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ON A CERTAIN NILPOTENT EXTENSION OVER $\boldsymbol{Q}$ OF DEGREE 64 AND THE 4-TH MULTIPLE RESIDUE SYMBOL

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Abstract. In this paper, we introduce the 4-th multiple residue symbol $[p_1, p_2, p_3, p_4]$ for certain four prime numbers p_i's, which extends the Legendre symbol $\left(\frac{p_1}{p_2}\right)$ and the Rédei triple symbol $[p_1, p_2, p_3]$ in a natural manner. For this we construct concretely a certain nilpotent extension K over $\boldsymbol{Q}$ of degree 64, where ramified prime numbers are p_1, p_2 and p_3, such that the symbol $[p_1, p_2, p_3, p_4]$ describes the decomposition law of p_4 in the extension $K/\boldsymbol{Q}$. We then establish the relation of our symbol $[p_1, p_2, p_3, p_4]$ and the 4-th arithmetic Milnor invariant $\mu_2(1234)$ (an arithmetic analogue of the 4-th order linking number) by showing $[p_1, p_2, p_3, p_4] = (-1)^{\mu_2(1234)}$.

Introduction. As is well known, for two odd prime numbers p_1 and p_2, the Legendre symbol $\left(\frac{p_1}{p_2}\right)$ describes the decomposition law of p_2 in the quadratic extension $\boldsymbol{Q}(\sqrt{p_1})/\boldsymbol{Q}$. In 1939, L. Rédei ([R]) introduced a triple symbol with the intention of a generalization of the Legendre symbol and Gauss' genus theory. For three prime numbers $p_i \equiv 1 \pmod 4$ ($i = 1, 2, 3$) with $\left(\frac{p_i}{p_j}\right) = 1$ ($1 \le i \ne j \le 3$), the Rédei triple symbol $[p_1, p_2, p_3]$ describes the decomposition law of p_3 in a Galois extension over $\boldsymbol{Q}$ where all ramified prime numbers are p_1 and p_2 and the Galois group is the dihedral group D_8 of order 8.

Although a meaning of the Rédei symbol had been obscure for a long time, in 2000, M. Morishita ([Mo1, 2, 3]) interpreted the Rédei symbol as an arithmetic analogue of a mod 2 triple linking number, following the analogies between knots and primes. In fact, he introduced arithmetic analogue $\mu_2(12\cdots n) \in \boldsymbol{Z}/2\boldsymbol{Z}$ of Milnor's link invariants (higher order linking numbers) for prime numbers $p_1, \dots, p_n$ such that

$$\left(\frac{p_1}{p_2}\right) = (-1)^{\mu_2(12)}, \quad [p_1, p_2, p_3] = (-1)^{\mu_2(123)}.$$

Since it is difficult to compute arithmetic Milnor invariants by the definition, it is desirable to construct Galois extensions $K_n/\boldsymbol{Q}$ concretely such that $[p_1, \dots, p_n] = (-1)^{\mu_2(12\cdots n)}$ describes the decomposition law of p_n in $K_n/\boldsymbol{Q}$, just as in the cases of the Legendre symbol where K_2 is a quadratic extension and the Rédei triple symbol where K_3 is a dihedral extension of degree 8. As we shall explain in Subsection 2.1, link theory suggests that the desired extension $K_n/\boldsymbol{Q}$ should be a Galois extension such that all ramified prime numbers

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are $p_1, \dots, p_{n-1}$ and the Galois group is the nilpotent group

$$N_n(\boldsymbol{F}_2) = \left\{ \left(\begin{array}{cccc} 1 & * & \cdots & * \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & 1 \end{array} \right) \middle| * \in \boldsymbol{F}_2 \right\}$$

consisting of $n \times n$ unipotent upper-triangular matrices over $\boldsymbol{F}_2$. Note that $N_2(\boldsymbol{F}_2) = \boldsymbol{Z}/2\boldsymbol{Z}$ and $N_3(\boldsymbol{F}_2) = D_8$.

The purpose of this paper is to construct concretely such an extension $K_n/\boldsymbol{Q}$ for $n = 4$ in a natural manner extending Rédei's dihedral extension. We then introduce the 4-th multiple residue symbol $[p_1, p_2, p_3, p_4]$ describing the decomposition law of p_4 in $K_4/\boldsymbol{Q}$ and prove that it coincides with the 4-th Milnor invariant $\mu_2(1234)$,

$$[p_1, p_2, p_3, p_4] = (-1)^{\mu_2(1234)}.$$

NOTATION. For a number field k, we denote by $\mathcal{O}_k$ the ring of integers of k. For a group G and $d \in \boldsymbol{N}$, we denote by $G^{(d)}$ the d-th term of the lower central series of G defined by $G^{(1)} := G, G^{(d+1)} := [G, G^{(d)}]$. For a ring R, $R^\times$ denotes the group of invertible elements of R.

1. Rédei's dihedral extension and triple symbol. In this section, we recall the construction of Rédei's dihedral extension and triple symbol ([R]), which will be used later. We also give some basic properties of Rédei's dihedral extension and triple symbol.

