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# On a Canonical Treatment of the Einstein Equations in an Expanding Universe

## —— *Isotropic Coordinate Conditions* ——

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A cosmological version of Arnowitt, Deser and Misner's second coordinate conditions in their canonical treatment of general relativity is proposed in such a way that it is compatible with the effect of cosmic expansion. In terms of these coordinate conditions, Nariai and Kimura's canonical formalism of the cosmological gravitational field is refined so as to be appropriate in various isotropic problems. It is shown that the refined formalism is suitable to deal with the problem of a spherically symmetric matter-distribution in the universe.

## § 1. Introduction

It is very difficult in general to solve Einstein's gravitational equations because of their nonlinearity. In addition, their dynamical and constraint parts mix with each other, so that the dynamical behavior of a general gravitational field cannot be pursued so easily. In view of this, Arnowitt, Deser and Misner (we abbreviate as ADM in what follows)[1a] proposed a canonical formalism for the gravitational field by dividing its field variables and their equations into dynamical and constraint parts. Their original motivation was to search for the quantization of Einstein's gravitational field, but their canonical formalism is also useful to the examination of a classical gravitational field.

On the other hand, by virtue of astronomical observations of pulsars, X-ray sources and the 3K background radiation, the existence of a strong gravitational field such as those for neutron stars, black holes and an early stage of the big-bang universe became clear for the past fifteen years.

As regards the problem of gravitational collapse attacked originally by Oppenheimer and Snyder,[2] Misner and Sharp[3] developed the ADM-like formalism for a spherical gravitational collapse and it was applied by various authors[4] to their numerical works. An extension of these works to the case of non-spherical gravitational collapse (in which the emission of gravitational waves plays an important role) was performed by Smarr,[5] Piran[6] and Nakamura-Maeda-Miyama-Sasaki.[7] As regards the dynamical problem in a homogeneous and isotropic universe, there are three works to be picked up, i.e., Lifshitz's linear theory[8] of the gravitational instability in the universe, Tomita's extension[9] to the second order and Nariai and Kimura's canonical formalism[10] (an ADM-like approach to a cosmological gravitational field). Moreover, the dynamics of a homogeneous and anisotropic universe was developed by the Soviet group consisting of Lifshitz, Khalatnikov and Belinskii,[11] and Misner's group[12] relying on the ADM formalism.

The purpose of this paper is to refine Nariai and Kimura's canonical formalism by introducing an extended version of ADM's isotropic coordinate conditions[1b] which is reconciled with the effect of cosmic expansion. It is shown that such a refinement is suitable to search for the problem of an inhomogeneous and spherical distribution of matter immersed in the isotropic expanding universe.

In § 2, the refinement in question is performed in terms of a suitable set of isotropic coordinate conditions. It is shown in § 3 that the refined formalism leads easily to the Schwarzschild-de Sitter space-time as an isotropic and vacuum solution of the Einstein equations with the cosmological term.

## § 2. The refinement of Nariai and Kimura's canonical formalism in an isotropic expanding universe

Let us consider an isotropic expanding universe specified by the metric

$$ds^2=({}^4g_{\mu\nu})_b dx^\mu dx^\nu=-dt^2+f^2(t)(dx^2+dy^2+dz^2) \tag{2·1}^{*)}$$

and by Friedmann's cosmological equations

$$\begin{cases}\rho_b=2(3h^2-\Lambda),\\ p_b+\rho_b=-4\dot{h}\,,\end{cases} \tag{2·2}^{*)}$$

where $f(t)$ is a scale factor such that $h\equiv \dot{f}/f$ (Hubble's expansion parameter), $\Lambda$ the cosmological constant, and $\rho_b$, $p_b$ stand for the background density and pressure, respectively.

