(*) in § 2 because $f_n[\mathrm{Fr}(R)] = f_n(C)$ and this set lies in V. Accordingly $f_n(R) \subset V_\sigma(y)$.

Now for n sufficiently large, say $n > N_1 > N$, we have $x_n \epsilon R$ and $y_n = f(x_n) \epsilon V_\sigma(y)$. However this gives $z_n \epsilon f_{k_n}(R) \subset V_\sigma(y)$ so that $\varrho(y_n, z_n) < 2\sigma = \alpha$, contrary to relation (i) (see first paragraph of the proof in § 4). Thus the supposition of non-uniform convergence on A leads to a contradiction.

7. Conclusion. Since for a real continuous function on the whole real axis, or on a connected open set of real numbers, monotoneity of the function is equivalent to quasi-openness of the mapping generated by the function, the theorem just proven gives at once the result: *any sequence of monotone non-increasing continuous real functions on the whole real axis which converges at an everywhere dense set to a function $f(x)$ which is continuous for all real x, necessarily converges almost uniformly to $f(x)$.*

It seems likely, however, that our theorem for quasi-open mappings may be of greater interest in connection with sequences of functions of a complex variable or of mappings on surfaces and other more complex spaces. The setting provided by a closed algebra of complex valued functions seems of special interest and it is proposed to study this in a later paper.

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ON A CERTAIN DISTANCE OF SETS AND THE CORRESPONDING DISTANCE OF FUNCTIONS

BY

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It is well known that the measure of the symmetric difference of two sets can be considered as a distance of sets (so called *distance of Fréchet-Nikodym-Aronszajn*): $\varrho(A, B) = \mu(A \dot{-} B)$. This distance is a particular case of the distance in the space of Lebesgue integrable functions.

This paper is devoted to the study of another distance of sets, defined by the formula

$$\sigma(A, B) = \frac{\mu(A \dot{-} B)}{\mu(A+B)} = \frac{\varrho(A, B)}{\mu(A+B)},$$

and the corresponding distance of functions.

The distance σ seems to be useful in several practical applications and especially in some biological problems (see n° 3 and our paper on a systematical distance of biotopes [2]).

1. SETS

1.1. Metric ϱ. Let $(X, \boldsymbol{M}, \mu)$ be a σ-finite σ-measure space. Let us denote by $\boldsymbol{M}_0$ the class of all sets $A \epsilon \boldsymbol{M}$ with $\mu(A) < \infty$, and by ϱ_μ the well-known distance of sets $A, B \epsilon \boldsymbol{M}_0$:

$$\varrho_\mu(A, B) = \mu(A \dot{-} B),$$

where $A \dot{-} B$ denotes the symmetric difference of A and B.

The index μ will be omitted in this case and in other analogous ones, when no misunderstanding is possible.

Let us recall the fundamental properties of ϱ (see e. g. [1], p. 168 and 169):

(i) *$(\boldsymbol{M}_0, \varrho)$ is a metric space when we identify any two sets, the symmetric difference of which is of measure μ zero.*

(ii) *None of the numbers*: $\varrho(A_1+B_1, A_2+B_2)$, $\varrho(A_1B_1, A_2B_2)$, $\varrho(A_1-B_1, A_2-B_2)$, $\varrho(A_1\dot{-}B_1, A_2\dot{-}B_2)$ *surpasses* $\varrho(A_1, A_2)+\varrho(B_1, B_2)$. *None of the numbers* $|\mu(A_1)-\mu(A_2)|$ *and* $\varrho(X-A_1, X-A_2)$ *surpasses* $\varrho(A_1, A_2)$.

In other words, all set-theoretical finite operations and the real function μ are Lipschitzian (with coefficients 1) in $(\boldsymbol{M}_0, \varrho)$. Thus, we can state a fortiori that

(iii) *The finite set-theoretical operations are continuous in* $(\boldsymbol{M}_0, \varrho)$.

(iv) μ *is a real function continuous in* $(\boldsymbol{M}_0, \varrho)$.