1.1. The Rédei extension. Let p_1 and p_2 be distinct prime numbers satisfying

(1.1.1) $$p_i \equiv 1 \pmod 4 \ (i = 1, 2), \quad \left(\frac{p_1}{p_2}\right) = \left(\frac{p_2}{p_1}\right) = 1.$$

We set $k_i = \boldsymbol{Q}(\sqrt{p_i})$ $(i = 1, 2)$. It follows from this assumption (1.1.1) that we have the following Lemma.

LEMMA 1.1.2 ([A, Lemma 1.1]). There are integers x, y, z satisfying the following conditions:

(1) $x^2 - p_1 y^2 - p_2 z^2 = 0$.

(2) $\text{g.c.d}(x, y, z) = 1, \quad y \equiv 0 \pmod 2, \quad x - y \equiv 1 \pmod 4$.

Furthermore, for a given prime ideal $\mathfrak{p}_2$ of $\mathcal{O}_{k_1}$ lying over p_2, we can find integers x, y, z which satisfy (1), (2) and $(x + y\sqrt{p_1}) = \mathfrak{p}_2^m$ for an odd positive integer m.

Let $\boldsymbol{a} = (x, y, z)$ be a triple of integers satisfying the conditions (1), (2) in Lemma 1.1.2. Then let $\alpha = x + y\sqrt{p_1}$ and set

(1.1.3) $$k_{\boldsymbol{a}} = \boldsymbol{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{\alpha}).$$

The following theorem was proved by L. Rédei ([R]).

THEOREM 1.1.4 ([R]). (1) *The field $k_{\boldsymbol{a}}$ is a Galois extension over $\boldsymbol{Q}$ whose Galois group is the dihedral group of order* 8.

(2) *Let* $d(k_1(\sqrt{\alpha})/k_1)$ *be the relative discriminant of the extension* $k_1(\sqrt{\alpha})/k_1$. *Then we have* $N_{k_1/\boldsymbol{Q}}(d(k_1(\sqrt{\alpha})/k_1)) = (p_2)$. *In particular, all prime numbers ramified in* $k_{\boldsymbol{a}}/\boldsymbol{Q}$ *are* p_1 *and* p_2 *with ramification index* 2.

The fact that $k_{\boldsymbol{a}}$ is independent of the choice of $\boldsymbol{a}$ was also shown in [R]. The author gave an alternative proof of this fact in [A], based on a proof communicated by D. Vogel ([V2]).

THEOREM 1.1.5 ([A, Corollary 1.5]). *A field* $k_{\boldsymbol{a}}$ *is independent of the choice of* $\boldsymbol{a} = (x, y, z)$ *satisfying* (1) *and* (2) *in Lemma* 1.1.2*, namely, it depends only on a set* $\{p_1, p_2\}$.

DEFINITION 1.1.6. By Proposition 1.1.5, we denote by $k_{\{p_1,p_2\}}$ the field $k_{\boldsymbol{a}} = \boldsymbol{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{\alpha})$ given by (1.1.3) and call $k_{\{p_1,p_2\}}$ the *Rédei extension* over $\boldsymbol{Q}$ associated to a set $\{p_1, p_2\}$ satisfying (1.1.1).

The following theorem shows that the Rédei extension $k_{\{p_1,p_2\}}/\boldsymbol{Q}$ is characterized by the information on the Galois group and the ramification given in Theorem 1.1.4.

THEOREM 1.1.7 ([A, Theorem 2.1]). *Let* p_1 *and* p_2 *be prime numbers satisfying the condition* (1.1.1). *Then the following conditions on a number field* K *are equivalent*:

(1) K *is the Rédei extension* $k_{\{p_1,p_2\}}$.

(2) K *is a Galois extension over* $\boldsymbol{Q}$ *such that the Galois group is the dihedral group* D_8 *of order* 8 *and prime numbers ramified in* $K/\boldsymbol{Q}$ *are* p_1 *and* p_2 *with ramification index* 2.

1.2. The Rédei triple symbol. Let p_1, p_2 and p_3 be three prime numbers satisfying

$$p_i \equiv 1 \pmod 4 \ (i = 1, 2, 3), \quad \left(\frac{p_i}{p_j}\right) = 1 \ (1 \le i \ne j \le 3). \tag{1.2.1}$$

Let $k_{\{p_1,p_2\}}$ be the Rédei extension over $\boldsymbol{Q}$ associated to a set $\{p_1, p_2\}$ (Definition 1.1.6).

DEFINITION 1.2.2. We define *Rédei triple symbol* $[p_1, p_2, p_3]$ by

$$[p_1, p_2, p_3] = \begin{cases} 1 & \text{if } p_3 \text{ is completely decomposed in } k_{\{p_1,p_2\}}/\boldsymbol{Q}, \\ -1 & \text{otherwise.} \end{cases}$$

The following theorem is a reciprocity law for the Rédei triple symbol:

THEOREM 1.2.3 ([R], [A, Theorem 3.2]). *We have*

$$[p_1, p_2, p_3] = [p_i, p_j, p_k]$$

for any permutation $\{i, j, k\}$ *of* $\{1, 2, 3\}$.

