ON A CERTAIN FAMILY OF ASYMMETRIC RIEMANN SURFACES WITH THE CYCLIC AUTOMORPHISM GROUP

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Abstract

A compact Riemann surface X of genus $g \geq 2$ is called *asymmetric* or *pseudo-real* if it admits an anticonformal automorphism but no anticonformal involution. The order $d = \#(\delta)$ of an anticonformal automorphism δ of such a surface is divisible by 4. In the particular case where $d = 4$, δ is a pseudo-symmetry and the surface is called *pseudo-symmetric*.

A Riemann surface X is said to be *p-hyperelliptic* if it admits a conformal involution ρ for which the orbit space $X/\langle \rho \rangle$ has genus p. This notion is the particular case of so called *cyclic (q,n)-gonal* surface which is defined as the one admitting a conformal automorphism ϕ of prime order n such that X/ϕ has genus q. We are interested in possible values of n and q for which an asymmetric surface of given genus $g \geq 2$ is (q,n)-gonal, and possible values of p for which the surface is p-hyperelliptic. Up till now, this problem was solved in the case where the surface is asymmetric and pseudo-symmetric. If an asymmetric Riemann surface X is not pseudo-symmetric then any anticonformal automorphism of X has order divisible by $2^s n$ for $s \geq 3$ and $n = 1$ or n being an odd prime. In this paper we give the necessary and sufficient conditions on the existence of an asymmetric Riemann surface with the full automorphism group being $G = Z_{2^s n}$, and we study (q,n)-gonal automorphisms and p-hyperelliptic involutions in G.

1. Introduction

A Riemann surface is called *asymmetric*, if it admits an anticonformal automorphism but no anticonformal involution. The term *pseudo-real* is also used, see for example [5]. Asymmetric Riemann surfaces appear naturally when we consider an involution $\iota : \mathcal{M}_g \to \mathcal{M}_g$ on the moduli space of Riemann surfaces of genus g mapping a Riemann surface onto its complex conjugate.

Then $\imath$ preserves symmetric surfaces while asymmetric surfaces are outside the set Fix$(\imath)$.

In the literature there are many results concerning asymmetric surfaces. We would like to mention some of them which were a starting point for this paper. In [8] asymmetric surfaces with cyclic automorphism groups are studied, and in [5] authors prove that there exists an asymmetric Riemann surface of any genus $g \geq 2$, they give the sharp bound on the order of its automorphism group, and they study asymmetric surfaces of genera 2 and 3. Also, the minimal genus problem for cyclic actions Z_n for arbitrary n on pseudo-real surfaces was completely solved in [1] by Gromadzki and Baginski.

An orientation reversing automorphism which has order 4 is called a *pseudo-symmetry* and the corresponding surface—*pseudo-symmetric*. Not every asymmetric surface is pseudo-symmetric—it is enough to consider an asymmetric surface with the full automorphism group being Z_8. Also, not every pseudo-symmetric surface is asymmetric.

In this paper we study the degree of hyperellipticity of asymmetric Riemann surfaces with the full automorphism group being a cyclic group of order $2^s n$ for a prime integer n and $s \geq 3$. A compact Riemann surface X of genus $g \geq 2$ is called *p-hyperelliptic* if it admits a conformal involution (called a *p-hyperelliptic involution*) such that the orbit space under the action of this involution is an orbifold of genus p. In particular cases, when $p = 0$ and $p = 1$, the surface is called *hyperelliptic* and *elliptic-hyperelliptic* respectively. It was proved in [10], that for $g > 4p + 1$, the p-hyperelliptic involution is unique and hence central in the group of all automorphisms of the surface. Using this result, groups of automorphisms of hyperelliptic, elliptic-hyperelliptic and 2-hyperelliptic surfaces were determined respectively in [6], [16] and [17]. In the asymmetric case, the hyperelliptic surfaces were studied in [15], whilst in [3] the authors determined the defining equations for such surfaces; the special case of hyperelliptic asymmetric pseudo-symmetric surfaces is also treated there. A p-hyperelliptic surface is the particular case of so-called cyclic (q, n)-gonal surface, which is defined as the one admitting a conformal automorphism δ of prime order n such that X/δ has genus q.