Next, we shall consider a general gravitational field specified by the metric tensor ${}^4g_{\mu\nu}$ and the source matter as a perfect fluid whose density, pressure and four-velocity are $\rho$, $p$ and $u^\mu$ (${}^4g_{\mu\nu}u^\mu u^\nu=-1$), respectively. According to Nariai and Kimura's canonical formalism[10] in the isotropic universe with the metric (2·1), the Lagrangian densities for ${}^4g_{\mu\nu}$ and its source matter are, respectively, given by

$$\mathcal{L}_g=-g_{ij}\partial_t\pi^{ij}+Ng^{-1/2}\left\{g(R-2\Lambda)+\frac{1}{2}\pi^2-\pi_{ij}\pi^{ij}\right\}$$

*) The metric tensor ${}^4g_{\mu\nu}$ and its background value $({}^4g_{\mu\nu})_b$ are assumed to be of the signature $(-+++)$ with $x^0\equiv t$ and $x^i\equiv(x, y, z)$. Moreover, the system of units such as $c=2\kappa=1$ ($\kappa$ is Einstein's gravitation constant) is used.

$$+2N_i\pi^{ij}{}_{|j}-2\left\{g^{1/2}N^{|i}+N_j\left(\pi^{ij}-\frac{1}{2}g^{ij}\pi\right)\right\}_{,i} \tag{2·3}$$

and

$$\mathcal{L}_m=g^{1/2}[(p+\rho)\beta u_0+N\{p-(p+\rho)\beta^2\}-(p+\rho)\beta N_i u^i]. \tag{2·4}$$

In the above expressions, various symbols are defined by

$$\left\{\begin{array}{l} N\equiv(-{}^4g^{00})^{-1/2}, \quad N_i\equiv{}^4g_{0i}\,, \\ g_{ij}\equiv{}^4g_{ij}\,, \quad g^{ij}\equiv{}^4g^{ij}+N^iN^j/N^2, \quad g\equiv\det(g_{ij})\,, \\ \pi^{ij}\equiv(-{}^4g)^{1/2}\{{}^4\Gamma^0_{mn}-g^4_{mn}\Gamma^0_{pq}g^{pq}\}g^{im}g^{jn}, \quad \pi\equiv g_{ij}\pi^{ij}, \\ \beta\equiv-Nu^0=-(u_iu^i+1)^{1/2}, \quad u_i\equiv g_{ij}u^j, \end{array}\right. \tag{2·5}$$

and $R$ is the three-dimensional curvature scalar formed from $g_{ij}$. Moreover, the symbols " | " and " , " denote covariant differentiation with respect to the metric tensor $g_{ij}$ and ordinary one, respectively.

Taking the variation of the action $I=\int(\mathcal{L}_g+\mathcal{L}_m)d^4x$ with respect to the dynamical variables ($\pi^{ij}$, $g_{ij}$) and the constraint variables ($N$, $N_i$), we obtain the dynamical and constraint equations in the following way:

$$\partial_t g_{ij}=2Ng^{1/2}\left(\pi_{ij}-\frac{1}{2}g_{ij}\pi\right)+N_{i|j}+N_{j|i}\,, \tag{2·6}$$

$$\begin{aligned}\partial_t\pi^{ij}=&-Ng^{1/2}\left\{R^{ij}-\frac{1}{2}g^{ij}R+\left(\Lambda-\frac{1}{2}p\right)g^{ij}\right\}+g^{1/2}(N^{|ij}-g^{ij}N^{|m}{}_{|m})\\ &-2Ng^{-1/2}\left(\pi^{im}\pi_m{}^j-\frac{1}{2}\pi\pi^{ij}\right)+\frac{1}{2}Ng^{-1/2}\left(\pi^{mn}\pi_{mn}-\frac{1}{2}\pi^2\right)g^{ij}\\ &+\{(\pi^{ij}N^m)_{|m}-N^i{}_{|m}\pi^{mj}-N^j{}_{|m}\pi^{mi}\}\end{aligned} \tag{2·7}$$

and

$$g(R-2\Lambda)+\frac{1}{2}\pi^2-\pi_{ij}\pi^{ij}+g\{p-(p+\rho)\beta^2\}=0\,, \tag{2·8}$$