2. Milnor invariants. In this section, we recall the arithmetic analogues of Milnor invariants of a link introduced by M. Morishita ([Mo1, 2, 3]) and clarify a meaning of the Rédei extension and the Rédei triple symbol in Section 1 from the viewpoint of the analogy between knot theory and number theory. The underlying idea is based on the following analogies between knots and primes (cf. [Mo4]):

knot $\mathcal{K} : S^1 \hookrightarrow \boldsymbol{R}^3$	prime $\mathrm{Spec}(\boldsymbol{F}_p) \hookrightarrow \mathrm{Spec}(\boldsymbol{Z})$
link $\mathcal{L} = \mathcal{K}_1 \cup \cdots \cup \mathcal{K}_r$	finite set of primes $S = \{p_1, \ldots, p_r\}$
$X_{\mathcal{L}} = \boldsymbol{R}^3 \setminus \mathcal{L}$	$X_S = \mathrm{Spec}(\boldsymbol{Z}) \setminus S$
link group $G_{\mathcal{L}} = \pi_1(X_{\mathcal{L}})$	Galois group with restricted ramification $G_S = \pi_1^{\text{ét}}(X_S) = \mathrm{Gal}(\boldsymbol{Q}_S/\boldsymbol{Q})$ $\boldsymbol{Q}_S$: maximal extension over $\boldsymbol{Q}$ unramified outside $S \cup \{\infty\}$

In the following, we firstly explain Milnor invariants of a link and their meaning in nilpotent coverings of S^3 ([Mi2], [Mu]). We then discuss their arithmetic analogues for prime numbers where the Rédei triple symbol is interpreted as an arithmetic analogues of a triple Milnor invariant. The analogy also suggests that a natural generalization of the Legendre and Rédei symbols, called a multiple residue symbol $[p_1, \ldots, p_n]$, should describe the decomposition law of p_n in a certain nilpotent extension over $\boldsymbol{Q}$ unramified outside $p_1, \ldots, p_{n-1}$ and ∞ (∞ being the infinite prime).

2.1. Milnor invariants of a link. Let $\mathcal{L} = \mathcal{K}_1 \cup \cdots \cup \mathcal{K}_r$ be a link with r components in $\boldsymbol{R}^3$ and let $X_{\mathcal{L}} = \boldsymbol{R}^3 \setminus \mathcal{L}$ and $G_{\mathcal{L}} := \pi_1(X_{\mathcal{L}})$ be the link group of $\mathcal{L}$. Let F be the free group on the words $x_1, \ldots, x_r$ where x_i represents a meridian of $\mathcal{K}_i$. The following theorem is due to J. Milnor.

THEOREM 2.1.1 ([Mi2, Theorem 4]). *For each $d \in \boldsymbol{N}$, there is $y_i^{(d)} \in F$ such that*

$$G_{\mathcal{L}}/G_{\mathcal{L}}^{(d)} = \langle x_1, \ldots, x_r \mid [x_1, y_1^{(d)}] = \cdots = [x_r, y_r^{(d)}] = 1,\ F^{(d)} = 1 \rangle\,,$$
$$y_j^{(d)} \equiv y_j^{(d+1)} \mod F^{(d)}\,,$$

where $y_j^{(d)}$ is a word representing a longitude of $\mathcal{K}_j$ in $G_{\mathcal{L}}/G_{\mathcal{L}}^{(d)}$.

Let $\mathbf{Z}\langle\langle X_1, \ldots, X_r\rangle\rangle$ be the algebra of non-commutative formal power series of variables $X_1, \ldots, X_r$ over $\mathbf{Z}$, and let

$$M : F \longrightarrow \mathbf{Z}\langle\langle X_1, \ldots, X_r\rangle\rangle^{\times}$$

be the Magnus homomorphism defined by

$$M(x_i) := 1 + X_i,\ M(x_i^{-1}) := 1 - X_i + X_i^2 - \cdots,\ 1 \le i \le r\,.$$

For $f \in F$, $M(f)$ has the form

$$M(f) = 1 + \sum_{n=1}^{\infty} \sum_{1 \le i_1, \ldots, i_n \le r} \mu(i_1 \cdots i_n; f) X_{i_1} \cdots X_{i_n}\,,$$

where the coefficients $\mu(i_1 \cdots i_n; f)$ are called the *Magnus coefficients*.

Let $\mathbf{Z}[F]$ be the group algebra of F over $\mathbf{Z}$ and let $\varepsilon_{\mathbf{Z}[F]} : \mathbf{Z}[F] \to \mathbf{Z}$ be the augmentation map. We note that the Magnus coefficients can be written in terms of the Fox derivative introduced in [F]:

$$\mu(i_1 \cdots i_n;\, f) = \varepsilon_{\mathbf{Z}[F]}\left(\frac{\partial^n f}{\partial x_{i_1} \cdots \partial x_{i_n}}\right).$$

For the word $y_j^{(d)}$ in Theorem 2.1.1, we set

$$\mu^{(d)}(i_1 \cdots i_n j) := \mu(i_1 \cdots i_n;\, y_j^{(d)}).$$

Since $\mu(i_1 \cdots i_n; f) = 0$ for $f \in F^{(d)}$ if $d > n$, by Theorem 2.1.1, $\mu^{(d)}(I)$ is independent of d if $d \geq |I|$, where $|I|$ denotes the length of a multi-index I. Define $\mu(I) := \mu^{(d)}(I)$ $(d \gg 1)$. For a multi-index I with $|I| \geq 2$, we define $\Delta(I)$ to be the ideal of $\mathbf{Z}$ generated by $\mu(J)$ where J runs over cyclic permutations of proper subsequences of I. If $|I| = 1$, we set $\mu(I) := 0$ and $\Delta(I) := 0$. The *Milnor $\overline{\mu}$-invariant* is then defined by

$$\overline{\mu}(I) := \mu(I) \mod \Delta(I).$$

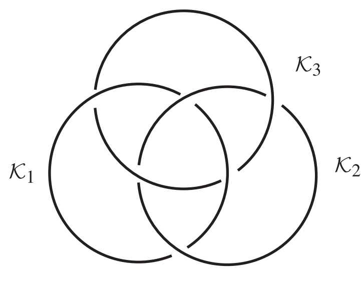
The fundamental results, due to Milnor, are as follows.