This paper is a continuation of [11], where we studied asymmetric Riemann surfaces with the full automorphism group Z_{4n}. We gave the necessary and sufficient conditions on the existence of such a surface, we determined the degree of its hyperellipticity, and we found all integers q for which the surface is (q, n)-gonal. Let us notice that an asymmetric Riemann surface whose full automorphism group is a cyclic group $Z_{2^s n}$ is pseudo-symmetric only for $s = 2$. So in this paper we consider surfaces which are not pseudo-symmetric.

2. Preliminaries

In this chapter we give the basic definitions concerning theory of non-euclidean crystallographic groups, called *NEC groups*, which are applied in the paper. The NEC groups are discrete and cocompact subgroups of the group $\mathscr{G}$

of all isometries of the hyperbolic plane $\mathcal{H}$. For any NEC group Λ, its algebraic structure is described by the so-called *signature*:

$$(1) \qquad s(\Lambda) = (h; \pm; [m_1, \ldots, m_r]; \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\}),$$

where the brackets $(n_{i1}, \ldots, n_{is_i})$ are called the *period cycles* and the integers n_{ij} are the *link periods*, m_i are called the *proper periods* and h is the *orbit genus* of Λ.

A presentation for the group Λ with signature (1) is given by the following generators, called *canonical generators*:

$x_1, \ldots, x_r, e_i, c_{ij}$, for $1 \le i \le k$, $0 \le j \le s_i$, and $a_1, b_1, \ldots, a_h, b_h$ if the sign is $+$ or $d_1, \ldots, d_h$ otherwise,

and relations:

$x_i^{m_i}$ for $1 \le i \le r$, $c_{ij-1}^2, c_{ij}^2, (c_{ij-1}c_{ij})^{n_{ij}}, c_{i0}e_i^{-1}c_{is_i}e_i$, for $1 \le i \le k, 1 \le j \le s_i$ and

$$x_1 \cdots x_r e_1 \cdots e_k a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_h b_h a_h^{-1} b_h^{-1} \quad \text{or} \quad x_1 \cdots x_r e_1 \cdots e_k d_1^2 \cdots d_h^2,$$

according to whether the sign is $+$ or $-$. The last relation is called the *long relation*. The elements x_i are elliptic transformations, a_i, b_i hyperbolic translations, d_i glide reflections and c_{ij} hyperbolic reflections.

An abstract group with the presentation given above can be realized as an NEC group Λ if and only if the value

$$\eta h + k - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}\left(1 - \frac{1}{n_{ij}}\right),$$

is positive, where $\eta = 2$ or 1 according to the sign being $+$ or $-$. The above expression times 2π is equal to the hyperbolic area $\mu(\Lambda)$ of a fundamental region of Λ. The Riemann-Hurwitz formula says that for any subgroup $\tilde{\Lambda}$ of an NEC group Λ with finite index,

$$[\Lambda : \tilde{\Lambda}] = \frac{\mu(\tilde{\Lambda})}{\mu(\Lambda)}.$$

If an NEC group has no orientation reversing elements, then it is the classical Fuchsian group and has a signature of the form

$$(h; +; [m_1, \ldots, m_r]; \{-\}),$$

which is often written as $(h; m_1, \ldots, m_r)$.

Any NEC group Λ with the signature (1) has so-called *canonical Fuchsian subgroup* Λ^+ consisting of all orientation preserving elements which by [14], has signature

$$(2) \qquad (\eta h + k - 1; m_1, m_1, \ldots, m_r, m_r, n_{11}, \ldots, n_{ks_k}).$$

A signature σ is called *maximal*, if for every NEC group Λ' with a signature σ' containing an NEC group Λ with the signature σ and having the same

Teichmüller dimension, the equality $\Lambda = \Lambda'$ holds. If the above condition does not hold, then the pair (σ, σ') is called a *normal* or *non-normal pair* according to if Λ is a normal subgroup of Λ' or not. The complete lists of normal and non-normal pairs are given in [2] and [9] (see also [13] and [4]). An NEC group Λ is called *maximal* if there does not exist another NEC group containing it properly. Also, for any maximal signature σ, there exists a maximal NEC group with the signature σ.