Following proposition is important:

(v) *Every sequence of sets* $A_j \epsilon \boldsymbol{M}_0$, *fundamental with respect to* ϱ, *contains a subsequence* A_{k_j} *such that*

$$\mu(\limsup_j A_{k_j} - \liminf_j A_{k_j}) = 0$$

(where lim sup *and* lim inf *are to be read in the set-theoretical sense).*

(vi) *If*

$$A_* = \liminf_i A_j, \qquad A^* = \limsup_i A_j$$

and

$$\mu(A^*-A_*) = 0,$$

then

$$\lim_j \varrho(A_j, A_*) = 0 = \lim_j \varrho(A_j, A^*).$$

It results easily from (v) and (vi) that

(vii) *The metric space* $(\boldsymbol{M}_0, \varrho)$ *is complete.*

1.2. Metric σ. Let us define a new distance of sets belonging to $\boldsymbol{M}_0$:

$$\sigma_\mu(A, B) = \begin{cases} \dfrac{\mu(A\dot{-}B)}{\mu(A+B)} & \text{if} \quad \mu(A+B) > 0, \\ 0 & \text{if} \quad \mu(A+B) = 0. \end{cases}$$

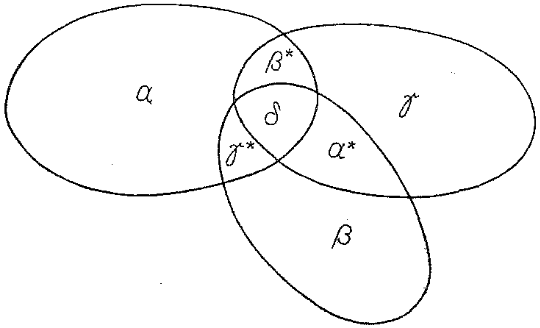
(i) $(\boldsymbol{M}_0, \sigma)$ *is a metric space (when we identify any two sets, the symmetric difference of which is of measure μ zero).*

It follows directly from the definition that $\sigma(A, B) = 0$ if and only if $\mu(A\dot{-}B) = 0$, and that σ is symmetric.

In order to prove that

$$\sigma(A, B)+\sigma(B, C) \geqslant \sigma(A, C)$$

let us denote by $\alpha, \alpha^*, \beta, \beta^*, \gamma, \gamma^*$ and δ the values of μ for the atoms of A, B and C:



Then, the inequality in question may be written as follows:

$$\frac{\alpha+\alpha^*+\beta+\beta^*}{\alpha+\alpha^*+\beta+\beta^*+\gamma^*+\delta} + \frac{\beta+\beta^*+\gamma+\gamma^*}{\beta+\beta^*+\gamma+\gamma^*+\alpha^*+\delta} \geqslant \frac{\alpha+\alpha^*+\gamma+\gamma^*}{\alpha+\alpha^*+\gamma+\gamma^*+\beta^*+\delta}.$$

After reducing the fractions above to a common denominator and ordering the terms according to the powers of δ, we have only to prove an inequality of the form

$$P\delta^2+Q\delta+R \geqslant p\delta^2+q\delta+r.$$

An easy verification gives $P \geqslant p$, $Q \geqslant q$ and $R \geqslant r$, and, since $\delta \geqslant 0$, the inequality is proved.

It follows easily from the definition of σ that

(ii) $\sigma(A, B) \leqslant 1$ *for* $A, B \epsilon \boldsymbol{M}_0$, *and* $\sigma(A, B) = 1$ *if and only if* $\mu(AB) = 0$ *and* $\mu(A+B) > 0$.

In particular

(iii) *If* $\mu(A) = 0$ *and* $\mu(B) > 0$, *then* $\sigma(A, B) = 1$.

Therefore the empty set (with all sets of measure μ zero) forms an isolated point of the space $(\boldsymbol{M}_0, \sigma)$.

Thus the identity transformation of $(\boldsymbol{M}_0, \varrho)$ onto $(\boldsymbol{M}_0, \sigma)$ is not homeomorphic for some measures. Nevertheless, we shall prove that the empty set is a unique discontinuity point of this mapping:

(iv) *If $A \in M_0$ and $A_j \in M_0$ for $j = 1, 2, \ldots$, then the relation*

$$(\alpha) \qquad \lim_j \sigma(A_j, A) = 0$$

implies

$$(\beta) \qquad \lim_i \varrho(A_j, A) = 0$$

and if, moreover, $\mu(A) > 0$, then (β) *implies* (α).

Since

$$1 \geqslant \sigma(A_j, A) = \frac{\mu(A_j \dot{-} A)}{\mu(A_j + A)} = 1 - \frac{\mu(A_j A)}{\mu(A_j + A)} \geqslant 1 - \frac{\mu(A)}{\mu(A + A_j)},$$

the relation (α) implies $\mu(A + A_j) \to \mu(A)$ and consequently the sequence $\mu(A + A_j)$ is bounded: $0 \leqslant \mu(A + A_j) < a$, whence

$$\sigma(A_j, A) = \frac{\varrho(A_j, A)}{\mu(A_j + A)} \geqslant \frac{\varrho(A_j, A)}{a} \geqslant 0.$$

Thus (α) implies (β).