THEOREM 2.1.2 ([Mi2, Theorems 5, 6]). (1) $\overline{\mu}(ij) = \mathrm{lk}(\mathcal{K}_i, \mathcal{K}_j)$ $(i \neq j)$.
(2) *If* $2 \leq |I| \leq d$, $\overline{\mu}(I)$ *is a link invariant of* $\mathcal{L}$.
(3) (Shuffle relation) *For any* $I, J\,(|I|, |J| \geq 1)$ *and* i $(1 \leq i \leq r)$, *we have*

$$\sum_{H \in \mathrm{PSh}(I,J)} \overline{\mu}(Hi) \equiv 0 \mod \text{g.c.d}\{\Delta(Hi) \mid H \in \mathrm{PSh}(I, J)\}$$

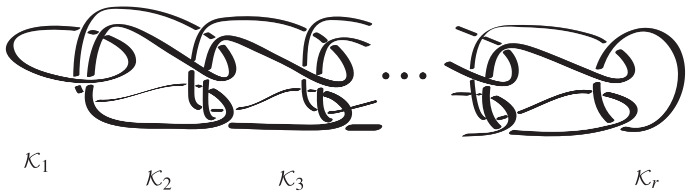
where $\mathrm{PSh}(I, J)$ *stands for the set of results of proper shuffles of* I *and* J (cf. [CFL]).
(4) (Cyclic symmetry). $\overline{\mu}(i_1 \cdots i_n) = \overline{\mu}(i_2 \cdots i_n i_1) = \cdots = \overline{\mu}(i_n i_1 \cdots i_{n-1})$.

EXAMPLE 2.1.3. For a multi-index I $(|I| \geq 2)$, $\overline{\mu}(I) = \mu(I)$ is an integral link invariant if $\mu(J) = 0$ for all multi-index J with $|J| < |I|$. For example, let $\mathcal{L} = \mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}_3$ be the following *Borromean rings*:



Then $\mu(I) = 0$ if $|I| \leq 2$ and hence $\mu(I) \in \mathbf{Z}$ for $|I| = 3$. In fact, we have $\mu(ijk) = \pm 1$ if ijk is a permutation of 123 and $\mu(ijk) = 0$ otherwise.

Move generally, let $\mathcal{L} = \mathcal{K}_1 \cup \cdots \cup \mathcal{K}_r$ be the following link, called the *Milnor link* ([Mi1, 5]). We easily see that the link obtained by removing any one component $\mathcal{K}_i$ from $\mathcal{L}$ is trivial. So $\mu(I) = 0$ if $|I| \leq n - 1$ and $\mu(I) \in \mathbf{Z}$ if $|I| = n$. For instance, $\mu(12 \cdots n) = 1$.



Next, we recall that Milnor invariants may be regarded as invariants associated to nilpotent coverings of S^3. For a commutative ring R, let $N_n(R)$ be the group consisting of n by n unipotent uppertriangular matrices. For a multi-index $I = (i_1 \cdots i_n)(n \geq 2)$, we define the map $\rho_I : F \to N_n(\mathbf{Z}/\Delta(I))$ by

$$\rho_I(f) := \begin{pmatrix} 1 & \varepsilon\left(\frac{\partial f}{\partial x_{i_1}}\right) & \varepsilon\left(\frac{\partial^2 f}{\partial x_{i_1}\partial x_{i_2}}\right) & \cdots & \varepsilon\left(\frac{\partial^{n-1} f}{\partial x_{i_1}\cdots\partial x_{i_{n-1}}}\right) \\ 0 & 1 & \varepsilon\left(\frac{\partial f}{\partial x_{i_2}}\right) & \cdots & \varepsilon\left(\frac{\partial^{n-2} f}{\partial x_{i_2}\cdots\partial x_{i_{n-1}}}\right) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & \varepsilon\left(\frac{\partial f}{\partial x_{i_{n-1}}}\right) \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \mod \Delta(I),$$

where we set $\varepsilon = \varepsilon_{\mathbf{Z}[F]}$ for simplicity. It can be shown by the property of the Fox derivative that ρ_I is a homomorphism.

