(2.15) \quad |(U_i * H_1)(E)| \leq \frac{c_1 c_1(k) k^{3/2} \bar{v}_3^2}{6n^{3/2}} \frac{\tau_i^3}{i^{1/2}}, \quad i = 1, 2, \dots, n - 2.$$

Furthermore,

$$(2.16) \quad \begin{aligned} |(U_{n-1} * H_1)(E)| &\leq \frac{1}{n^{1/2}} \sum_{u=1}^k \sup_x |\partial_u^1 g_{n-1}(x)| \int_{R^k} |x_u| |\bar{P} - N_1|(dx), \\ |\partial_u^1 g_{n-1}(x)| &\leq \sup_{E \in \mathcal{E}} |(P_{(n)}^{n-1} - N_{n^{1/2}}^{n-1})(E)| \int_{R^k} |\partial_u^1 \varphi_{\tau_{n-1}}(x)| dx, \end{aligned}$$

by the induction assumption

$$(2.17) \quad \sup_{E \in \mathcal{E}} |(P_{(n)}^{n-1} - N_{n^{1/2}}^{n-1})(E)| \leq c_1(k) \bar{v}_3 (n-1)^{-1/2} \leq \sqrt{2} c_1(k) \bar{v}_3 n^{-1/2},$$

and it is easy to calculate that, for $\tau > 0$

$$(2.18) \quad \int_{R^k} |\partial_u^1 \varphi_\tau(x)| dx \leq \left(\frac{2}{\pi}\right)^{1/2} \tau.$$

In this way, using (2.13), we have

$$(2.19) \quad |(U_{n-1} * H_1)(E)| \leq \frac{2}{\sqrt{\pi}} \frac{c_1(k)k^{1/2}\bar{v}_3^2 T}{n}.$$

From formulas (2.12), (2.15), (2.19), and (23) of [16], and Lemma 2.1 it follows now that for any $T > 0$

$$(2.20) \quad \sup_{E \in \mathcal{E}} |\bar{P}_n(E) - N_1(E)| \leq \alpha_1 \frac{\bar{v}_3}{n^{1/2}} + \beta_1 \frac{c_1(k)\bar{v}_3^2 T}{n} + \gamma_1 T^{-1},$$

where

$$(2.21) \quad \alpha_1 = \frac{2\sqrt{2}}{3} c_1 k^{3/2}, \quad \beta_1 = \frac{2\sqrt{3}}{3} c_1 k^{3/2} + \frac{4}{\sqrt{\pi}} k^{1/2},$$

$$\gamma_1 = 24 \left[\frac{\Gamma[(k+1)/2]}{\Gamma(k/2)} \right]^2.$$

By placing $T = (\gamma_1 n / \beta_1 c_1(k))^{1/2} (1/\bar{v}_3)$, in (2.20), we obtain the desired result in which we can take $c_1(k)$ to be

$$(2.22) \quad c_1(k) = \beta_1 \gamma_1 \left[1 + \left(1 + \frac{\alpha_1}{\beta_1 \gamma_1} \right)^{1/2} \right]^2 = c' k^{5/2} (1 + o(1)) \text{ as } k \rightarrow \infty.$$

The lemma is proved.

LEMMA 2.3. Let I be the $(k \times k)$ -identity matrix and let $\nu_I(\cdot)$ be defined by formula (3.2) (that is, $\nu_I(E)$ is the distance from 0 to the boundary of the set E). For arbitrary $E \subset R^k, x \in R^k, \lambda \in R^1$

$$(2.23) \quad |\nu_I(E) - \nu_I(E + x)| \leq |x|, \quad \nu_I(\lambda E) = |\lambda| \nu_I(E).$$

The proof of the lemma is elementary and we shall omit it.

LEMMA 2.4. For any probability measure $P \in \mathcal{P}_k$ with the identity covariance matrix I , for arbitrary $T \geq 1$

$$(2.24) \quad \sup_{E \in \mathcal{E}} \nu_I^3(E) |P(E) - N_1(E)| \leq 6 \sup_{E \in \mathcal{E}} \nu_I^3(E) |(P - N_1) * N_T|(E) + c_2(k) T^{-1},$$

where $\nu_I(\cdot)$ is the same as in the preceding lemma, and $c_2(k) \leq ck^{5/2}$.

This lemma is essentially proved in [14].

LEMMA 2.5. For all $s, t = 0, 1, 2, 3; \tau > 0$,

$$(2.25) \quad \int_{R^k} |x|^s |\partial^t \varphi_\tau(x)| dx \leq ck^{s/2} \tau^{t-s},$$

where ∂^t is any partial derivative of t th order with respect to x .

The assertion of the lemma is verified by simple calculation.

LEMMA 2.6. Let E be an arbitrary subset of R^k and let $\nu_\tau(\cdot)$ be defined as in Lemma 2.3. We put

$$(2.26) \quad \bar{R}(x, z) = \int_{R^k} \nu_\tau(E - y) (\varphi_1(x - y) - \varphi_1(z - y)) dy.$$

Then, for all $\tau > 0$, $x, z \in R^k$

$$(2.27) \quad |\bar{R}(x, z)| \leq |x - z|,$$

$$(2.28) \quad |\partial^t \bar{R}(x, z)| \leq ck^{1/2}, \quad t = 1, 2, 3,$$

$$(2.29) \quad \int_{R^k} |\partial^t [\bar{R}^s(x, z) \varphi_\tau(x - z)]| dz \\ \leq \begin{cases} ck^{s/2} \tau^{-s}, & t = 0, s = 0, 1, 2, 3, \\ ck^{s/2} \tau^{-s} (\tau + \dots + \tau^t), & t, s = 1, 2, 3, \\ c\tau^t, & s = 0, t = 0, 1, 2, 3, \end{cases}$$

where ∂^t is any partial derivative of t th order with respect to x .

