ON A CERTAIN CLASS OF ORTHOGONAL POLYNOMIALS

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Introduction. E. H. Hildebrandt has demonstrated the following theorem[1]: *If y is a non-identically zero solution of the Pearsonian Differential Equation,*

$$\frac{1}{y}\frac{dy}{dx} = \frac{a_0 + a_1 x}{b_0 + b_1 x + b_2 x^2} \equiv \frac{N}{D}, \qquad a_i, b_i \text{ real, then} \tag{1}$$

$$\frac{D^{n-k}}{y}\frac{d^n}{dx^n}(D^k y) \equiv P_n(k, x), \qquad n, k \text{ integers}, n \geq 0, \text{ is a} \tag{2}$$

polynomial in x of degree n at most. Hildebrandt has obtained various relations connecting the $P_n(k, x)$ and their derivatives as well as a recurrence relation.

If in (2) we set $k = n$ there results from a proper choice of N and D in (1), the classical Hermite, Laguerre, Jacobi and Legendre Polynomials. Many properties of these classical polynomials have been obtained by numerous investigators.[2]

One of the most important of these properties is that of orthogonality which can be stated as follows: *Consider a sequence of the classical polynomials* $\Phi_i(x) = x^i - S_i x^{i-1} + \cdots$. *There exists an interval* (a, b) *finite or infinite and a unique weight function* $\psi(x)$, *monotonic non-decreasing over* (a, b) *such that,*

$$\int_a^b \Phi_m(x)\Phi_n(x)\, d\psi(x) = 0, \qquad \text{for } n \neq m. \tag{3}$$

In the future we will refer to the type of orthogonality given by (3) *with* $\psi(x)$ *monotonic non-decreasing as orthogonality in the restricted sense.* In order to determine whether a given system of polynomials is orthogonal in the restricted sense we have the following theorem:[3]

Theorem 1. *In order that the sequence of polynomials* $\Phi_i(x) = x^i - S_i x^{i-1} +$

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[1] E. H. Hildebrandt, "Systems of polynomials connected with the Charlier expansions, etc.," *Annals of Math. Stat.*, Vol. 2(1931), pp. 379–439.

[2] For an account of these properties as well as an extensive bibliography the reader can refer to one of two treatises viz.: J. Shohat, *Théorie Générale des Polynomes Orthogonaux de Tchebichef*, Memoriale des Sciences Mathématiques, Fascicule 66, Paris, Gauthier Villars, 1936.

Gabor Szegö, *Orthogonal Polynomials*, Am. Math. Soc., Colloquium Publications, Vol. 23, 1939.

[3] J. Shohat, "The relation of the classical orthogonal polynomials to the polynomials of Appell," *Am. Jour. of Math.*, Vol. 58(1936), pp. 454–455.

$\cdots$, $i = 1, 2, 3, \cdots$ *with real coefficients be orthogonal in the restricted sense it is necessary and sufficient that there exist a recurrence relation,*

$$(4) \qquad \Phi_i(x) = (x - c_i)\Phi_{i-1}(x) - \lambda_i\Phi_{i-2}(x), \qquad \Phi_0 = 1, \qquad \Phi_1 = x - c_1,$$

c_i, λ_i *const. with all* $\lambda_i > 0$, $i \geq 2$.

With Shohat[4] *we will say that a system of polynomials* $\Phi_i(x) = x^i - S_i x^{i-1} + \cdots$, $i = 1, 2, 3, \cdots$, *with real coefficients is orthogonal in the general sense if there exists at least one weight function* $\psi(x)$, *of bounded variation over* (a, b) *such that* (3) *is satisfied.* In connection with generalized orthogonality we have the following theorem:[4]

THEOREM 2. *In order that the system* $\Phi_i(x)$, $i = 1, 2, 3, \cdots$ *be orthogonal in the general sense it is necessary and sufficient that relation* (4) *be satisfied with all* $\lambda_i \neq 0$.