THEOREM 2.1.6 ([Mo4, Theorem 8.8], [Mu]). (1) *The homomorphism ρ_I factors through the link group $G_{\mathcal{L}}$. Furthermore it is surjective if $i_1, \ldots, i_{n-1}$ are all distinct.*

(2) *Suppose that $i_1, \ldots, i_{n-1}$ are all distinct. Let $X_I \to X_{\mathcal{L}}$ be the Galois covering corresponding to* $\mathrm{Ker}(\rho_I)$ *whose Galois group* $\mathrm{Gal}(X_I/X_{\mathcal{L}}) = N_n(\mathbf{Z}/\Delta(I))$. *When $\Delta(I) \neq 0$, let $M_I \to S^3$ be the Fox completion of $X_I \to X_{\mathcal{L}}$, a Galois covering ramified over the link $\mathcal{K}_{i_1} \cup \cdots \cup \mathcal{K}_{i_{n-1}}$. For a longitude β_{i_n} of $\mathcal{K}_{i_n}$, one has*

$$\rho_I(\beta_{i_n}) = \begin{pmatrix} 1 & 0 & \cdots & 0 & \overline{\mu}(I) \\ 0 & 1 & \cdots & & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}$$

and hence the following holds:

$$\overline{\mu}(I) = 0 \Longleftrightarrow \mathcal{K}_{i_n} \textit{is completely decomposed in } M_I \to S^3.$$

2.2. Milnor invariants for prime numbers. Let $S = \{p_1, \ldots, p_r\}$ be a set of r distinct odd prime numbers and let $G_S := \pi_1^{\text{ét}}(\mathrm{Spec}(\mathbf{Z}) \setminus S)$. In order to get the analogy of the link case, we consider the maximal pro-2 quotient, denoted by $G_S(2)$, of G_S which is the Galois group of the maximal pro-2 extension $\boldsymbol{Q}_S(2)$ over $\boldsymbol{Q}$ which is unramified outside

$S \cup \{\infty\}$. Here we fix an algebraic closure $\overline{\boldsymbol{Q}}$ of $\boldsymbol{Q}$ containing $\boldsymbol{Q}_S(2)$. We also fix an algebraic closure $\overline{\boldsymbol{Q}}_{p_i}$ of $\boldsymbol{Q}_{p_i}$ and an embedding $\overline{\boldsymbol{Q}} \hookrightarrow \overline{\boldsymbol{Q}}_{p_i}$ for each i. Let $\boldsymbol{Q}_{p_i}(2)$ be the maximal pro-2 extension of $\boldsymbol{Q}_{p_i}$ contained in $\overline{\boldsymbol{Q}}_{p_i}$. Then we have

$$\boldsymbol{Q}_{p_i}(2) = \boldsymbol{Q}_{p_i}(\zeta_{2^n},\ \sqrt[2^n]{p_i} \mid n \geq 1)$$

where $\zeta_{2^n} \in \overline{\boldsymbol{Q}}$ is primitive 2^n-th root of unity such that $\zeta_{2^t}^{2^s} = \zeta_{2^{t-s}}$ $(t \geq s)$. The local Galois group $\mathrm{Gal}(\boldsymbol{Q}_{p_i}(2)/\boldsymbol{Q}_{p_i})$ is then topologically generated by the monodromy τ_i and the extension of the Frobenius automorphism σ_i defined by

$$\begin{aligned} &\tau_i(\zeta_{2^n}) = \zeta_{2^n}\,, \quad \tau_i(\sqrt[2^n]{p_i}) = \zeta_{2^n}\sqrt[2^n]{p_i}\,, \\ &\sigma_i(\zeta_{2^n}) = \zeta_{2^n}^{p_1}\,, \quad \sigma_i(\sqrt[2^n]{p_i}) = \sqrt[2^n]{p_i} \end{aligned} \tag{2.2.1}$$

and τ_i, σ_i are subject to the relation $\tau_i^{p_i-1}[\tau_i, \sigma_i] = 1$.

The embedding $\overline{\boldsymbol{Q}} \hookrightarrow \overline{\boldsymbol{Q}}_{p_i}$ induces the embedding $\boldsymbol{Q}_S(2) \hookrightarrow \boldsymbol{Q}_{p_i}(2)$ and hence the homomorphism $\eta_i : \mathrm{Gal}(\boldsymbol{Q}_{p_i}(2)/\boldsymbol{Q}_{p_i}) \to G_S$. We denote by the same τ_i, σ_i the images of τ_i, σ_i under η_i. Let $\hat{F}$ denote the free pro-2 group on the words $x_1, \ldots, x_r$ where x_i represents τ_i. The following theorem, due to H. Koch, may be regarded as an arithmetic analogue of Milnor's Theorem 2.1.1.

THEOREM 2.2.2 ([K2, Theorem 6.2]). *The pro-2 group $G_S(2)$ has the following presentation*:

$$G_S(2) = \langle x_1, \ldots, x_r \mid x_1^{p_1-1}[x_1, y_1] = \cdots = x_r^{p_r-1}[x_r, y_r] = 1 \rangle\,,$$

where $y_j \in \hat{F}$ is the pro-2 word which represents σ_j.