Since

$$\sigma(A_j, A) = \frac{\varrho(A_j, A)}{\mu(A_j + A)} \leqslant \frac{\varrho(A_j, A)}{\mu(A)},$$

the relations $\mu(A) > 0$ and (β) imply (α). Theorem (iv) is thus proved.

Let us prove now the following lemma:

(v) *If A_j form a fundamental sequence with respect to σ and if $\mu(A_j) < 0$, then there are positive numbers a and b such that $a < \mu(A_j) < b$ $(j = 1, 2, \ldots)$.*

It suffices to prove that the relations $\mu(A_j) \to 0$ and $\mu(A_j) \to \infty$ are impossible.

We have for every fixed positive integer j_0

$$(*) \qquad \sigma(A_j, A_{j_0}) = 1 - \frac{\mu(A_j A_{j_0})}{\mu(A_j + A_{j_0})}.$$

If $\lim_j \mu(A_j) = 0$, then $\lim_j \mu(A_j A_{j_0}) = 0$, and $\lim_j \mu(A_j + A_{j_0}) = \mu(A_{j_0})$, whence, by (*),

$$(**) \qquad \lim_j \sigma(A_j, A_{j_0}) = 1.$$

If $\lim_j \mu(A_j) = \infty$, then

$$\frac{\mu(A_j A_{j_0})}{\mu(A_j + A_{j_0})} \leqslant \frac{\mu(A_{j_0})}{\mu(A_j + A_{j_0})} \to 0,$$

whence, in view of (*), we obtain (**) again.

But formula (**) contradicts the hypothesis that A_j form a sequence fundamental with respect to σ.

Lemma (v) implies

(vi) *The space (M_0, σ) is complete.*

In fact, if A_j is a fundamental sequence in (M_0, σ), then, by (v), A_j is fundamental in (M_0, ϱ). By 1.1 (vii) there exists such an $A \in M$ that $\varrho(A_j, A) \to 0$, whence, by 1.1 (iv), $\mu(A_j) \to \mu(A)$. It follows from (v) that $\mu(A) > 0$, whence, in view of (iv), $\sigma(A_j, A) \to 0$, q. e. d.

Let us denote by M_0^+ the class of all sets $A \in M$ with $0 < \mu(A) < \infty$. Propositions (iv), 1.1 (iii), and 1.1 (iv) imply the continuity of finite set-theoretical operations and of μ in (M_0^+, σ). It is worth noticing that, moreover, these operations are Lipschitzian (with coefficients 1):

(vii) $\sigma(A_1 + B_1, A_2 + B_2) \leqslant \sigma(A_1, A_2) + \sigma(B_1, B_2)$.

In the case $\mu(A_1 + A_2) \neq 0 \neq \mu(B_1 + B_2)$ it follows from 1.1 (ii) that

$$\sigma(A_1 + B_1, A_2 + B_2) = \frac{\varrho(A_1 + B_1, A_2 + B_2)}{\mu(A_1 + B_1 + A_2 + B_2)} \leqslant \frac{\varrho(A_1, A_2) + \varrho(B_1, B_2)}{\mu(A_1 + B_1 + A_2 + B_2)}$$

$$\leqslant \frac{\varrho(A_1, A_2)}{\mu(A_1 + A_2)} + \frac{\varrho(B_1, B_2)}{\mu(B_1 + B_2)} = \sigma(A_1, A_2) + \sigma(B_1, B_2).$$

In the case $\mu(A_1 + A_2) = 0$ (and, analogously, if $\mu(B_1 + B_2) = 0$) we have $\sigma(A_1 + B_1, A_2 + B_2) = \sigma(B_1, B_2)$ and $\sigma(A_1, A_2) = 0$. Thus (vii) is proved.

By arguments analogous to the first part of the preceding proof, we get

(viii) *If $\mu(A_1 B_1) \neq 0 \neq \mu(A_2 B_2)$, then*

$$\sigma(A_1 B_1, A_2 B_2) \leqslant \sigma(A_1, A_2) + \sigma(B_1, B_2).$$

(ix) *If $\mu(A_1 - B_1) \neq 0 \neq \mu(A_2 - B_2)$, then*

$$\sigma(A_1 - B_1, A_2 - B_2) \leqslant \sigma(A_1, A_2) + \sigma(B_1, B_2).$$

2. *FUNCTIONS*

2.1. Metric ϱ. Let us denote by $\mathcal{L}_\mu$ the class of all μ-integrable real functions (defined on X) and by ϱ_μ the well-known distance (cf. e. g. [1], p. 98) of functions belonging to $\mathcal{L}_\mu$:

$$\varrho_\mu(f, g) = \int |f(x) - g(x)| \, d\mu(x)$$

(where the integral is extended over X). Obviously

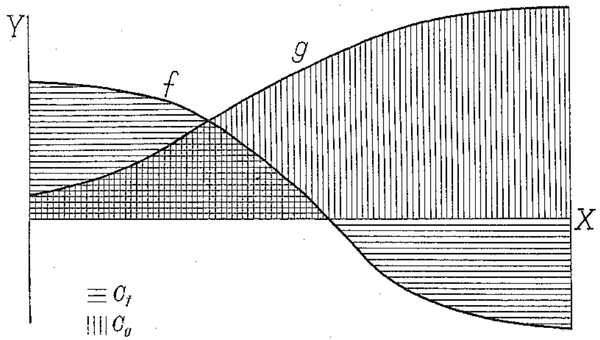
(i) *$(\mathcal{L}_\mu, \varrho_\mu)$ is a metric space (when we identify any two functions which are equal μ-almost everywhere).*

It is well-known that the distance ϱ of sets may be treated as a special case of the distance ϱ of functions. Namely, it χ_A denotes the characteristic function of the set A, we have the following obvious relation:

(ii) $\varrho_\mu(A, B) = \varrho_\mu(\chi_A, \chi_B)$ *for* $A, B \in \boldsymbol{M}_0$.

It is possible to formulate also a converse relation, which permits to say that the distance of functions is, in a certain sense, a case of the distance of sets.

To this purpose, let us denote by Y the real axis, by N the smallest σ-field of sets containing as elements all sets of the form $A \times B$, where $A \in \boldsymbol{M}$ and B is Lebesgue measurable. Let ν be a σ-measure in N, which is the direct product of μ and of the Lebesgue measure in Y.



Next, denote by C_f for $f \in \mathcal{L}_\mu$ the set of points lying in $X \times Y$ between the graph of f and the X-axis:

$$C_f = \{(x, y)\colon x \in X, \text{ and } 0 \leqslant y \leqslant f(x) \text{ or } f(x) \leqslant y \leqslant 0\}.$$

Then

(iii) $\varrho_\mu(f, g) = \varrho_\nu(C_f, C_g)$ *for* $f, g \in \mathcal{L}_\mu$.

In fact, the intersection of $C_f \dot{-} C_g$ by every vertical line $x = x_0$ is an interval of length $|f(x_0) - g(x_0)|$, whence, by the theorem of Fubini,

$$\int |f(x) - g(x)|\, d\mu(x) = \nu(C_f \dot{-} C_g).$$

2.2. Metric σ. We define the metric σ_μ in $\mathcal{L}_\mu$ as follows:

$$\sigma_\mu(f, g) = \frac{\int |f(x) - g(x)|\, d\mu(x)}{\int \max\big(|f(x)|, |g(x)|, |f(x) - g(x)|\big)\, d\mu(x)}$$

and $\sigma_\mu(f, g) = 0$ if $f(x) = 0 = g(x)$ μ-almost everywhere.

Let us remark that the number $\max(|a|, |b|, |a-b|)$ has a simple geometrical meaning: it is the length of the shortest closed interval containing the points: 0, a, and b.

In the case of non-negative functions the definition of σ_μ admits of a simpler form:

$$\sigma_\mu(f, g) = \frac{\int |f(x) - g(x)|\, d\mu(x)}{\int \max(|f(x)|, |g(x)|)\, d\mu(x)}.$$

Let us prove now two propositions analogous to 2.1 (ii) and 2.1 (iii).

(i) $\sigma_\mu(A, B) = \sigma_\mu(\chi_A, \chi_B)$.

In fact

$$\mu(A \dot{-} B) = \int |\chi_A(x) - \chi_B(x)|\, d\mu(x),$$
$$\mu(A + B) = \int \max\big(\chi_A(x), \chi_B(x)\big)\, d\mu(x).$$

(ii) $\sigma_\mu(f, g) = \sigma_\nu(C_f, C_g)$.

The intersection of $C_f + C_g$ by the vertical line $x = x_0$ is an interval of length

$$\max\big(|f(x_0)|, |g(x_0)|, |f(x_0) - g(x_0)|\big),$$

whence, by the theorem of Fubini,

$$\int \max\big(|f(x)|, |g(x)|, |f(x) - g(x)|\big)\, d\mu(x) = \nu(C_f + C_g).$$

Consequently, in view of 2.1 (iii), formula (ii) is proved.