It is the purpose of this paper to investigate the orthogonality properties of the general polynomials $P_n(n, x)$ given by (2). In Part 1 a general recurrence relation is derived which applies to all the polynomials $P_n(k, x)$. In Part 2 all the different types of orthogonal polynomials $P_n(n, x)$ are determined by making use of the general recurrence relation derived in Part 1. We also show, following lines laid down by Hahn[5], that the only systems of polynomials with simple zeros which are orthogonal in either the restricted or the general sense and whose derivatives are orthogonal in either sense are the systems considered in Part 2.

1. **The general recurrence relation.** From (2) we can write,

$$(5) \qquad P_{n-1}(k, x) = \frac{D^{n-k-1}}{y}\frac{d^{n-1}}{dx^{n-1}} D^k y \equiv \frac{D^{n-k-1}}{y}\frac{d^{n-1}}{dx^{n-1}}[D \cdot D^{k-1} y].$$

Apply Leibnitz Formula to the right side and make use of (2). There results,

$$(6) \qquad \begin{aligned} P_{n-1}(k, x) &= P_{n-1}(k-1, x) + (n-1)D'P_{n-2}(k-1, x) \\ &\quad + \frac{(n-1)(n-2)}{1 \cdot 2} D''DP_{n-3}(k-1, x). \end{aligned}$$

From Hildebrandt's paper we have,[6]

$$(7) \quad P_{n+1}(k+1, x) = [N + (k+1)D']P_n(k, x) + n[N' + (k+1)D'']DP_{n-1}(k, x).$$

Decrease k and n each by one in (7) and obtain a relationship which we number (8). Again decrease n by one in (8) and get a relation which we number (9).

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[4] J. Shohat, "Sur les polynomes orthogonaux généralisés," *Comptes Rendus*, Vol. 207 (1938), p. 556.

[5] Wolfgang Hahn, "Über die Jacobischen polynome und zwei verwandte polynomklassen," *Math. Zeits.*, Vol. 39(1934–35), pp. 634–638.

[6] E. H. Hildebrandt, loc. cit. p. 407.

From (6), (7), (8) and (9) eliminate $P_{n-1}(k, x)$, $P_{n-2}(k-1, x)$, and $P_{n-3}(k-1, x)$. There results,

$$
\begin{aligned}
(10)\qquad [2N' + (2k - n + 1)D''][N' + kD'']P_{n+1}(k+1, x) \\
&= \{[2N' + (2k - n + 1)D''][N' + kD''][N + (k+1)D'] \\
&+ n[N' + (k+1)D''][2N'D' + kD'D'' - ND'']\}P_n(k, x) \\
&+ n[N' + (k+1)D'']\{2(N' + kD'')^2 D \\
&- (N + kD')(2N'D' + kD'D'' - ND'')\}P_{n-1}(k-1, x).
\end{aligned}
$$

In (10) decrease n and k each by one and replace N and D by their values from (1). Thus we get,

$$
\begin{aligned}
(11)\quad [a_1 + (2k - n)b_2][a_1 + 2(k-1)b_2]P_n(k, x) \\
&= \{[a_1 + (2k-2)b_2][a_1 + 2kb_2][a_1 + (2k-1)b_2]x \\
&+ [a_1 + (2k-2)b_2][a_1 + (2k-n)b_2][a_0 + kb_1] \\
&+ (n-1)[a_1 + 2kb_2][a_1b_1 + (k-1)b_1b_2 - a_0b_2]\}P_{n-1}(k-1, x) \\
&+ (n-1)[a_1 + 2kb_2]\{b_0[a_1 + (2k-2)b_2]^2 \\
&- [a_0 + (k-1)b_1][a_1b_1 + (k-1)b_1b_2 - a_0b_2]\}P_{n-2}(k-2, x).
\end{aligned}
$$

In this recurrence formula the $P_n(k, x)$ have in general a coefficient of x^n different from one. *Polynomials which have one for the coefficient of x^n we will refer to in the future as normalized.* Let us now transform (11) for normalized $P_n(k, x)$. Theorem 1 deals with polynomials normalized in the above sense. Let us write,

$P_n(k, x) = a_{n,k}x^n - b_n x^{n-1} + \cdots$. In (4) set, $\Phi_n(x) = P_n(k, x)/a_{n,k}$.