Set $e_S := \max\{e \mid p_i \equiv 1 \mod 2^e\ (1 \leq i \leq r)\}$ and fix $m = 2^e$ $(1 \leq e \leq e_S)$. Let $\mathbf{Z}_2\langle\langle X_1, \ldots, X_r \rangle\rangle$ be the algebra of non-commutative formal power series of variables $X_1, \ldots, X_r$ over $\mathbf{Z}_2$, the ring of 2-adic integers, and let

$$\hat{M} : \hat{F} \longrightarrow \mathbf{Z}_2\langle\langle X_1, \ldots, X_r \rangle\rangle^\times$$

be the pro-2 Magnus embedding ([K1, 4.2]). For $f \in \hat{F}$, $\hat{M}(f)$ has the from

$$\hat{M}(f) = 1 + \sum_{1 \leq i_1, \ldots, i_n \leq r} \hat{\mu}(i_1 \cdots i_n;\, f) X_{i_1} \cdots X_{i_n}\,,$$

where the coefficients $\hat{\mu}(i_1 \cdots i_n;\, f)$ are called the 2*-adic Magnus coefficients*. We let

$$M_2 : \hat{F} \longrightarrow \boldsymbol{F}_2\langle\langle X_1, \ldots, X_r \rangle\rangle^\times$$

be the mod 2 Magnus embedding defined by composing $\hat{M}$ with the natural homomorphism $\mathbf{Z}_2\langle\langle X_1, \ldots, X_r \rangle\rangle^\times \longrightarrow \boldsymbol{F}_2\langle\langle X_1, \ldots, X_r \rangle\rangle^\times$.

Let $\mathbf{Z}_2[[\hat{F}]]$ be the complete group algebra over $\mathbf{Z}_2$ and let $\varepsilon_{\mathbf{Z}_2[[\hat{F}]]} : \mathbf{Z}_2[[\hat{F}]] \to \mathbf{Z}_2$ be the augmentation map. In terms of the pro-2 Fox free derivative ([I], [O]), the 2-adic Magnus coefficients are written as

$$\hat{\mu}(i_1 \cdots i_n;\, f) = \varepsilon_{\mathbf{Z}_2[[\hat{F}]]}\left(\frac{\partial^n f}{\partial x_{i_1} \cdots \partial x_{i_n}}\right).$$

For the word y_j in Theorem 2.2.2, we set

$$\hat{\mu}(i_1 \cdots i_n j) := \hat{\mu}(i_1 \cdots i_n; y_j)$$

and we set, for a multi-index I,

$$\mu_m(I) := \hat{\mu}(I) \mod m\,.$$

For a multi-index with I with $1 \leq |I| \leq 2^{es}$, let $\Delta_m(I)$ be the ideal of $\mathbf{Z}/m\mathbf{Z}$ generated by $\binom{2^{es}}{t}$ $(1 \leq t \leq |I|)$ and $\mu_m(J)$ (J running over cyclic permutation of proper subsequences of I). The Milnor $\overline{\mu}_m$-invariant is then defined by

$$\overline{\mu}_m(I) := \mu_m(I) \mod \Delta_m(I)\,.$$

The following analogue of Theorem 2.1.2 is due to Morishita.

THEOREM 2.2.3 ([Mo3, Theorems 1.2.1, 1.2.5]). (1) *$\zeta_m^{\mu_m(ij)} = \left(\frac{p_j}{p_i}\right)_m$ where ζ_m is the primitive m-th root of unity given in* (2.2.1) *and $\left(\frac{p_j}{p_i}\right)_m$ is the m-th power residue symbol in $\boldsymbol{Q}_{p_i}$.*

(2) *If $2 \leq |I| \leq 2^{es}$, $\overline{\mu}_m(I)$ is an invariant depending only on S.*

(3) *Let r be an integer such that $2 \leq r \leq 2^{es}$. For multi-indices I, J such that $|I|+|J| = r-1$, we have, for any $1 \leq i \leq n$,*

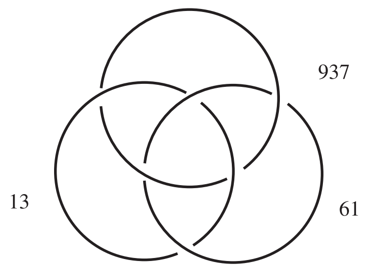
$$\sum_{H \in \mathrm{PSh}(I,J)} \overline{\mu}_m(Hi) \equiv 0 \mod \mathrm{g.c.d}\{\Delta(Hi) \mid H \in \mathrm{PSh}(I,J)\}\,.$$

EXAMPLE 2.2.4. Let $S = \{p_1, p_2, p_3\}$ be a triple of distinct prime numbers satisfying the condition (1.2.1) and let $m = 2$. Then $\mu_2(I) = 0$ if $|I| \leq 2$ and hence, for $|I| = 3$, $\Delta_2(I) = 0$ and $\overline{\mu}_2(I) = \mu_2(I) \in \mathbf{Z}/2\mathbf{Z}$. The following theorem interprets the Rédei triple symbol as a Milnor invariant.

THEOREM 2.2.4.1 ([Mo2, Theorem 3.2.5]). *Under the above assumption on $\{p_1, p_2, p_3\}$ we have*

$$[p_1, p_2, p_3] = (-1)^{\mu_2(123)}\,.$$

For example, D. Vogel ([V1, Example 3.14]) showed that for $S = \{13, 61, 937\}$ $\mu_2(I) = 0$ ($|I| \leq 2$), $\mu_2(I) = 1$ (I is a permutation of 123), $\mu_2(ijk) = 0$ (otherwise). In view of Example 2.1.3, this triple of prime numbers may be called the *Borromean primes*.