PROOF. The inequality (2.27) follows from Lemma 2.3:

$$(2.30) \quad |\bar{R}(x, z)| = \left| \int_{R^k} (\nu_\tau(E - x - y) - \nu_\tau(E - z - y)) \varphi_1(y) dy \right| \leq |x - z|.$$

For the proof of (2.28) we note that

$$(2.31) \quad \int_{R^k} \partial^t \varphi_\tau(x) dx = 0, \quad t = 1, 2, 3,$$

and, consequently, by Lemmas 2.3 and 2.5

$$(2.32) \quad |\partial^t \bar{R}(x, z)| = \left| \int_{R^k} \nu_\tau(E - y) \partial^t \varphi_1(x - y) dy \right| \\ = \left| \int_{R^k} (\nu_\tau(E - x) - \nu_\tau(E - y)) \partial^t \varphi_1(x - y) dy \right| \\ \leq \int_{R^k} |x| |\partial^t \varphi_1(x)| dx \leq ck^{1/2}.$$

Finally, we obtain inequality (2.29), by a simple computation using (2.27), (2.28), and Lemma 2.5.

LEMMA 2.7. Let E be an arbitrary measurable subset of R^k and let $\nu_\tau(\cdot)$ be defined as in Lemma 2.3. Set

$$(2.33) \quad \bar{\nu}(x) = \nu_\tau(E - x), \quad \bar{R}(x) = \int_{R^k} \bar{\nu}(y) \varphi_1(x - y) dy,$$

and for $\tau > 0$ let

$$(2.34) \quad h_\tau(x) = \begin{cases} N_\tau(E - x) & \text{if } 0 \notin E, \\ 1 - N_\tau(E - x) & \text{if } 0 \in E. \end{cases}$$

Then, for all $x \in R^k$

$$(2.35) \quad |\bar{R}(x) - \bar{z}(x)| \leq k^{1/2},$$

and for $|x| < \bar{z}(0)$

$$(2.36) \quad |\partial^t[\bar{R}^s(x)h_t(x)]| \leq \begin{cases} ck^{s/2}(1 + \tau^{-s}), & t = 0, s = 0, 1, 2, 3, \\ ck^{s/2}(\tau^{-(s-1)} + \dots + \tau^t), & s, t = 1, 2, 3, \\ c\tau^t, & s = 0, t = 0, 1, 2, 3. \end{cases}$$

PROOF. Inequality (2.35) follows immediately from Lemma 2.3:

$$(2.37) \quad |\bar{R}(x) - \bar{z}(x)| = \left| \int_{R^k} (\bar{z}(x+z) - \bar{z}(x))\varphi_1(z) dz \right| \leq \int_{R^k} |z|\varphi_1(z) dz \leq k^{1/2}.$$

Let us bound $|\bar{z}^s(x)\partial^t h_t(x)|$, $s, t = 0, 1, 2, 3$. Using the notation (3.6) for $t \neq 0$ because of equation (2.31) and Lemma 2.5, we have

$$(2.38) \quad \begin{aligned} |\bar{z}^s(x)\partial^t h_t(x)| &= \left| \bar{z}^s(x) \int_{E-x} \partial^t \varphi_t(y) dy \right| \\ &\leq \bar{z}^s(x) \int_{E-x} |\partial^t \varphi_t(y)| dy \leq \int_{R^k} |y|^s |\partial^t \varphi_t(y)| dy \\ &\leq ck^{s/2}\tau^{t-s} \end{aligned}$$

for all $x \in R^k$, $s = 0, 1, 2, 3$. From this, in particular, (2.36) follows for $s = 0$. Further, noting that if $|x| < \bar{z}(0)$, then $0 \in E \Rightarrow 0 \in E - x$, $0 \notin E \Rightarrow 0 \notin E - x$, and using Lemma 2.5, we have

$$(2.39) \quad |\bar{z}^s(x)h_t(x)| \leq \int_{R^k} |y|^s \varphi_t(y) dy \leq ck^{s/2}\tau^{-s}$$

for all x such that $|x| < \bar{z}(0)$ and all $s = 0, 1, 2, 3$. Since, because of (2.35)

$$(2.40) \quad \begin{aligned} \bar{R}^s(x) &= (\bar{z}(x) + \bar{R}(x) - \bar{z}(x))^s \leq 2^{s-1}(\bar{z}^s(x) + |\bar{R}(x) - \bar{z}(x)|^s) \\ &\leq 2^{s-1}(\bar{z}^s(x) + k^{s/2}), \end{aligned}$$

it follows from (2.38) and (2.39) that

$$(2.41) \quad |\bar{R}^s(x)\partial^t h_t(x)| \leq ck^{s/2}(1 + \tau^{t-s})$$

for $|x| < \bar{z}(0)$ and $s, t = 0, 1, 2, 3$. From (2.41), in particular, (2.36) follows for $t = 0$. Finally, the assertion (2.36) for $s, t = 1, 2, 3$ is obtained by a simple calculation using (2.41) and the inequality

$$(2.42) \quad |\partial^t \bar{R}(x)| \leq k^{1/2}, \quad t = 1, 2, 3,$$

which follows from (2.28).

The lemma is proved.

LEMMA 2.8. Let \bar{P}_n be the distribution of the normalized sum $\zeta_n = n^{-1/2} \sum_{i=1}^n \xi_i$ of independent, identically distributed random variables $\xi_i =$

$(\xi_{i,1}, \dots, \xi_{i,k})$ with distribution function $\bar{P} \in \mathcal{P}_k$ having the identity as covariance matrix. Then, for any measurable set $E \subset R^k$

$$(2.43) \quad M_n = \nu_1^3(E) |\bar{P}_n(E) - N_1(E)| \leq \bar{c}k^{1/2}(k + (k^{3/2} + \nu_3)n^{-1/2}),$$

where $\nu_1(\cdot)$ is the same as in Lemma 2.3, and ν_3 is defined by equation (2.14).