A torsion-free Fuchsian group Γ is called the *surface Fuchsian group*. It has a signature $(g; -)$, and $\mathscr{H}/\Gamma$ is a compact Riemann surface of genus g. Conversely, every compact Riemann surface X of genus $g \geq 2$ is isomorphic to such an orbit space for some surface Fuchsian group Γ. A finite group G is a group of automorphisms of X if and only if $G = \Lambda/\Gamma$ for some NEC group Λ containing Γ as a normal subgroup, and we say that G acts with the signature $s(\Lambda)$. If G is not the full automorphism group of X, then there exists another NEC group Λ' normalizing Γ such that $\Lambda \subsetneq \Lambda'$ with a finite index.

The formula on the number of fixed points of an automorphism of a Riemann surface was given by Macbeath in [12]. By $N_G(\langle g \rangle)$ we mean the normalizer in G of the subgroup generated by g.

THEOREM 2.1. *Let $G = \Delta/\Gamma$ be the group of orientation preserving automorphisms of a Riemann surface $X = \mathscr{H}/\Gamma$, and let $x_1, x_2, \ldots, x_r$ be the set of canonical elliptic generators of Δ with periods $m_1, \ldots, m_r$ respectively. Let $\theta : \Delta \to G$ be the canonical epimorphism. Then the number m of points of X fixed by $g \in G$ is given by the formula*

$$m = |N_G(\langle g \rangle)| \sum 1/m_i,$$

where the sum is taken over those i for which g is conjugate to a power of $\theta(x_i)$.

□

3. The degree of hyperellipticity

Let $X = \mathscr{H}/\Gamma$ be an asymmetric Riemann surface of genus $g \geq 2$, for a surface Fuchsian group Γ. Since X has anticonformal automorphisms but no symmetries, it follows that the group of automorphisms of X acts with a signature of the form $(h; -; [m_1, \ldots, m_r]; \{-\})$. Moreover, the order α of an anticonformal automorphism φ of X is divisible by 4, since if α is odd, then φ is a power of φ^2 and so it would be orientation preserving, while if $\alpha \equiv 2 \ (4)$ then $\varphi^{\alpha/2}$ would be a symmetry of X.

For a prime divisor n of α, the element $\rho = \varphi^{\alpha/n}$ is a (q, n)-gonal automorphism of X, where q is the genus of the orbifold $X/\langle \rho \rangle$. The group $\langle \rho \rangle$ acts with the signature $(q; n \stackrel{r}{\ldots}, n)$, where by the Riemann-Hurwitz formula $r = 2 + (2g - 2nq)/(n - 1)$. Thus according to Theorem 2.1, ρ has r fixed points.

We want to determine the possible values of q in the case when ρ is an involution. The task becomes much easier, if we consider a subgroup of $\langle \varphi \rangle$

generated by an anticonformal automorphism $\delta = \varphi^{\alpha/(2^s n)}$, where s is the greatest integer such that 2^s divides α, and n is an odd prime divisor of α or $n = 1$ if $\alpha = 2^s$. The possible degrees of hyperellipticity in the case $s = 2$ were given in [11], where we also proved that no cyclic group acts on an asymmetric Riemann surface with a non-maximal signature. Here we shall study surfaces which are not pseudo-symmetric and therefore we shall assume that $s \geq 3$. The notation introduced above will be consistently employed throughout the chapter, unless directly stated otherwise. First, we give the necessary and sufficient conditions on the existence of an asymmetric Riemann surface of genus g with the automorphism group $Z_{2^s n}$.

THEOREM 3.1. *A cyclic group $Z_{2^s n}$ is the full automorphism group of an asymmetric Riemann surface of genus g if and only if $g = nq + a(n-1)$ for some integers $a \geq -1$, $q \geq 0$, and there exist nonnegative integers k, γ, k_i, l_i, $i = 1, \ldots, s-1$ such that*

(I) $$a = 2^{s-1} k + \sum_{i=1}^{s-1} 2^{s-1-i} k_i - 1 \quad \text{for } n \neq 1$$