$$2\pi^{ij}{}_{|j}-g^{1/2}(p+\rho)\beta u^i=0\,. \tag{2·9}$$

According to ADM's canonical theory, all of the field variables $g_{ij}$ and $\pi^{ij}$ cannot be independent to each other, because of the existence of constraint equations like Eqs. (2·8) and (2·9), together with the indeterminacy of gauge characteristic in the Einstein theory. In other words, among the twelve field variables $g_{ij}$ and $\pi^{ij}$, truely dynamical ones are their transverse traceless parts $g_{ij}{}^{TT}$ and $\pi^{ij\,TT}$ such as $g^{TT}_{ij,j}=g^{TT}_{ii}\equiv0$, etc. Moreover, in the present case, we have to take into consideration the effect of cosmic expansion, as shown by Eqs. (2·1) and (2·2). In view of this, we performed in Ref. 10) the following decomposition of $g_{ij}$ and $\pi^{ij}$:

$$\begin{cases} g_{ij}=(f^2-2)\delta_{ij}+g_{ij}{}^{TT}+\frac{1}{2}\{\delta_{ij}g^T-(1/\nabla^2)g^T{}_{,ij}\}+g_{i,j}+g_{j,i}\,, \\ \pi^{ij}=-2hf\delta_{ij}+\pi^{ij\,TT}+\frac{1}{2}\{\delta_{ij}\pi^T-(1/\nabla^2)\pi^T{}_{,ij}\}+\pi^i{}_{,j}+\pi^j{}_{,i}\,, \end{cases} \tag{2·10}$$

where

$$\begin{cases} g^T\equiv g_{ii}-(1/\nabla^2)g_{ij,ij}-3(f^2-2), \\ g_i\equiv(1/\nabla^2)\Big\{g_{ij,j}-\frac{1}{2}(1/\nabla^2)g_{mn,mni}\Big\}, \end{cases} \quad \text{etc.}$$

in which the symbol $(1/\nabla^2)$ stands for the inverse of Laplacian operator $\nabla^2\equiv\partial^2/\partial x^i\partial x^i$ with an appropriate boundary condition. In the above decompositon, we have taken account of the coordinate conditions to be specified later.

On inserting Eqs. (2·8) and (2·9) into the original action $I$ (from which there arise Eqs. (2·6) and (2·7)) and making use of Eq. (2·10), we obtain such a reduced action as

$$I=\int d^4x\Big[\pi^{ij^{TT}}\partial_t g_{ij}{}^{TT}+\nabla^2 g^T\partial_t\Big\{-\frac{1}{2}(1/\nabla^2)\pi^T\Big\}-2\pi^{ij}{}_{,j}\partial_t g_i$$
$$+2hf^2(\pi^T+2\pi^i{}_{,i})+4\dot{h}f^3+2(g^T+2g_{i,i})\partial_t(hf)+g^{1/2}(p+\rho)\beta u_0\Big]. \tag{2·11}$$

In order that the above action may be reduced further to the required form $I=\int d^4x\{\pi^{ij^{TT}}\partial_t g_{ij}{}^{TT}-\mathcal{H}(g_{ij}{}^{TT},\pi^{ij^{TT}},t)\}$, we must impose a suitable set of coordinate conditions (the fixation of gauge). The coordinate conditions adopted in Ref. 10) were

$$-\frac{1}{2}(1/\nabla^2)\pi^T=\int^{x^0}dt/f(t), \qquad g_i=x^i. \tag{2·12}$$

However, under these coordinate conditions, the metric three-tensor $g_{ij}$ is not isotropic, even when $g_{ij}{}^{TT}=0$. (In the original ADM's theory, this led to a variety of complexities.)

The aim of our refinement is to remedy this weak point in Ref. 10). For that purpose, let us introduce the following coordinate conditions resembling those in Ref. 1b):

$$\begin{cases} -\frac{1}{2}(1/\nabla^2)(\pi^T+2\pi^i{}_{,i}-\chi)=\int^{x^0}dt/f(t), \\ g_i-\frac{1}{4}(1/\nabla^2)g^T{}_{,i}=x^i, \end{cases} \tag{2·13}$$