The proof of this lemma is essentially contained in [14]. Let $\zeta_{n,j} = n^{-1/2} \sum_{i=1}^n \xi_{i,j}$. Using the inequality

$$(2.44) \quad \mathcal{E}|\zeta_{n,j}|^3 \leq 4(1 + n^{-1/2} \mathcal{E}|\xi_{1,j}|^3)$$

for one dimensional random variables (see [12], [14]), noting that

$$(2.45) \quad \mathcal{E}|\xi_1|^3 = \int_{R^k} |x|^3 (\bar{P} - N_1)(dx) + \int_{R^k} |x|^3 N_1(dx) \leq \nu_3 + ck^{3/2},$$

and defining \tilde{E} by formula (3.6), we have

$$(2.46) \quad \begin{aligned} M_n &= \nu_1^3(\tilde{E}) |(\bar{P}_n - N_1)(\tilde{E})| \leq \nu_1^3(\tilde{E}) (\bar{P}_n + N_1)(\tilde{E}) \\ &\leq \mathcal{E}|\zeta_n|^3 + \int_{R^k} |x|^3 N_1(dx) \leq k^{1/2} \mathcal{E} \left(\sum_{j=1}^k |\zeta_{n,j}|^3 \right) + ck^{3/2} \\ &\leq ck^{1/2} \left(k + n^{-1/2} \mathcal{E} \left(\sum_{j=1}^k |\xi_{1,j}|^3 \right) \right) \leq ck^{1/2} (k + n^{-1/2} \mathcal{E}|\xi_1|^3) \\ &\leq \bar{c}k^{1/2} (k + (k^{3/2} + \nu_3)n^{-1/2}). \end{aligned}$$

3. Formulation and proof of the theorem

THEOREM 3.1. *If $\{\xi_i\}$, for $i = 1, 2, \dots$, is a sequence of independent, identically distributed random variables with distribution $P \in \mathcal{P}_k$, P_n is the distribution of the normalized sum $n^{-1/2} \sum_{i=1}^n \xi_i$, and Q is the normal distribution with the same first and second moments as P , then for any $E \in \mathcal{C}$ with boundary ∂E*

$$(3.1) \quad |P_n(E) - Q(E)| \leq c(k) \frac{\bar{\nu}_3}{1 + \nu_\Delta^3(E)} n^{-1/2}, \quad n = 1, 2, \dots,$$

where $\bar{\nu}_3, \Delta$ are the same as in Lemma 2.2, and

$$(3.2) \quad \nu_\Delta(E) = \inf_{x \in \partial E} (\Delta^{-1}x, x)^{1/2}.$$

The constant $c(k) \leq ck^5$.

PROOF. Pursuing the same reasoning as we did at the beginning of the proof of Lemma 2.2 and recalling that in the notation introduced there

$$(3.3) \quad \nu_\Delta(E) = \inf_{x \in \partial E} |Ax| = \inf_{x \in \partial(AE)} |x| = \nu_1(AE),$$

we see that for the proof of the theorem it is sufficient to obtain the bound

$$(3.4) \quad |\bar{P}_n(E) - N_1(E)| \leq c(k) \frac{\bar{\nu}_3}{1 + \nu_1^3(E)} n^{-1/2}, \quad n = 1, 2, \dots,$$

where

$$(3.5) \quad \bar{v}_3 = |\bar{P} - N_1|(S_1) + \int_{S_1^c} |x|^3 |\bar{P} - N_1|(dx).$$

For any set $E \subset R^k$ let

$$(3.6) \quad \tilde{E} = \begin{cases} E, & \text{if } 0 \notin E, \\ E^c, & \text{if } 0 \in E. \end{cases}$$

Obviously, for any measurable $E \subset R^k$

$$(3.7) \quad \begin{aligned} \nu_1^3(E) |\bar{P}(E) - N_1(E)| &= \nu_1^3(\tilde{E}) |\bar{P}(\tilde{E}) - N_1(\tilde{E})| \\ &\leq \nu_1^3(\tilde{E}) |\bar{P} - N_1|(\tilde{E}) \leq \nu_3 \leq \bar{\nu}_3, \end{aligned}$$

and, therefore, taking into account (2.11), we have

$$(3.8) \quad |\bar{P}(E) - N_1(E)| \leq \frac{3}{2} \frac{\bar{\nu}_3}{1 + \nu_1^3(E)},$$

that is, (3.4) is true for $n = 1$. We shall show that if (3.4) is true for all values of n smaller than some fixed value, with constant $c(k)$ which will be made precise later, then (3.4) is true also for this fixed value of n with the same constant $c(k)$. In what follows we shall assume $n \geq 2$.

Throughout the proof of the theorem, E will denote a fixed, measurable, convex set. Let us define the probability measure $P_{(n)}$ by $P_{(n)}(\cdot) = \bar{P}(n^{1/2}\cdot)$. For brevity, let

$$(3.9) \quad H_i = P_{(n)}^i - N_{n^{1/2}}, \quad i = 1, 2, \dots, n.$$

In exactly the same way as in [16] (page 186), for arbitrary $T > 0$ we have

$$(3.10) \quad (\bar{P}_n - N_1) * N_T = H_n * N_T = \left(\sum_{i=1}^{n-1} U_i + nN_{\tau_0} \right) * H_1,$$

where

$$(3.11) \quad U_i = H_i * N_{\tau_i}, \quad \tau_i = \left(\frac{n - i - 1}{n} + T^{-2} \right)^{-1/2}, \quad i = 1, 2, \dots, n - 1.$$

Below, we shall assume $T \geq 1$. Furthermore, let

$$(3.12) \quad \begin{aligned} f_i(x) &= H_i(E - x), \quad i = 1, 2, \dots, n - 1, \quad \nu(x) = \nu_1(E - x), \\ R(x) &= \int_{R^k} \nu(y) \varphi_1(x - y) dy, \quad R(x, z) = R(x) - R(z), \\ R_1(x) &= R(0, x). \end{aligned}$$