(II) $$q = 1 + 2^{s-1} \left[\gamma - 1 + \sum_{i=1}^{s-1} (l_i + k_i) \right] - \sum_{i=1}^{s-1} 2^{s-1-i} (l_i + k_i),$$

(III) $k = k_1 = \cdots = k_{s-1} = 0$ *for $n = 1$,*
(IV) γ *has the same parity as g,*
(V) *if $\gamma = 0$, then $l_{s-1} \neq 0$ for $n = 1$; $k_{s-1} \neq 0$ or $kl_{s-1} \neq 0$ for $n \neq 1$, and none of the two cases holds*:
  (a) $\gamma = 1$, $l_{s-1} = k_{s-1} = 0$ *and* $\sum_{i=1}^{s-2} (l_i + k_i) + k = 1$,
  (b) $\gamma = 0$, $l_{s-1} + k_{s-1} = 1$ *and* $\sum_{i=1}^{s-2} (l_i + k_i) + k = 1$.

Proof. Suppose that δ is an anticonformal automorphism of order $2^s n$ of an asymmetric Riemann surface $X = \mathscr{H}/\Gamma$ of genus g, and let $G = \langle \delta \rangle$. Then there exists an NEC group Λ containing Γ as a normal subgroup with index $2^s n$ such that $G = \Lambda/\Gamma$. Since there is no period cycles in the signature of Λ and the conformal automorphisms in G have orders 2^i, n or $2^i n$ for some $i = 1, \ldots, s-1$, it follows that Λ has a signature of the form

(3) $$(\gamma + 1; -; [(2)^{l_1}, (4)^{l_2}, \ldots, (2^{s-1})^{l_{s-1}}, (n)^k, (2n)^{k_1}, \ldots, (2^{s-1} n)^{k_{s-1}}]; \{-\}),$$

where k, γ, l_i and k_i are nonnegative integers, and the symbol $(m)^r$ denotes r periods equal to m. In the particular case when $n = 1$, $k = k_1 = \cdots = k_{s-1} = 0$. By the Riemann-Hurwitz formula,

(4) $$g = 1 + 2^{s-1} \left[n\gamma - n + n \sum_{i=1}^{s-1} (l_i + k_i) + k(n-1) \right] - \sum_{i=1}^{s-1} 2^{s-1-i} (nl_i + k_i).$$

According to (2), the canonical Fuchsian subgroup Λ^+ of Λ has the signature

$$(\gamma; (2)^{2l_1}, (4)^{2l_2}, \ldots, (2^{s-1})^{2l_{s-1}}, (n)^{2k}, (2n)^{2k_1}, \ldots, (2^{s-1}n)^{2k_{s-1}}).$$

The group $G^+ = \Lambda^+/\Gamma = Z_{2^{s-1}n}$ is generated by the conformal automorphism δ^2. If $n \neq 1$, then $\rho = \delta^{2^s} \in G^+$ is a (q, n)-gonal automorphism of X with $m = 2 + (2g - 2nq)/(n-1)$ fixed points, where q is the genus of $X/\langle\rho\rangle$. On the other hand, by Macbeath's theorem, $m = 2^s k + \sum_{i=1}^{s-1} 2^{s-i}k_i$ and so

$$(n-1)2^{s-1}k = (n-1)\left(m/2 - \sum_{i=1}^{s-1} 2^{s-1-i}k_i\right)$$

$$= n - 1 + g - nq - (n-1)\sum_{i=1}^{s-1} 2^{s-1-i}k_i.$$

By substituting the last equation to (4), we get (II). Thus $g - nq = (n-1)(2^{s-1}k + \sum_{i=1}^{s-1} 2^{s-1-i}k_i - 1) = a(n-1)$ for a given by (I).

By inspecting lists in [2] and [9], we check that the signature (3) is non-maximal if and only if it has a form $(2; -; [t]; \{-\})$ or $(1; -; [t, u]; \{-\})$ for some $u \in \{2^{s-1}, 2^{s-1}n\}$ and $t \in \{n, 2^i n, 2^i\}$, where $i \neq s - 1$. However, it was proved in [4] that no cyclic group acts on an asymmetric Riemann surface with listed signatures, as they in fact give rise to actions on symmetric surfaces. Thus the cases (a) and (b) must be excluded.