where

$$\chi \equiv 6hf(1-f^{-1}g^{1/6}). \tag{2·14}$$

Then we can reduce Eq. (2·10) to

$$\begin{cases} g_{ij}=f^2(t)\phi^4\delta_{ij}+g_{ij}{}^{TT}, \quad \phi\equiv\left(1+\frac{1}{2}f^{-2}g^T\right)^{1/4}, \\ \pi^{ij}=-2hf\delta_{ij}+\pi^{ij^{TT}}+\tilde{\pi}^{ij}, \end{cases} \tag{2·15}$$

where

$$\tilde{\pi}^{ij}\equiv\frac{1}{2}\{\delta_{ij}\pi^T-(1/\nabla^2)\pi^T{}_{,ij}\}+\pi^i{}_{,j}+\pi^j{}_{,i} \tag{2·16}$$

which, together with Eq. (2·13), leads to $\tilde{\pi}^{ii}=\chi$. The first of Eq. (2·15) shows the isotropy of $g_{ij}$ when $g_{ij}{}^{TT}=0$. By making use of Eq. (2·13), we can further reduce Eq. (2·11) to the required form

$$I=\int d^4x(\pi^{ij^{TT}}\partial_t g_{ij}{}^{TT}-\mathcal{H}), \tag{2·17}$$

where

$$\mathcal{H}\equiv-\mathcal{T}^0_0\equiv-f^{-1}\nabla^2 g^T$$
$$-3(\dot{h}f+h^2f)g^T-4\dot{h}f^3+\frac{1}{2}\dot{\chi}g^T-2hf^2\chi-g^{1/2}(p+\rho)\beta u_0. \tag{2·18}$$

Similarly, the total generator $G$ obtained from the variation of $I$ at the end point is of the same as Eq. (4·8) in Ref. 10), i.e.,

$$G=\int d^3x\{\pi^{ij^{TT}}\delta g_{ij}{}^{TT}+\mathcal{T}_\mu{}^0\delta x^\mu\}, \tag{2·19}$$

where $\mathcal{T}_\mu{}^0\equiv(-\mathcal{H},-2\pi^{ij}{}_{,j})$.

So far as the linear approximation to the dynamical equations for $g_{ij}{}^{TT}$ and $\pi^{ij^{TT}}$ (which corresponds to $\mathcal{H}$ being quadratic with respect to these dynamical variables) is concerned, the refined formalism is equivalent to the old one in Ref. 10).

## § 3. Derivation of the Schwarzschild-de Sitter solution

As already mentioned, our refined formalism obtained in § 2 is useful to look into the dynamical problem of a spherical distribution of matter in the isotropic expanding universe universe with the metric (2·1). To exemplify the situation as simply as possible, we shall consider an isotropic and vacuum universe with a positive $\Lambda$, so that we must have $p=\rho=0$ in Eqs. (2·6)~(2·9) (which further provide us with $p_b=\rho_b=0$ in Eq. (2·2)). Accordingly the only background universe is the de Sitter universe specified by

$$f(t)=e^{ht} \quad \text{with } h\equiv \dot{f}/f=(\Lambda/3)^{1/2}. \tag{3·1}$$

Since we are considering an isotropic case such as $g_{ij}{}^{TT}=\pi^{ij^{TT}}=0$, Eq. (2·15) is reduced to

$$\begin{cases} g_{ij}=f^2(t)\phi^4\delta_{ij}\,, \\ \pi^{ij}=-2hf\delta_{ij}+\tilde{\pi}^{ij}, \end{cases} \tag{3·2}$$

where it follows from Eqs. (2·14), (2·16) and (3·2) that

$$\tilde{\pi}^{ii}=\chi=6hf(1-\phi^2), \tag{3·3}$$

in which $\phi=\phi(t,r)$ with $r\equiv(x^2+y^2+z^2)^{1/2}$. Similarly, we must have

$$N=N(t,r), \qquad N_i=0. \tag{3·4}$$

On inserting Eq. (3·2)~(3·4) and $p=\rho=0$ into Eqs. (2·8) and (2·9), we obtain

$$g(R-2\Lambda)+\frac{1}{2}\pi^2-\pi_{ij}\pi^{ij}=-f^4\{\phi^7\nabla^2\phi+(c_{ij})^2\}=0, \tag{3·5}$$

$$\pi^{ij}{}_{|j}=\phi^{-4}c_{ij,j}=0\,, \tag{3·6}$$

where

$$c_{ij}\equiv\phi^4\{\tilde{\pi}^{ij}-2hf(1-\phi^2)\delta_{ij}\}, \quad \text{so that } c_{ii}=0. \tag{3·7}$$