Thus we get,

$$(12)\qquad P_n(k, x) = (A_n x - B_n)P_{n-1}(k-1, x) - \gamma_n P_{n-2}(k-2, x)$$

where

$$\gamma_n \equiv \frac{a_{n,k}}{a_{n-2,k-2}}\lambda_n, \qquad A_n \equiv \frac{a_{n,k}}{a_{n-1,k-1}}, \quad \text{and} \quad B_n \equiv \frac{a_{n,k}}{a_{n-1,k-1}}C_n.$$

Relation (12) is essentially of the same form as (11). Each of these is to be reduced to form (4).

From a previous paper by the author[7] we have,

$$(13)\qquad P'_{n+1}(k, x) = (n+1)[N' + \tfrac{1}{2}(2k - n)D'']P_n(k, x).$$

$n - 1$ successive applications of this relation give us, $[P_0(k, x) \equiv 1]$, that the coefficient of x^n in $P_n(k, x)$ is,

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[7] Frank S. Beale, "On the polynomials related to Pearson's differential equation," *Annals of Math. Stat.*, Vol. 8(1937), p. 207 (2).

$$a_{n,k} = \prod_{i=0}^{n-1} [a_1 + (2k - n + 1 + i)b_2]. \tag{14}$$

By employing (14) in (12) we see that (12) or (11) reduces to form (4) where,

$$\begin{aligned} c_n = &- \frac{[a_1 + (2k - n)b_2][a_0 + kb_1]}{[a_1 + 2kb_2][a_1 + (2k - 1)b_2]} \\ &- (n - 1) \frac{[a_1 b_1 + (k - 1)b_1 b_2 - a_0 b_2]}{[a_1 + (2k - 1)b_2][a_1 + (2k - 2)b_2]}. \end{aligned} \tag{15}$$

$$\lambda_n = - (n - 1) \frac{[a_1 + (2k - n - 1)b_2]\{b_0 [a_1 + (2k - 2)b_2]^2 - [a_0 + (k - 1)b_1][a_1 b_1 + (k - 1)b_1 b_2 - a_0 b_2]\}}{[a_1 + (2k - 3)b_2][a_1 + (2k - 2)b_2]^2[a_1 + (2k - 1)b_2]} \tag{16}$$

Equation (16) together with Theorems 1 and 2 can now be applied to the polynomials $P_n(k, x)$.

From (14) *it is seen that* $P_n(k, x)$ *is of degree* n *provided that none of the factors of the product vanishes. This condition we assume to hold here for all* n.

We can now obtain a recurrence relation for the qth derivatives of $P_n(k, x)$. A repeated application of (13) leads to,

$$\frac{d^q}{dx^q} P_n(k, x) = P_{n-q}(k, x) \prod_{i=0}^{q-1} (n - i) [a_1 + (2k - n + i + 1)b_2], \tag{17}$$

where $P_n(k, x)$ is not normalized in the above sense. By considering the right side of (17) together with (14) we see that (17) can be divided by

$$a_{n-q,k} \prod_{i=0}^{q-1} (n - i) [a_1 + (2k - n + i + 1)b_2]$$

and thus normalize the polynomials on both the right and left sides of (17). Consequently the recurrence relation for normalized $d^q[P_n(k, x)]/dx^q$, $n = 0, 1, 2, \cdots$, is identical with the recurrence relation for normalized $P_{n-q}(k, x)$ as given by (4), (15) and (16) when we replace n by $n - q$ in these latter.

2. The different types of orthogonal $P_n(n, x)$. Suppose first that $b_2 \neq 0$ in (1). A transformation on x with real coefficients can be affected which changes (1) into either,

$$\frac{1}{y}\frac{dy}{dx} = \frac{(\alpha - \beta) + (-\alpha - \beta)x}{1 - x^2} \quad \text{or} \tag{18}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{-2mx - q}{a^2 + x^2}. \tag{19}$$