The action of G on X is induced by an epimorphism $\theta : \Lambda \to G$ with torsion-free kernel Γ. Any such an epimorphism maps elliptic generators $x_i \in \Lambda$ of orders m_i to $\delta^{2^s n \gamma_i / m_i}$ for some integers γ_i co-prime with m_i. Thus the product of all elliptic generators is mapped to δ^T for some even integer T. The glide reflection generators d_i of Λ are mapped to δ^{t_i} for some odd integers t_i. Thus the long relation is preserved if and only if the sum of $\gamma + 1$ odd integers t_i is equal to $-T/2$ modulo $2^s n$. The parity of the last integer depends only on the parity of $l_{s-1} + k_{s-1}$ and so by (4), is different from the parity of g. Thus γ must have the same parity as g. Let us also notice, that for $\gamma = 0$, the restriction of θ to Λ^+ is an epimorphism onto $Z_{2^{s-1}n}$ under condition that $l_{s-1} \neq 0$ for $n = 1$, and $k_{s-1} \neq 0$ or both k and l_{s-1} are different from 0 for $n \neq 1$.

Conversely, suppose that $g = nq + a(n-1)$ for some integers $a \geq -1$ and $q \geq 0$, and there exist nonnegative parameters $l_1, \ldots, l_{s-1}$, $k_1, \ldots, k_{s-1}$, k and γ such that the conditions (I)–(V) are satisfied, and neither case (a) nor (b) holds. Then an NEC group Λ with the signature (3) is maximal. Let $\theta : \Lambda \to Z_{2^s n} = \langle\delta\rangle$ be a homomorphism which maps the elliptic generators x_i of orders m_i to $\delta^{2^s n/m_i}$ and the glide reflection generators d_i to δ^{t_j}, for $(t_1, t_2, \ldots, t_{\gamma-2}, t_{\gamma-1}, t_\gamma, t_{\gamma+1}) = (1, -1, \ldots, -1, 1, -1, -T/2)$ or $(1, -1, \ldots, 1, -1, -1, -T/2 + 1)$, according to if g is even or odd. Then θ is an epimorphism with torsion-free kernel Γ. By the Riemann-Hurwitz formula, the orbit space $X = \mathcal{H}/\Gamma$ has genus g, and $G = \Lambda/\Gamma$ is the full automorphism group of X. Since G contains anticonformal automorphisms but no symmetries, it follows that X is asymmetric. $\square$

In the next theorem we exceptionally let the parameter s have value 2.

THEOREM 3.2.   *Suppose that $G = Z_{2^s n} = \langle \delta \rangle$ is the automorphism group of an asymmetric Riemann surface X of genus g, for $g \neq n-1, 2n-1, 2n-2$, and let $r = g \mod n$.   If $n > 1$ and $g \geq (n-r)(n-1)$, then X is a n-sheeted covering of an orbifold of genus q ramified over m points, where*

(5) $q = [g - (n-r)(n-1)]/n - c(n-1) \quad \text{and} \quad m = 2[n(c+1) + 1 - r],$

for some integer $c \geq -1$ for which q and m are nonnegative.   Moreover, for any such c, there exists an asymmetric Riemann surface of genus g with the full automorphism group Z_{4n}.

Proof.   Suppose that $X/\langle \delta^{2^s} \rangle$ has genus q.   Then δ^{2^s} is a (q,n)-gonal automorphism with $m = 2 + (2g - 2nq)/(n-1)$ fixed points.   By the proof of Theorem 3.1, m is even and so $g = nq + a(n-1)$ for $a = m/2 - 1$.   Writing g in the form

$$g = [(g-r)/n + 1 - (n-r)]n + (n-r)(n-1),$$

we can take $q = [g - (n-r)(n-1)]/n$ and $a = n - r$.   However, there are other possibilities: $q = [g - (n-r)(n-1)]/n - c(n-1)$ and $a = n - r + cn$ for any integer $c \geq -1$ for which $q \geq 0$ and $a \geq -1$.