Theorem (ii) permits to reduce problems on the distance between functions to the analogous problems on sets. First of all, the triangle property of functions results from the same property of sets (1.2 (ii)). Therefore:

(iii) *$(\mathcal{L}_\mu, \sigma_\mu)$ is a metric space (when we identify any two functions equal μ-almost everywhere).*

It follows directly from 1.2 (iv), 2.1 (iii) and 2.2 (ii) that, in $\mathcal{L}_\mu$, the convergences with respect to ϱ_μ and σ_μ to a function f essentially different from zero are equivalent:

(iv) *If $f, f_j \in \mathcal{L}_\mu$ for $j = 1, 2, \ldots$ and $\mu(C_f) > 0$, then $\lim_j \varrho_\mu(f_j, f) = 0$ if and only if $\lim_j \sigma_\mu(f_j, f) = 0$.*

Denote now by N_0 the class of all sets $E \in N$ with $\nu(E) < \infty$ and by C the class of all sets of the form C_f, where $f \in \mathfrak{L}_\mu$. We shall prove that

(iv) *The set C is closed in the space (N_0, σ_ν).*

Suppose that a sequence $A_j = C_{f_j}$ is convergent with respect to σ^ν to a set A. It is to prove that there is a function f such that $\nu(A \dot{-} C_f) = 0$. We may assume of course $\nu(A) > 0$.

On account of 1.2 (iv) $\lim_j \varrho_\nu(A_j, A) = 0$. Then, by 1.1 (v), there exists a subsequence $A_{k_j} = C_{f_{k_j}}$ such that

$$\nu(\limsup_j A_{k_j} - \liminf_j A_{k_j}) = 0.$$

Putting $f(x) = \limsup_j f_{k_j}(x)$ we easily get

$$\liminf_j A_{k_j} \subset C_f \subset \limsup_j A_{k_j},$$

whence, by 1.1 (vi), $\lim_j \varrho_\nu(A_j, C_f) = 0$, and consequently $\nu(A \dot{-} C_f) = 0$, q. e. d.

Propositions (iv), (ii) and 1.2 (vi) (applied to the measure ν) imply

(v) *The space $(\mathfrak{L}_\mu, \sigma_\mu)$ is complete.*

3. BIOTOPES

An especially simple case of σ arises if the space X contains only a finite number of elements $\{a, b, c, \ldots, k\}$, M being the class of all subsets of X and $\mu(E)$ the number of elements of E. Then the integration in the formulae of the preceding section reduces to the ordinary addition.

The distance σ pertaining to the case above may be applied e. g. in the study of biotopes, where it can be employed as a quantitative characteristic of the qualitative difference of two biotopes.

For instance, to characterize numerically the difference of two forests $\mathfrak{A}$ and $\mathfrak{B}$ we may proceed as follows: we consider the set A of all species growing in $\mathfrak{A}$, and the set B of all species growing in $\mathfrak{B}$; the distance $\sigma(A, B)$ will be the characteristic sought for. If there are no species common to $\mathfrak{A}$ and $\mathfrak{B}$, the distance will assume its greatest value: it will be equal to 1; if the forests are identical as to the species they contain, the distance will assume its smallest value, which is 0.

If we were interested not only which species can be found in the forests but also, how often they appear, we would take instead of A and B functions defined in $A+B$: function f may be the quantitative characteristic of the species growing in $\mathfrak{A}$, function g the analogous characteristic for $\mathfrak{B}$. The distance $\sigma(f, g)$ will now be the quantitative characteristic of the biotopical difference between the two forests considered.

Let us take, for instance, two real forests of Lower Silesia: In $\mathfrak{A}$ we have red pines (a), oaks (b), birches (c), and alders (d); in $\mathfrak{B}$ oaks, birches, alders and black pines (e). We can symbolize by 1 the presence and by 0 the absence of a species in a forest, which leads to the following table:

	a	b	c	d	e	μ
A	1	1	1	1	0	
B	0	1	1	1	1	
$A \dot{-} B$	1	0	0	0	1	2
$A+B$	1	1	1	1	1	5

We easily read that $\mu(A \dot{-} B) = 2$, $\mu(A+B) = 5$, thus $\varrho(A, B) = 2/5$.

The same two forests give another table if frequencies of the species are taken into account:

	a	b	c	d	e	$\int$
f	4	2	3	1	0	
g	0	2	1	2	5	
$\lvert f-g \rvert$	4	0	2	1	5	12
$\max(f, g)$	4	2	3	2	5	16

Thus $\sigma(f, g) = 3/4$.

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