Using the representation

$$(3.13) \quad \nu(0) = q(z) + R(x, z) + R_1(x),$$

where

$$(3.14) \quad q(z) = \nu(z) + (\nu(0) - R(0) - \nu(z) + R(z)),$$

we have, for all i

$$\begin{aligned}
 (3.15) \quad \iota^3(0)(U_i \star H_1)(E) &= \iota^3(0) \int_{R^k} \left[\int_{R^k} f_i(x+y)\varphi_{\tau_i}(y) dy \right] H_1(dx) \\
 &= \int_{R^k} \left[\int_{R^k} \iota^3(0)f_i(z)\varphi_{\tau_i}(x-z) dz \right] H_1(dx) \\
 &= \sum_{\substack{j_1, j_2, j_3 \geq 0 \\ j_1 + j_2 + j_3 = 3}} \frac{3!}{j_1! j_2! j_3!} I_i(j_1, j_2, j_3),
 \end{aligned}$$

where

$$\begin{aligned}
 (3.16) \quad I_i(j_1, j_2, j_3) &= \int_{R^k} \left[\int_{R^k} q^{j_1}(z)f_i(z)R^{j_2}(x, z)\varphi_{\tau_i}(x-z) dz \right] R_1^{j_3}(x)H_1(dx).
 \end{aligned}$$

We shall now concern ourselves with bounding $I_i = (j_1, j_2, j_3)$. First of all, since by Lemma 2.7

$$(3.17) \quad |\iota(0) - R(0) - \iota(z) + R(z)| \leq 2k^{1/2},$$

then, taking into account the obvious inequality

$$(3.18) \quad \iota^j(z) \leq 1 + \iota^3(z), \quad j = 0, 1, 2, 3,$$

we have

$$\begin{aligned}
 (3.19) \quad |q^{j_1}(z)| &\leq 2^{j_1-1}(\iota^{j_1}(z) + |\iota(0) - R(0) - \iota(z) + R(z)|^{j_1}) \\
 &\leq 2^{j_1-1}(1 + \iota^3(z) + (2k^{1/2})^{j_1}), \quad j_1 = 0, 1, 2, 3.
 \end{aligned}$$

Furthermore, because

$$(3.20) \quad H_i(\cdot) = (P_{(n)}^i - N_{n^{1/2}}^i)(\cdot) = (\bar{P}_i - N_1) \left(\left(\frac{n}{i} \right)^{1/2} \cdot \right),$$

and $(n/i)^{1/2}(E - z) \in \mathcal{C}$, it follows from Lemma 2.3 and the induction assumption that

$$\begin{aligned}
 (3.21) \quad |(1 + \iota^3(z))f_i(z)| &\leq \left(1 + \iota^3 \left[\left(\frac{n}{i} \right)^{1/2} (E - z) \right] \right) \left| (\bar{P}_i - N_1) \left[\left(\frac{n}{i} \right)^{1/2} (E - z) \right] \right| \\
 &\leq c(k)\bar{v}_3 i^{-1/2},
 \end{aligned}$$

and from Lemma 2.2 it follows that

$$(3.22) \quad |f_i(z)| \leq c_1(k)\bar{v}_3 i^{-1/2}.$$

Combining (3.19), (3.21) and (3.22), we obtain

$$(3.23) \quad |q^{j_1}(z)f_i(z)| \leq 2^{j_1-1}(c(k) + (2k^{1/2})^{j_1}c_1(k))\bar{v}_3 i^{-1/2}$$

for all $j_1 = 0, 1, 2, 3; i = 1, 2, \dots, n - 1$.

For the bound of $I_i(j_1, j_2, j_3)$ with $j_3 = 0$ we expand the function

$$(3.24) \quad f_{j_1, j_2, i}(x) = \int_{\mathbb{R}^k} q^{j_1}(z) f_i(z) R^{j_2}(x, z) \varphi_{\tau_i}(x - z) dz$$

in a Taylor's series to terms of third order and use the relations

$$(3.25) \quad \int_{\mathbb{R}^k} H_1(dx) = \int_{\mathbb{R}^k} x_u H_1(dx) = \int_{\mathbb{R}^k} x_u x_v H_1(dx) = 0,$$

which follow from the fact that the corresponding first and second moments of \bar{P} and N_1 are the same. We have

$$(3.26) \quad \begin{aligned} & |I_i(j_1, j_2, 0)| \\ &= \frac{1}{6} \left| \int_{\mathbb{R}^k} \left(\sum_{u, v, w=1}^k \partial_{u, v, w}^3 f_{j_1, j_2, i}(\partial x) x_u x_v x_w \right) H_1(dx) \right| \\ &\leq \frac{1}{6} \sup_{x, u, v, w} \left| \partial_{u, v, w}^3 f_{j_1, j_2, i}(x) \right| \left| \int_{\mathbb{R}^k} \left(\sum_{u=1}^k |x_u| \right)^3 \right| H_1(dx) \\ &\leq \frac{k^{3/2}}{6} \sup_{x, u, v, w} \left| \partial_{u, v, w}^3 f_{j_1, j_2, i}(x) \right| \frac{\bar{v}_3}{n^{3/2}}. \end{aligned}$$

Furthermore, since for $T \geq 1$

$$(3.27) \quad 1 \leq \sqrt{2} \tau_i, \quad i = 0, 1, \dots, n - 1,$$

it follows from (3.23) and Lemma 2.6 that

$$(3.28) \quad \begin{aligned} & \left| \partial_{u, v, w}^3 f_{j_1, j_2, i}(x) \right| \\ &\leq \sup_z \left| q^{j_1}(z) f_i(z) \right| \left| \int_{\mathbb{R}^k} \left| \partial_{u, v, w}^3 (R^{j_2}(x, z) \varphi_{\tau_i}(x - z)) \right| dz \right| \\ &\leq ck^{j_2/2} (c(k) + k^{j_1/2} c_1(k)) \bar{v}_3 i^{-1/2} \lambda_{j_2}(\tau_i), \end{aligned}$$

where

$$(3.29) \quad \lambda_j(\tau_i) = \begin{cases} \tau_i^3, & j = 0, 1 \\ \tau_i, & j = 2, 3. \end{cases}$$