Now for a given such c, let $\gamma = q \mod 2$, and let $k = 0$ or -1 according to if a is even or odd.   Then there exists an NEC group Λ with the signature

$$(\gamma + 1; -; [2, \overset{q-2\gamma-k}{\dots}, 2, n, \overset{(a-k)/2}{\dots}, n, 2n, \overset{k+1}{\dots}, 2n]; \{-\})$$

which is maximal for $(\gamma, q, a, k) \neq (1, 1, 1, -1)$, $(0, 0, 1, -1)$ and $(0, 0, 2, 0)$.   In the exceptional cases, $g = 2n - 1$, $n - 1$ and $2n - 2$ respectively, however these values of g are rejected by assumption.   Let $\theta : \Lambda \to G = Z_{4n} = \langle \delta \rangle$ be an epimorphism defined by

$$\theta(x_i) = \delta^{2n}, \qquad i = 1, \dots, k_1 = q - 2\gamma - k,$$
$$\theta(x_{k_1+i}) = \delta^4, \qquad i = 1, \dots, k_2 = (a-k)/2,$$
$$\theta(x_{k_1+k_2+i}) = \delta^2, \quad i = 1, \dots, k_3 = k + 1,$$

and $\theta(d_1) = \delta^{-l}$ if q is even, or $\theta(d_1) = \delta$ and $\theta(d_2) = \delta^{-1-l}$ if q is odd, where $l = nq - nk + a + 1 - 2n\gamma$.   Then the kernel Γ of θ is a surface Fuchsian group with the signature $(g; -)$ and $X = \mathscr{H}/\Gamma$ is an asymmetric Riemann surface X of genus g with the full automorphism group $Z_{4n} = \Lambda/\Gamma$. $\square$

THEOREM 3.3.   *For any integer g such that $g \geq 2^{s-1}n(2^{s-1} + n)$, there exists an asymmetric Riemann surface of genus g with the full automorphism group $Z_{2^s n}$.*

Proof. Suppose that $g \geq 2^{s-1}n(2^{s-1} + n)$, and let $r = g \mod n$,

$$r' = (g - r)/n + 1 \mod 2^{s-1} \quad \text{and} \quad b = [(g - r)/n + 1 - r']/2^{s-1}.$$

Then $g \geq 2^{s-1}n(2^{s-1} + n) - (n - r)$ and so $(g - r)/n + 1 \geq 2^{s-1}(2^{s-1} + n) \geq 2^{s-1}(2^{s-1} + n - 1) + r'$. Thus

$$b = [(g - r)/n + 1 - r']/2^{s-1} \geq 2^{s-1} + n - 1 \geq (2^{s-1} - r') + (n - r) - 1.$$

For b satisfying the last inequality, there exist parameters a, q, γ, k, k_i, l_i, $i = 1, \ldots s - 1$ for which the conditions (I)–(V) of Theorem 3.1 are satisfied and neither case (a) nor (b) holds. Indeed, since

$$(6) \qquad g = [(g - r)/n + 1 - (n - r)]n + (n - r)(n - 1)$$

$$= [(b + 1)2^{s-1} - (2^{s-1} - r') - (n - r)]n + (n - r)(n - 1),$$

it follows that $g = nq + a(n - 1)$ for $q = (b + 1)2^{s-1} - (2^{s-1} - r') - (n - r)$ and $a = n - r$. The remaining parameters can be chosen as follows:

for b odd

$n \neq 1$: $\gamma = b + 1 - (n - r) - (2^{s-1} - r')$, $k = 0$, $k_{s-1} = n - r + 1$,
        $l_{s-1} = 2^{s-1} - r'$ and $k_i = l_i = 0$ for $i \neq s - 1$,

$n = 1$: $\gamma = b - (2^{s-1} - r')$, $l_{s-1} = 2^{s-1} - r' + 2$, $l_i = 0$ for $i \neq s - 1$,

for b even

$n \neq 1$: $\gamma = b + 2 - (n - r) - (2^{s-1} - r')$, $k = 0$, $k_{s-1} = n - r - 1$,
        $k_{s-2} = 1$, $l_{s-1} = 2^{s-1} - r'$, $l_{s-2} = 0$, and $k_i = l_i = 0$
        for $i \neq s - 1, s - 2$,

$n = 1$: $\gamma = b + 1 - (2^{s-1} - r')$, $l_{s-1} = 2^{s-1} - r'$, $l_{s-2} = 1$,
        $l_i = 0$ for $i \neq s - 1, s - 2$.