(A) Equation (18) together with (2) for $k = n$ defines the generalized Jacobi Polynomials (normalized in the above sense),

$$J_n(x, \alpha, \beta) = \frac{1}{a_{n,n}} (1 + x)^{-\alpha}(1 - x)^{-\beta} \frac{d^n}{dx^n} [(1 + x)^{n+\alpha}(1 - x)^{n+\beta}]$$

where $1/a_{n,n}$ is given by (14). If in (16) we set $k = n$ and make proper replacements for constants as (18) and (1) show we have,

$$\lambda_n = 4(n-1)\frac{(\alpha+\beta+n-1)(\alpha+n-1)(\beta+n-1)}{(\alpha+\beta+2n-3)(\alpha+\beta+2n-2)^2(\alpha+\beta+2n-1)}, \quad n \geq 2. \tag{20}$$

From Theorem 1 and this value of λ_n we conclude that if $\alpha > -1, \beta > -1$, the sequence $\{J_n(x, \alpha, \beta)\}$ is orthogonal in the restricted sense—a well-known result. From Theorem 2 we can similarly conclude that if neither α, β, nor $(\alpha + \beta)$ equals $-j$, j a positive integer, the sequence $\{J_n(x, \alpha, \beta)\}$ is orthogonal in the general sense.

(A_1) If in (18) we set $\alpha = \beta = 0$ we obtain a differential equation which together with (2) for $k = n$ leads to the Legendre Polynomials, (normalized in above sense), $P_n(x) = \dfrac{n!}{(2n)!}\dfrac{d^n}{dx^n}(x^2 - 1)^n$. Setting $\alpha = \beta = 0$ in (20) leads to $\lambda_n = \dfrac{(n-1)^2}{(2n-3)(2n-1)}$, $n \geq 2$. Thus from Theorem 1 we conclude that the Legendre Polynomials are orthogonal in the restricted sense, a result well known.

(B) Equation (19) together with (2) for $k = n$ leads to a class of polynomials (normalized in above sense), mentioned by Romanovsky.[8]

$$R_n(x, m, q, a) = \frac{1}{a_{n,n}}(a^2 + x^2)^m \exp\left(\frac{q}{a}\tan^{-1}\frac{x}{a}\right)\frac{d^n}{dx^n}\left[(a^2 + x^2)^{n-m}\exp -\frac{q}{a}\tan^{-1}\frac{x}{a}\right]$$

where again $1/a_{n,n}$ is given by (14). In (16) set $k = n$ and make the proper replacements of constants and,

$$\lambda_n = \frac{n-1}{4}\,\frac{(2m-n+1)\{4a^2(m-n+1)^2 + q^2\}}{(2m-2n+3)(m-n+1)^2(2m-2n+1)}, \qquad n \geq 2.$$

From Theorem 2 it now follows that the sequence $\{R_n(x, m, q, a)\}$ is orthogonal in the general sense if $m \neq j/2$, j a positive integer. There is no set of parameters m, q, a which assures orthogonality in the restricted sense.

In connection with Romanovsky's note there appear to be several discrepancies. For the weight functions given there under types IV and V, the nth moments for sufficiently large n do not exist over the intervals there considered. Type V is the special case of type IV for $a = 0$. Type VI is none other than Jacobi Polynomials so that the orthogonality relations given there for this case are incorrect. In all three types listed certain of the recurrence relations for the polynomials are in error.

(B_1) We note here one special sub-class of R_n. Take $m = q = 0$ and $a = 1$ in (19). We obtain from (2) and (14) a system of normalized polynomials analogous to the Legendre Polynomials namely, $\phi_n(x) = \dfrac{n!}{(2n)!}\dfrac{d^n}{dx^n}(x^2 + 1)^n$.

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[8] V. Romanovsky, "Sur quelques classes nouvelles de polynomes orthogonaux," *Comptes Rendus*, Vol. 188(1929), pp. 1023–1025.

It is easy to verify for these that,

$$\int_{-i}^{i} \phi_n(x)\phi_m(x)\,dx = 0, \qquad m \neq n, \qquad i = \sqrt{-1}.$$

(C) Suppose that in (1), $b_2 = 0$, $b_1 \neq 0$. A linear transformation with real coefficients changes (1) into, $\frac{1}{y}\frac{dy}{dx} = \frac{\alpha - x}{x}$. This equation together with (2) and (14) for $k = n$ defines the generalized Laguerre Polynomials, (normalized in above sense), $L_n(x, \alpha) = (-1)^n x^{-\alpha} e^x \frac{d^n}{dx^n}[x^{n+\alpha}e^{-x}]$. Setting $k = n$ and making proper replacements in (16) we get, $\lambda_n = (n - 1)(\alpha + n - 1)$, $n \geq 2$. From Theorem 1 we see that if $\alpha > -1$ the L_n are orthogonal in the restricted sense, a well-known result. From Theorem 2 we can say that if $\alpha \neq -j$, j a positive integer, the polynomials are orthogonal in the general sense.