Finally, we give an analogue of Theorem 2.1.6 for prime numbers. Let $I = (i_1 \cdots i_n)$, $2 \leq n \leq l^{es}$ and assume $\Delta_m(I) \neq \boldsymbol{Z}/m\boldsymbol{Z}$. We define the map $\rho_{(m,I)} : \hat{F} \to N_n((\boldsymbol{Z}/m\boldsymbol{Z})/\Delta_m(I))$ by

$$\rho_{(m,I)}(f) := \begin{pmatrix} 1 & \varepsilon\left(\frac{\partial f}{\partial x_{i_1}}\right)_m & \varepsilon\left(\frac{\partial^2 f}{\partial x_{i_1}\partial x_{i_2}}\right)_m & \cdots & \varepsilon\left(\frac{\partial^{n-1} f}{\partial x_{i_1}\cdots\partial x_{i_{n-1}}}\right)_m \\ & 1 & \varepsilon\left(\frac{\partial f}{\partial x_{i_2}}\right)_m & \cdots & \varepsilon\left(\frac{\partial^{n-2} f}{\partial x_{i_2}\cdots\partial x_{i_{n-1}}}\right)_m \\ & & \ddots & \ddots & \vdots \\ & 0 & & 1 & \varepsilon\left(\frac{\partial f}{\partial x_{i_{n-1}}}\right)_m \\ & & & & 1 \end{pmatrix} \bmod \Delta_m(I)\,,$$

where we set $\varepsilon(\alpha)_m = \varepsilon_{\boldsymbol{Z}[[\hat{F}]]}(\alpha) \bmod m$ for $\alpha \in \boldsymbol{Z}_l[[\hat{F}(l)]]$. It can be shown by the property of the pro-2 Fox derivative that $\rho_{(m,I)}$ is a homomorphism.

THEOREM 2.2.5 ([Mo3, Theorem 1.2.7]). (1) *The homomorphism $\rho_{(m,I)}$ factors through the Galois group $G_S(2)$. Further it is surjective if $i_1, \ldots, i_{n-1}$ are all distinct.*

(2) *Suppose that $i_1, \ldots, i_{n-1}$ are all distinct. Let $K_{(m,I)}$ be the extension over $\boldsymbol{Q}$ corresponding to* $\mathrm{Ker}(\rho_{(m,I)})$. *Then $K_{(m,I)}/\boldsymbol{Q}$ is a Galois extension unramified outside $p_{i_1}, \ldots, p_{i_{n-1}}$ and ∞ with Galois group* $\mathrm{Gal}(K_{(m,I)}/\boldsymbol{Q}) = N_n((\boldsymbol{Z}/m\boldsymbol{Z})/\Delta_m(I))$. *For a Frobenius automorphism σ_{i_n} over p_{i_n}, one has*

$$\rho_{(m,I)}(\sigma_{i_n}) = \begin{pmatrix} 1 & 0 & \cdots & 0 & \overline{\mu}_m(I) \\ & 1 & \cdots & & 0 \\ & & \ddots & & \vdots \\ & 0 & & 1 & 0 \\ & & & & 1 \end{pmatrix}$$

and hence the following holds:

$$\overline{\mu}_m(I) = 0 \Longleftrightarrow p_{i_n} \textit{ is completely decomposed in } K_{(m,I)}/\boldsymbol{Q}.$$

EXAMPLE 2.2.6. Let $m = 2$ and $K = K_{(2,I)}$. For $S = \{p_1, p_2\}$, $p_i \equiv 1 \pmod 4$ $(i = 1, 2)$ and $I = (12)$, we have

$$K = \boldsymbol{Q}(\sqrt{p_1}),\ \mathrm{Gal}(K/\boldsymbol{Q}) = N_2(\boldsymbol{F}_2) = \boldsymbol{Z}/2\boldsymbol{Z},\ (-1)^{\mu_2(12)} = \left(\frac{p_1}{p_2}\right).$$

For $S = \{p_1, p_2, p_3\}$ satisfying the condition (1.2.1) and $I = (123)$, we have

$$K = k_{\{p_1,p_2\}},\ \mathrm{Gal}(K/\boldsymbol{Q}) = N_3(\boldsymbol{F}_2) = D_8,\ (-1)^{\mu_2(123)} = [p_1, p_2, p_3]\,.$$

Theorem 2.2.5 suggests a problem to construct concretely a Galois extension $K_n/\boldsymbol{Q}$ unramified outside $p_1, \ldots, p_{n-1}$ and ∞ with Galois group $N_n(\boldsymbol{F}_2)$ and to introduce the multiple residue symbol $[p_1, \ldots, p_n]$, as a generalization of the Legendre symbol and the Rédei triple symbol, which should describe the decomposition law of p_n in the extension $K_n/\boldsymbol{Q}$ and coincide with $(-1)^{\mu_2(12\cdots n)}$. In the next section, we solve this problem for the case $n = 4$.

3. Construction of an $N_4(\boldsymbol{F}_2)$-extension and the 4-th multiple residue symbol. In this section, under certain conditions on three prime numbers p_1, p_2, p_3, we construct concretely a Galois extension K over $\boldsymbol{Q}$ where all ramified prime numbers are p_1, p_2 and p_3 and the Galois group is $N_4(\boldsymbol{F}_2)$, and introduce the 4-th multiple residue symbol $[p_1, p_2, p_3, p_4]$ which describes the decomposition law of p_4 in $K/\boldsymbol{Q}$. We then show that $[p_1, p_2, p_3, p_4]$ coincides with $(-1)^{\mu_2(1234)}$, where $\mu_2(1234)$ is the 4-th arithmetic Milnor invariant defined in 2.2. We keep the same notations as in the previous sections.