Let us notice that by (6), g has a different parity than the number $r + r'$ and so γ has the same parity as g. Thus by Theorem 3.1, there exists an asymmetric Riemann surface of genus g with the full automorphism group $Z_{2^s n}$. $\square$

Remark 3.4. Suppose that $G = Z_{2^s n} = \langle \delta \rangle$ acts on an asymmetric Riemann surface $X = \mathscr{H}/\Gamma$ of genus g with the signature (3). Then X is p-hyperelliptic for

$$(7) \quad p = 1 + \left[n\gamma - n + n \sum_{i=1}^{s-1}(l_i + k_i) + k(n - 1) \right]2^{s-2} - \sum_{i=1}^{s-1} 2^{s-1-i}(nl_i + k_i),$$

and if $n \neq 1$, then X is (q, n)-gonal for q given by (II).

Proof. If $\Lambda^+ \leq \Lambda$ is the canonical Fuchsian subgroup, then $G^+ = \Lambda^+/\Gamma = Z_{2^{s-1}n}$ is the group of conformal automorphisms of X. By the proof of Theorem 3.1, $\delta^{2^s} \in G^+$ is (q, n)-gonal automorphism of X for q given by (II). Moreover, $\rho = \delta^{2^{s-1}n} \in G^+$ is p-hyperelliptic involution of X with $m = 2g + 2 - 4p$ fixed points, where p is the genus of $X/\langle \rho \rangle$. By Macbeath's theorem, $m =$

$\sum_{i=1}^{s-1} 2^{s-i}(nl_i + k_i)$. Thus

$$(8) \qquad p = \left[g + 1 - \sum_{i=1}^{s-1} 2^{s-1-i}(nl_i + k_i) \right] \Big/ 2,$$

and so by (4), we get (7).   □

For any even nonnegative integer u, there exists an asymmetric Riemann surface X with (q,n)-gonal automorphism and p-hyperelliptic involution having the same even number u of fixed points.

THEOREM 3.5.   *Let $n \neq 1$.   (i) Then for any even integer $u > 0$, there exists an asymmetric Riemann surface of genus $g = 1 + (2^s n - 1)\dfrac{u}{2}$ which is n-sheeted covering of an orbifold of genus $q = 1 + (2^s - 1)\dfrac{u}{2}$ with u ramification points and which is two-sheeted covering of an orbifold of genus $p = 1 + (2^{s-1}n - 1)\dfrac{u}{2}$ also with u ramification points.*

(ii) There exists an asymmetric Riemann surface of genus $g = 2^s n + 1$ which is unramified n-sheeted covering of an orbifold of genus $q = 1 + 2^s$ and which is unramified two-sheeted covering of an orbifold of genus $p = 1 + 2^{s-1}n$.

Proof.   (i) For a given nonnegative even integer u, let Λ be a maximal NEC group with the signature

$$(u/2 + 2; -; [2^{s-1}n, \overset{u}{.^{\underline{2}}.}, 2^{s-1}n]; \{-\}),$$

and let $G = \langle \delta \rangle$ be a cyclic group of order $2^s n$.   Then there exists an epimorphism $\theta : \Lambda \to G$ given by $\theta(x_i) = \delta^2$ for $i = 1, \ldots, \dfrac{u}{2}$, and $\theta(d_j) = \delta^{t_j}$, for $(t_1, \ldots, t_{u/2+2}) = (1, -1, \ldots, 1, -1, -u/2 + 1, -1)$ or $(1, -1, \ldots, 1, -1, -u/2)$ according to if $u \equiv 0$ (4) or $u \equiv 2$ (4). The kernel Γ of θ is a surface Fuchsian group which by the Riemann-Hurwitz formula has the signature $(g; -)$, for $g = 1 + (2^s n - 1)\dfrac{u}{2}$, and $G = \Lambda/\Gamma$ is the automorphism group of an asymmetric Riemann surface $X = \mathcal{H}/\Gamma$ of genus g.   By Remark 3.4, X has p-hyperelliptic involution and (q,n)-gonal automorphism for $q = 1 + (2^s - 1)\dfrac{u}{2}$ and $p = 1 + (2^{s-1}n - 1)\dfrac{u}{2}$.   The last elements have $2g + 2 - 4p$ and $2 + (2g - 2nq)/(n - 1)$ fixed points respectively, and these numbers are equal.