Expanding the function $f_{j_1, j_2, i}(x)$ in a Taylor's series up to terms of first order and reasoning analogously, we also have

$$(3.30) \quad |I_i(j_1, j_2, 0)| \leq k^{1/2} \sup_{x, u} \left| \partial_u^1 f_{j_1, j_2, i}(x) \right| \bar{v}_3 n^{-1/2},$$

and

$$(3.31) \quad \left| \partial_u^1 f_{j_1, j_2, n-1}(x) \right| \leq ck^{j_2/2} (c(k) + k^{j_1/2} c_1(k)) \bar{v}_3 n^{-1/2} T^{1-j_2}.$$

From (3.26), (3.28), (3.30), and (3.31) we obtain

$$(3.32) \quad |I_i(j_1, j_2, 0)| \leq J_i(j_1, j_2)$$

where

$$(3.33) \quad J_i(j_1, j_2) = ck^{(3+j_2)/2}(c(k) + k^{j_1/2}c_1(k))\bar{v}_3^2 n^{-3/2} i^{-1/2} \lambda_{j_2}(\tau_i)$$

for $i = 1, 2, \dots, n - 2$ and

$$(3.34) \quad J_{n-1}(j_1, j_2) = ck^{(1+j_2)/2}(c(k) + k^{j_1/2}c_1(k))\bar{v}_3^2 n^{-1} T^{1-j_2}.$$

In order to bound $I_i(j_1, j_2, j_3)$ with $j_3 = 1$ we write the product $f_{j_1, j_2, i}(x)R_1(x)$ in the form (with f in place of $f_{j_1, j_2, i}$ for brevity)

$$(3.35) \quad \begin{aligned} fR_1 = f(0) & \left(R_1(0) + \sum_{u=1}^k \partial_u^1 R_1(0)x_u + \frac{1}{2} \sum_{u,v=1}^k \partial_{u,v}^2 R_1(0)x_u x_v \right. \\ & \left. + \frac{1}{6} \sum_{u,v,w=1}^k \partial_{u,v,w}^3 R_1(\mathfrak{G}_1 x)x_u x_v x_w \right) \\ & + \sum_{u=1}^k \partial_u^1 f(0)x_u \left(R_1(0) + \sum_{v=1}^k \partial_v^1 R_1(0)x_v \right. \\ & \left. + \frac{1}{2} \sum_{v,w=1}^k \partial_{v,w}^2 R_1(\mathfrak{G}_2 x)x_v x_w \right) \\ & + \frac{1}{2} \sum_{u,v=1}^k \partial_{u,v}^2 f(\mathfrak{G}_3 x)x_u x_v R_1(x), \end{aligned}$$

($|\mathfrak{G}_\varkappa| \leq 1, \varkappa = 1, 2, 3$). Using a calculation analogous to that in (3.26), and taking into account that, according to Lemma 2.6, $|R_1(x)| \leq |x|$, we obtain

$$(3.36) \quad \begin{aligned} |I_i(j_1, j_2, 1)| & \leq \frac{k^{3/2}}{2} (|f_{j_1, j_2, i}(0)| \sup_{x,u,v,w} |\partial_{u,v,w}^3 R_1(x)| \\ & \quad + |\partial_u^1 f_{j_1, j_2, i}(0)| \sup_{x,u,v} |\partial_{u,v}^2 R_1(x)| \\ & \quad + \sup_{x,u,v} |\partial_{u,v}^2 f_{j_1, j_2, i}(x)| \bar{v}_3 n^{-3/2}). \end{aligned}$$

We now use Lemma 2.6, (3.23), and (3.27) as in (3.28), to bound $|\partial^t f_{j_1, j_2, i}(x)|$. Then taking into account that $|\partial^t R_1(x)| \leq ck^{1/2}$ (from Lemma 2.6), we have

$$(3.37) \quad |I_i(j_1, j_2, 1)| \leq J_i(j_1, j_2 + 1), \quad i = 1, 2, \dots, n - 2.$$

To obtain inequality (3.37) for $i = n - 1$, it is sufficient to use the relations

$$(3.38) \quad \begin{aligned} |I_i(j_1, j_2, 1)| & \leq \sup_x |f_{j_1, j_2, i}(x)| \int_{R^k} |x| |H_1|(dx) \\ & \leq \sup_x |f_{j_1, j_2, i}(x)| \bar{v}_3 n^{-1/2}, \end{aligned}$$

and then to bound $|f_{j_1, j_2, n-1}(x)|$ as has just been indicated.

To bound $I_i(j_1, j_2, 2)$, we use the following representation of the product $f_{j_1, j_2, i}R_1^2$ (with $f_{j_1, j_2, i} = f$ for brevity)

$$(3.39) \quad fR_1^2 = f(0) \left(R_1^2(0) + 2 \sum_{u=1}^k (R_1 \partial_u^1 R_1)(0)x_u + \sum_{u,v=1}^k \left[(R_1 \partial_{u,v}^2 R_1)(\vartheta_1 x) + (\partial_u^1 R_1 \partial_v^1 R_1)(0) + \sum_{w=1}^k (\partial_{u,w}^2 R_1 \partial_v^1 R_1 + \partial_u^1 R_1 \partial_{v,w}^2 R_1)(\vartheta_2 \vartheta_1 x) \vartheta_1 x_w \right] x_u x_v + \left(\sum_{u=1}^k \partial_u^1 f(\vartheta_3 x)x_u \right) R_1^2(x) \right),$$

where $|\vartheta_i| \leq 1, i = 1, 2, 3$. Performing the calculation, analogous to (3.26), with the use of the inequality $|R_1(x)| \leq |x|$, we have

$$(3.40) \quad |I_i(j_1, j_2, 2)| \leq k^{3/2} [|f_{j_1, j_2, i}(0)| (\sup_{x,u,v} |\partial_{u,v}^2 R_1(x)| + 2 \sup_{x,u,v,w} |(\partial_u^1 R_1 \partial_{v,w}^2 R_1)(x)|) + \sup_{x,u} |\partial_u^1 f_{j_1, j_2, i}(x)|] \bar{v}_3 n^{-3/2}.$$