Equations (3·6) and (3·7) show the transverse traceless property of $c_{ij}$, i.e., $c_{ij}=c_{ij}{}^{TT}$. This means that $c_{ij}=0$ in the present case, so that Eq. (3·5) is reduced to

$$\nabla^2\phi=0. \tag{3·8}$$

Since the metric three-tensor $g_{ij}=f^2(t)\phi^4(t,r)\delta_{ij}$ should tend to the background one for the de Sitter universe, i.e., $(g_{ij})_b=f^2(t)\delta_{ij}$, when $r\to\infty$, the required solution of Eq. (3·8) is given by

$$\phi=1+C(t)/r\,, \tag{3·9}$$

where $C(t)$ is an arbitrary function of $t$. Then we can also rewrite Eq. (3·2) as

$$\begin{cases} g_{ij}=f^2\phi^4\delta_{ij}\,, \\ \pi^{ij}=-2hf\phi^2\delta_{ij}\,. \end{cases} \tag{3·10}$$

Next we shall turn to the dynamical equations (2·6) and (2·7) with $p=\rho=0$. On inserting Eq. (3·10) into Eqs. (2·6) and (2·7) with $i\neq j$, we obtain

$$N_{i|j}+N_{j|i}=0\,, \tag{3·11}$$

$$N_{|ij}-\{-2\phi^{-1}\phi_{,ij}+6\phi^{-2}\phi_{,i}\phi_{,j}\}N=0\,. \tag{3·12}$$

By virtue of Eqs. (3·4) and (3·9), we easily see that Eq. (3·11) holds identically and that Eq. (3·12) gives rise to

$$N=\frac{1-C(t)/r}{1+C(t)/r}=2/\phi-1\,. \tag{3·13}$$

Then it follows from Eq. (2·6) with $i=j$ that

$$\partial_t(f^2\phi^4)=2(2/\phi-1)hf^2\phi^4,$$

which, together with Eq. (3·1), leads to

$$\phi-1\propto f^{-1}(t). \tag{3·14}$$

In order that Eqs. (3·9) and (3·14) may be consistent to each other, we must have

$$\phi=1+\psi,\quad \psi\equiv\frac{m}{2f(t)r}\,, \tag{3·15}$$

where $m$ is a positive constant to be identified with the central mass. Now it is not so difficult to examine that Eq. (2·7) with $i=j$ is automatically satisfied by Eqs. (3·10), (3·13) and (3·15), together with $p=N_i=0$.

The obtained solution is specified by the metric

$$ds^2=-\left(\frac{1-\psi}{1+\psi}\right)^2dt^2+f^2(t)(1+\psi)^4\delta_{ij}dx^idx^j, \tag{3·16}$$

which, together with Eqs. (3·1) and (3·15), represents the Schwarzschild-de Sitter solution.[13]

## § 4. Concluding remarks

Since the metric three-tensor $g_{ij}$ given by Eq. (2·15) consists of the isotropic part ($\propto\delta_{ij}$) and the dynamical one $g_{ij}{}^{TT}$, our refined formalism is suitable to search for cosmological counterparts of Arnowitt and Deser's work[14] for various conditions of flatness in general relativity and of Choquet-Bruhat and Deser's work[15] on the stability of flat space. These problems will be dealt with in a separate paper.

Here, it would not be useless to point out that Nariai's model (when $\Lambda=0$)[16] for a spherical gravitational collapse with pressure gradient also deals with a spherically symmetric space-time specified by Eq. (3·16) with $\psi^{-1}=f(t)(1+ar^2)^{1/2}$ ($a=\text{const}>0$) in which $f(t)$ is a regular and positive function of $t$.

Moreover, the problem worked out in § 3 will also be extensible to the case where the background space-time is a more realistic (e.g., the big-bang) universe rather than the de Sitter one.

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