(D) If in (1), $b_1 = b_2 = 0$, $b_0 \neq 0$ we can perform a linear transformation on x with real coefficients and get, $\frac{1}{y}\frac{dy}{dx} = hx$. This differential equation together with (2) and (14) gives a set of normalized polynomials $G_n(x) = \frac{1}{h^n} e^{-hx^2/2} \frac{d^n}{dx^n} e^{hx^2/2}$ Taking $k = n$ and making proper substitutions for constants in (16) we get $\lambda_n = -(n - 1)/h$, $n \geq 2$. If h is negative it follows from Theorem I that the sequence $\{G_n(x)\}$ is orthogonal in the restricted sense. In fact, $G_n(x) \equiv H_n(x) \equiv$ Hermite Polynomials.

On the other hand, if h is positive we have from Theorem 2 orthogonality in the general sense. In fact, it can be easily verified for this case that,

$$\int_{-i\infty}^{i\infty} e^{hx^2/2} G_n(x)G_m(x)\,dx = 0, \qquad m \neq n, \qquad i = \sqrt{-1}.$$

(E) The only remaining possibility for (1) not so far discussed occurs when $N \equiv$ constant and D is linear. In this case it has been shown that $P_n(k, x)$ of (2) reduces to a constant.[9]

E. H. Hildebrandt has shown[10] that the polynomials $P_n(n, x)$ of (2) satisfy a differential equation of the form,

$$(21) \quad (b_0 + b_1 x + b_2 x^2)\frac{d^2y}{dx^2} + [a_0 + b_1 + (a_1 + 2b_2)x]\frac{dy}{dx} - n[a_1 + (n + 1)b_2]y = 0, \qquad n = 1, 2, 3, \cdots.$$

Moreover with the coefficients of d^2y/dx^2 and dy/dx in (21) he has shown that for (21) to have a polynomial solution of degree n the coefficient of y must be of the form given in (21).

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[9] Frank S. Beale, loc. cit. p. 209, Theorem I_9.

[10] Loc. cit. pp. 404–405.

From (16) we can say that for $k = n$ and an orthogonal sequence $P_n(n, x)$, $n = 0, 1, 2, \cdots$ we have,

$$a_1 + (n - 1)b_2 \neq 0, \tag{22}$$

$$b_0[a_1 + (2n - 2)b_2]^2 - [a_0 + (n - 1)b_1][a_1b_1 + (n - 1)b_1b_2 - a_0b_2] \neq 0, \tag{23}$$

where n is an integer ≥ 2. Considering for (21) a solution of the type $y = \sum_{i=0}^{\infty} c_i x^i$ we readily show that if (22) and (23) are satisfied, (21) possesses for each n a single polynomial solution of degree n. Two solutions which differ merely by a constant factor are regarded as the same solution. This polynomial solution of (21) must be $P_n(n, x)$.

By employing theorems from a previous paper by the author[11] we can show that if (22) and (23) are satisfied, the zeros of the polynomials of section II are simple whether these zeros are real or complex.

Hahn has shown[12] that if a set of normalized polynomials and their derivatives satisfy a relation of the form (4) with $\lambda_i \neq 0$ and if the zeros of the polynomials are all simple then the polynomials must necessarily satisfy an equation of form (21). Since in this paper we have considered all possible values of a_i, $(i = 0, 1)$, and b_i, $(i = 0, 1, 2)$, which lead to orthogonal polynomials, it follows that the only systems of polynomials with simple zeros and orthogonal in either restricted or general sense whose derivatives in turn are orthogonal in either sense are the systems of section 2.

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[11] Loc. cit. pp. 207–209, Theorems I_1 to I_{10}.

[12] Loc. cit. pp. 634–636.