Now bounding $|\partial^t f_{j_1, j_2, i}(x)|$ similarly as in (3.28), using (3.23), (3.27) and Lemma 2.6, and noting that by Lemma 2.6, $|\partial^t R_1| \leq ck^{1/2}, t = 1, 2$, we obtain

$$(3.41) \quad |I_i(j_1, j_2, 2)| \leq J_i(j_1, j_2 + 2), \quad i = 1, 2, \dots, n - 2.$$

For $i = n - 1$, in exactly the same way as in the case of $j_3 = 1$, we deduce

$$(3.42) \quad |I_{n-1}(j_1, j_2, 2)| \leq J_{n-1}(j_1, j_2 + 1).$$

Finally, $I_i(0, 0, 3)$ is bounded quite simply. Using the inequality $|R_1(x)| \leq |x|$ and (3.23), we have for all $i = 1, 2, \dots, n - 1$,

$$(3.43) \quad |I_i(0, 0, 3)| \leq \sup_x |f_i(x)| \int_{R^k} |x|^3 |H_1|(dx) \leq c(c(k) + c_1(k)) \bar{v}_3^2 i^{-1/2} n^{-3/2}.$$

We now note that for $m \geq 0$

$$(3.44) \quad \sum_{i=1}^{n-2} \tau_i^m i^{-1/2} \leq \int_1^{n-1} \frac{dx}{(x-1)^{1/2} \left(\frac{n-x-1}{n} + T^{-2} \right)^{m/2}} \leq \begin{cases} \pi n^{1/2}, & m = 0, 1 \\ 2\sqrt{3} T n^{1/2}, & m = 3. \end{cases}$$

Combining (3.15), (3.32), (3.37), (3.41), (3.42), (3.43) we obtain

$$(3.45) \quad \left| \iota^3(0) \left[\left(\sum_{i=1}^{n-1} U_i \right) * H_1 \right] (E) \right| \leq ck^2 [(c(k) + kc_1(k))T + k(c(k) + c_1(k))] \bar{v}_3^2 n^{-1}.$$

Passing on now to the bound for $L = \iota^3(0)(N_{\tau_0} * H_1)(E)$, let us define h_{τ_0} by means of equation (2.34). For $\iota(0) \leq 1$, using bounds similar to (3.26), applying Lemma 2.7, and noting that

$$(3.46) \quad 2^{-1/2} \leq \tau_0 \leq 2^{1/2}$$

(remember that $T \geq 1$), we obtain

$$(3.47) \quad |L| \leq \frac{k^{3/2}}{6} \sup_{x, u, v, w} |\partial_{u, v, w}^3 h_{\tau_0}(x)| \bar{v}_3 n^{-3/2} \leq ck^{3/2} \bar{v}_3 n^{-3/2}.$$

Now consider the case where $\iota(0) > 1$. We have

$$(3.48) \quad L = \iota^3(0) \left(\int_{|x| < \iota(0)} + \int_{|x| \geq \iota(0)} \right) h_{\tau_0}(x) H_1(dx) = I_1 + I_2$$

and

$$(3.49) \quad |I_2| \leq \iota^3(0) \int_{|x| \geq \iota(0)} |H_1|(dx) \leq \int_{\mathbb{R}^k} |x|^3 |H_1|(dx) \leq \bar{v}_3 n^{-3/2}.$$

In order to bound I_1 , using the representation

$$(3.50) \quad \iota(0) = \iota(0) - R(0) + R(x) + R_1(x),$$

we write I_1 in the form

$$(3.51) \quad I_1 = \sum_{\substack{j_1, j_2, j_3 \geq 0 \\ j_1 + j_2 + j_3 = 3}} \frac{3!}{j_1! j_2! j_3!} I(j_1, j_2, j_3),$$

where

$$(3.52) \quad I(j_1, j_2, j_3) = (\iota(0) - R(0))^{j_1} \int_{|x| < \iota(0)} R^{j_2}(x) h_{\tau_0}(x) R_1^{j_3}(x) H_1(dx),$$

and we put

$$(3.53) \quad f_{j_1, j_2}(x) = (\iota(0) - R(0))^{j_1} R^{j_2}(x) h_{\tau_0}(x).$$

First, let us bound $I(j_1, j_2, j_3)$ with $j_3 = 0$. Expanding the function $f_{j_1, j_2}(x)$ in a Taylor's series up to terms of third order and, taking into account (3.25), we have

$$(3.54) \quad |I(j_1, j_2, 0)| \leq \left| f_{j_1, j_2}(0) \right| \int_{|x| \geq \iota(0)} |H_1|(dx) + \sum_{u=1}^k \left| \partial_u^1 f_{j_1, j_2}(0) \right| \int_{|x| \geq \iota(0)} |x_u| |H_1|(dx) + \frac{1}{2} \sum_{u, v=1}^k \left| \partial_{u, v}^2 f_{j_1, j_2}(0) \right| \int_{|x| \geq \iota(0)} |x_u x_v| |H_1|(dx)$$

$$+ \frac{1}{6} \sup_{\substack{|x| < \iota(0) \\ u, v, w}} \left| \partial_{u, v, w}^3 f_{j_1, j_2}(x) \right| \int_{\mathbb{R}^k} \left(\sum_{u=1}^k |x_u| \right)^3 |H_1|(dx).$$

Noting, further, that for $\iota(0) \geq 1$,

$$(3.55) \quad \int_{|x| \geq \iota(0)} \left(\sum_{u=1}^k |x_u| \right)^m |H_1|(dx) \leq k^{m/2} \int_{\mathbb{R}^k} |x|^3 |H_1|(dx) \leq k^{m/2} \bar{v}_3 n^{-3/2}, \quad m = 0, 1, 2,$$

and using Lemma 2.7 and formula (3.46), we obtain

$$(3.56) \quad |I(j_1, j_2, 0)| \leq \sup_{\substack{|x| < \iota(0) \\ t=0,1,2,3}} |\partial^t f_{j_1, j_2}(x)| k^{3/2} \bar{v}_3 n^{-3/2} \leq ck^3 \bar{v}_3 n^{-3/2}.$$