Similarly, we can prove the statement (ii) by taking an NEC group Λ with the signature $(4; -; [-]; \{-\})$ and an epimorphism $\theta : \Lambda \to G$ defined by $\theta(d_1) = \theta(d_2) = \delta$ and $\theta(d_3) = \theta(d_4) = \delta^{-1}$.   □

An asymmetric Riemann surface of genus g is p-hyperelliptic for some integer p in the range $0 \leq p \leq \dfrac{g}{2}$ or $1 \leq p \leq \dfrac{g+1}{2}$, according to if g is even of

odd. We will prove that there are infinite sequences of g for which the upper and lower bounds are attained.

THEOREM 3.6. *For any g such that $g \equiv 1$ $(2^s n)$, there exists an asymmetric Riemann surface X of genus g with the full automorphism group $G = Z_{2^s n}$, which acts with one of signatures $\left(\frac{g-1}{2^{s-1}n} + 2; -; [-]; \{-\} \right)$ or $(2; -; [2, \overset{\frac{g-1}{2^{s-2}n}}{\dots}, 2]; \{-\})$. In the first case, X is $\frac{g+1}{2}$-hyperelliptic and in the second one, X is elliptic-hyperelliptic.*

Proof. If $g \equiv 1$ $(2^s n)$, then $g = nq + a(n-1)$ for $q = (g-1)/n + 1$ and $a = -1$. Let $k = l_i = k_i = 0$ for $i = 1, \dots, s-1$ and $\gamma = (g-1)/(2^{s-1}n) + 1$ or let $\gamma = 1$, $k = l_i = k_i = 0$ for $i = 2, \dots, s-1$, $l_1 = (g-1)/(2^{s-2}n)$ and $k_1 = 0$. Then γ, a, k, γ, k_i and l_i satisfy the conditions (I)–(V) of Theorem 3.1 and none of the cases (a), (b) holds. Thus there exists an asymmetric Riemann surface X of genus g with the full automorphism group $Z_{2^s n}$ acting with the signature (3). By Remark 3.4, X is p-hyperelliptic, where $p = (g+1)/2$ for the first choice of parameters, and $p = 1$ for the second one. $\square$

THEOREM 3.7. *For any g such that $g \equiv 0$ $(2^s n)$ there exists an asymmetric Riemann surface X of genus g with the full automorphism group $Z_{2^s n}$, which acts with one of the signatures $\left(\frac{g}{2^{s-1}n} + 1; -; [2^{s-1}n]; \{-\} \right)$ or $(1; -; [2, \overset{\frac{g}{2^{s-2}n}}{\dots}, 2, 2^{s-1}n];$ $\{-\})$. In the first case, X is $\frac{g}{2}$-hyperelliptic and in the second one, X is hyperelliptic.*

Proof. Any integer g satisfying the congruence $g \equiv 0$ $(2^s n)$ can be written in the form $g = nq + a(n-1)$ for $q = g/n$ and $a = 0$. If $n \neq 1$, then by Theorem 3.1, a cyclic group $Z_{2^s n}$ acts on an asymmetric Riemann surface of genus g with the signature (3), where $k = l_i = k_i = 0$ for $i = 1, \dots, s-2$, $l_{s-1} = 0$, $k_{s-1} = 1$ and $\gamma = g/(2^{s-1}n)$, or $k = l_i = k_i = 0$ for $i = 2, \dots, s-2$, $l_1 = g/(2^{s-2}n)$, $k_1 = l_{s-1} = 0$, $k_{s-1} = 1$ and $\gamma = 0$. If $n = 1$, then we can assume that parameters k, γ, k_i and l_i, $i = 1, \dots, s-2$ have above values, $k_{s-1} = 0$ and $l_{s-1} = 1$. By Remark 3.4, X is p-hyperelliptic, where $p = g/2$ in the first case and $p = 0$ in the second one. $\square$

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