To bound $I(j_1, j_2, j_3)$ with $j_3 = 1$ (correspondingly with $j_3 = 2$), we represent the product $f_{j_1, j_2}(x)R_1^{j_3}(x) = fR_1$ in the form (3.35) (correspondingly, $f_{j_1, j_2}(x)R_1^{j_3}(x) = fR_1^2$ in the form (3.39)). Considerations, analogous to those applied to the bounding of $I(j_1, j_2, 0)$ with the use of (2.27) and (2.28), lead to the inequality

$$(3.57) \quad |I(j_1, j_2, j_3)| \leq ck^3 \bar{v}_3 n^{-3/2}, \quad j_3 = 1, 2.$$

Finally, according to (2.27),

$$(3.58) \quad |I(0, 0, 3)| \leq \int_{\mathbb{R}^k} |x|^3 |H_1|(dx) \leq \bar{v}_3 n^{-3/2}.$$

Combining (3.51), (3.56), and (3.58), we obtain

$$(3.59) \quad |I_1| \leq ck^3 \bar{v}_3 n^{-3/2},$$

which, together with (3.47)–(3.49), yields the bound

$$(3.60) \quad |L| \leq ck^3 \bar{v}_3 n^{-3/2}.$$

From (3.10), (3.45), (3.60), and Lemmas 2.2 and 2.4, we can now conclude that for $T \geq 1$

$$(3.61) \quad \sup_{E \in \mathcal{E}} (1 + \iota^3(E)) |\bar{P}_n(E) - N_1(E)| \leq ck^3 \bar{v}_3 n^{-1/2} + ck^2 [(c(k) + kc_1(k))T + k(c(k) + c_1(k))] \bar{v}_3^2 n^{-1} + c_2(k)T^{-1} \leq c_0 [k^3 \bar{v}_3 n^{-1/2} + ((k^2 c(k) + k^{11/2}) + (k^3 c(k) + k^{11/2})T) \bar{v}_3^2 n^{-1} + k^{5/2} T^{-1}].$$

We shall take the two constants c_1 and c_2 such that

$$(3.62) \quad \begin{aligned} c_1^2 &\leq c_2 \leq 2/3, \\ c_0(2c_2^{1/2} + c_2 + c_2^{3/2} + c_1 + c_1 c_2) &\leq 1, \\ \bar{c}(2c_2 c_1^{-1} + c_2) + c' c_2 &\leq 1, \end{aligned}$$

where c', \bar{c} are the constants of Lemmas 2.2 and 2.8 respectively, and we shall require $c(k)$ to satisfy the conditions

$$(3.63) \quad c_2^{-1}k^5 \leq c(k) \leq c_1^{-2}k^6.$$

Then, if $\bar{v}_3 n^{-1/2} \leq c_1 k^{-3}$, we have

$$(3.64) \quad T = \frac{n^{1/2}}{\bar{v}_3 c^{1/2}(k)} \geq 1,$$

and for this value of T the right side of (3.61) does not exceed

$$(3.65) \quad c(k)\bar{v}_3 n^{-1/2} c_0 \left[\frac{k^3}{c(k)} + \frac{k^2}{c^{1/2}(k)} + \frac{k^{11/2}}{c^{3/2}(k)} + \left(k^3 + \frac{k^{11/2}}{c(k)} \right) \frac{\bar{v}_3}{n^{1/2}} + \frac{k^{5/2}}{c^{1/2}(k)} \right] \leq c(k)\bar{v}_3 n^{-1/2}.$$

If $\bar{v}_3 n^{-1/2} > c_1 k^{-3}$, then by Lemmas 2.2 and 2.8

$$(3.66) \quad \begin{aligned} & \sup_{E \in \mathcal{G}} (1 + \nu_1^3(E)) |(\bar{P}_n - N_1)(E)| \\ & \leq \bar{c} k^{1/2} \left(\frac{k^4}{c_1} \frac{\bar{v}_3}{n^{1/2}} + \frac{k^{3/2}}{c_1} \frac{\bar{v}_3}{n^{1/2}} + \frac{\bar{v}_3}{n^{1/2}} \right) + c_1(k) \frac{\bar{v}_3}{n^{1/2}} \\ & \leq c(k) \frac{\bar{v}_3}{n^{1/2}} \left[\frac{k^5}{c(k)} \left(\bar{c} \left(\frac{2}{c_1} + 1 \right) + c' \right) \right] \\ & \leq c(k)\bar{v}_3 n^{-1/2}. \end{aligned}$$

The theorem has been proved.

REMARK 3.1. One can bound \bar{v}_3 (which enters into the formulation of the theorem and Lemma 2.2) by an expression depending only on the third pseudo moment ν_3 (see equation (2.14)).

In order to do this, we shall first prove an auxiliary inequality which is of interest in itself. Namely, we shall show that there exists a constant \hat{c} such that for any probability measure P on R^k

$$(3.67) \quad |P - N_1|(R^k) \leq \hat{c} k^{-3/2} \left(\int_{R^k} |x|^3 |P - N_1|(dx) \right)^{k/(k+3)}.$$

First, let us suppose that

$$(3.68) \quad 0 < v = |P - N_1|(R^k) < 2.$$

Let $R^k = R^+ \cup R^-$ be the Hahn-Jordan decomposition of the space R^k with respect to the signed measure $P - N_1$. We define the probability measure P' , putting

$$(3.69) \quad P'(E) = \frac{v}{2} \chi_E(0) + N_1(E \cap R^+) + P(E \cap E^-),$$

where $\chi_E(x)$ is the indicator of the set E . Clearly,

$$(3.70) \quad \int_{R^k} |x|^3 |P - N_1|(dx) \cong \int_{R^k} |x|^3 |P' - N_1|(dx) = \int_{R^k} |x|^3 (N_1 - P')(dx).$$

Furthermore, let a be a number such that

$$(3.71) \quad N_1(S_a) = \frac{v}{2},$$

where S_a is a sphere of radius a with center at zero. Because

$$(3.72) \quad N_1(S_a^c) = P'(S'_a) + P'(S_a^c) = 1 - \frac{v}{2},$$

where $S'_a = S_a - \{0\}$, it follows that

$$(3.73) \quad \int_{S_a} |x|^3 P'(dx) \leq a^3 P'(S'_a) = a^3 (N_1 - P')(S_a^c) \leq \int_{S_a^c} |x|^3 (N_1 - P')(dx),$$

and therefore,

$$(3.74) \quad \begin{aligned} \int_{S_a} |x|^3 N_1(dx) &= \int_{S_a} |x|^3 (N_1 - P')(dx) + \int_{S_a} |x|^3 P'(dx) \\ &\leq \int_{R^k} |x|^3 (N_1 - P')(dx). \end{aligned}$$

The inequality

$$(3.75) \quad \frac{|P - N_1|(R^k)}{\left(\int_{R^k} |x|^3 |P - N_1|(dx)\right)^{k/(k+3)}} \leq \frac{2N_1(S_a)}{\left(\int_{S_a} |x|^3 N_1(dx)\right)^{k/(k+3)}} = g(a, k)$$

follows from (3.68), (3.70), (3.71), and (3.74). It is not difficult to convince oneself that $g(a, k)$, as a function of a , does not increase as a increases, and that

$$(3.76) \quad g(+0, k) = \frac{2^{3(2-k)/2(3+k)}(k+3)^{k/(k+3)}}{(\Gamma(k/2))^{3/(k+3)}k} \leq ck^{-3/2}.$$

If $v = 2$, then the distribution P is singular with respect to N_1 and

$$(3.77) \quad \begin{aligned} \left(\int_{R^k} |x|^3 |P - N_1|(dx)\right)^{k/(k+3)} &\geq \left(\int_{R^k} |x|^3 N_1(dx)\right)^{k/(k+3)} \\ &\geq ck^{3/2}2 = ck^{3/2}|P - N_1|(R^k). \end{aligned}$$

Thus, (3.67) is proved.

Let us return now to the notation adopted in the theorem. If $v_3 \geq 1$, then, according to (3.67),

$$(3.78) \quad \begin{aligned} |\bar{P} - N_1|(S_1) &\leq \hat{c}k^{-3/2}v_3^{k/(k+3)} \leq \hat{c}k^{-3/2}v_3, \\ \int_{S_1^c} |x|^3 |\bar{P} - N_1|(dx) &\leq v_3, \end{aligned}$$

and, consequently,

$$(3.79) \quad \bar{v}_3 \leq (1 + \hat{c}k^{-3/2})v_3.$$

For $v_3 < 1$, according to (3.67),

$$(3.80) \quad \begin{aligned} |\bar{P} - N_1|(S_1) &\leq \hat{c}k^{-3/2}v_3^{k/(k+3)}, \\ \int_{S_1^c} |x|^3 |\bar{P} - N_1|(dx) &\leq v_3 \leq v_3^{k/(k+3)}, \end{aligned}$$

and, therefore,

$$(3.81) \quad \bar{v}_3 \leq (1 + \hat{c}k^{-3/2})v_3^{k/(k+3)}.$$

Thus,

$$(3.82) \quad \bar{v}_3 \leq (1 + \hat{c}k^{-3/2})\tilde{v}_3,$$

where

$$(3.83) \quad \tilde{v}_3 = \max(v_3, v_3^{k/(k+3)}).$$

It is appropriate to mention that \tilde{v}_3 cannot be bounded by a multiple of \bar{v}_3 . Indeed, let P_X , $X > 0$, be a probability measure on R^k , defined by

$$(3.84) \quad P_X(E) = N_1(E \cap S_X) + (2k)^{-1}N_1(S_X^c) \sum_{i=1}^k (\chi_E(y_i) + \chi_E(-y_i)),$$

where y_i is a point in R^k whose i th coordinate is

$$(3.85) \quad Y = \left(k \int_{S_X^c} x_i^2 N_1(dx) \right)^{1/2} (N_1(S_X^c))^{-1/2}$$

and the remaining are zero. A simple calculation shows that for P_X we have $\bar{v}_3 = v_3$ for $X \geq 1$, and that $v_3 \rightarrow 0$ as $X \rightarrow \infty$. Therefore, $\bar{v}_3 \bar{v}_3^{-1} = v_3^{-3/(k+3)} \rightarrow \infty$ as $X \rightarrow \infty$.

REMARK 3.2. Of course \bar{v}_3 can also be bounded in terms of the third absolute moments

$$(3.86) \quad \beta_3 = \int_{R^k} |x|^3 \bar{P}(dx), \quad \beta'_3 = \sum_{i=1}^k \int_{R^k} |x_i|^3 \bar{P}(dx).$$

In fact, because

$$(3.87) \quad \begin{aligned} \left(\int_{R^k} |x|^3 \bar{P}(dx) \right)^{1/3} &\geq \left(\int_{R^k} |x|^2 \bar{P}(dx) \right)^{1/2} = k^{1/2}, \\ \int_{R^k} |x|^3 N_1(dx) &\leq (k+1)k^{1/2}, \end{aligned}$$

we have

$$(3.88) \quad v_3 = \int_{R^k} |x|^3 |\bar{P} - N_1|(dx)$$

$$\leq \int_{R^k} |x|^3 (\bar{P} + N_1)(dx) \leq (2 + k^{-1})\beta_3,$$

so that, from (3.82) and the obvious inequality $\beta_3 \leq k^{1/2}\beta'_3$,

$$(3.89) \quad \bar{v}_3 \leq (2 + ck^{-1})\beta_3 \leq k^{1/2}(2 + ck^{-1})\beta'_3.$$

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