3.1. Construction of an $N_4(\boldsymbol{F}_2)$-extension. Let p_1, p_2 and p_3 be three prime numbers satisfying the conditions

$$(3.1.1)\qquad \begin{cases} p_i \equiv 1 \pmod 4 \ (i = 1, 2, 3)\,, & \left(\dfrac{p_i}{p_j}\right) = 1 \ (1 \le i \ne j \le 3)\,, \\ [p_i, p_j, p_k] = 1 \ (\{i, j, k\} = \{1, 2, 3\})\,. \end{cases}$$

We let

$$\begin{cases} k_i := \boldsymbol{Q}(\sqrt{p_i}) \ (i = 1, 2, 3), \ k_{ij} := k_i k_j = \boldsymbol{Q}(\sqrt{p_i}, \sqrt{p_j}) \ (1 \le i < j \le 3)\,, \\ k_{123} := k_1 k_2 k_3 = \boldsymbol{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3})\,. \end{cases}$$

For simplicity, we set $k := k_1$ in the following. Let $\mathfrak{p}_2$ be one of prime ideals of $\mathcal{O}_k$ lying over p_2. Then as in Lemma 1.1.2, we can find a triple of integers (x, y, z) with $\alpha = x + y\sqrt{p_1}$ satisfying (1), (2) in Lemma 1.1.2 such that

$$(\alpha) = \mathfrak{p}_2^m \ (m \text{ being an odd integer})\,, \quad k_{\{p_1, p_2\}} = \boldsymbol{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{\alpha})\,.$$

In the following, we fix such an α once and for all.

For a prime $\mathfrak{p}$ of k, we denote by $\left(\frac{\cdot,\cdot}{\mathfrak{p}}\right)$ the Hilbert symbol in the local field $k_{\mathfrak{p}}$, namely,

$$(a, k_{\mathfrak{p}}(\sqrt{b})/k_{\mathfrak{p}})\sqrt{b} = \left(\frac{a, b}{\mathfrak{p}}\right)\sqrt{b} \quad (a, b \in k_{\mathfrak{p}}^{\times})\,,$$

where $(\ , k_{\mathfrak{p}}(\sqrt{b})/k_{\mathfrak{p}}) : k_{\mathfrak{p}}^{\times} \to \mathrm{Gal}(k_{\mathfrak{p}}(\sqrt{b})/k_{\mathfrak{p}})$ is the norm residue symbol of local class field theory.

LEMMA 3.1.2. *For any prime $\mathfrak{p}$ of k, we have*

$$\left(\frac{\alpha, p_3}{\mathfrak{p}}\right) = 1\,.$$

PROOF. We consider the following five cases.

(Case 1) $\mathfrak{p}$ is prime to $\mathfrak{p}_2$, p_3, 2, ∞: Then we have $\alpha, p_3 \in U_{\mathfrak{p}}$, where $U_{\mathfrak{p}}$ is the unit group of $k_{\mathfrak{p}}$, and hence $\left(\frac{\alpha, p_3}{\mathfrak{p}}\right) = 1$.

(Case 2) $\mathfrak{p} = \mathfrak{p}_2$: Let π be a prime element of $k_{\mathfrak{p}_2}$. Write $\alpha = u_1\pi^{m_2}$, $u_1 \in U_{\mathfrak{p}_2}$. Then we have

$$\begin{aligned} \left(\frac{\alpha, p_3}{\mathfrak{p}_2}\right) &= \left(\frac{u_1, p_3}{\mathfrak{p}_2}\right)\left(\frac{\pi^{m_2}, p_3}{\mathfrak{p}_2}\right) \\ &= \left(\frac{\pi, p_3}{\mathfrak{p}_2}\right) \quad (u_1, p_3 \in U_{\mathfrak{p}_2}, m_2 \text{ is odd}) \end{aligned}$$

$$= \frac{(\pi, k_{\mathfrak{p}_2}(\sqrt{p_3})/k_{\mathfrak{p}_2})\sqrt{p_3}}{\sqrt{p_3}}\,.$$

Since $(\pi, k_{\mathfrak{p}_2}(\sqrt{p_3})/k_{\mathfrak{p}_2})$ is the Frobenius automorphism over p_2 in $k(\sqrt{p_3})/k$, $(\pi, k_{\mathfrak{p}_2}(\sqrt{p_3})/k_{\mathfrak{p}_2})(\sqrt{p_3}) = \sqrt{p_3}$ by $\left(\frac{p_1}{p_2}\right) = \left(\frac{p_3}{p_2}\right) = 1$.

(Case 3) $\mathfrak{p} \mid p_3$: Let ϖ be a prime element of $k_{\mathfrak{p}}$. Write $p_3 = u_2\varpi$, $u_2 \in U_{\mathfrak{p}}$. Then we have

$$\begin{aligned}\left(\frac{\alpha, p_3}{\mathfrak{p}}\right) &= \left(\frac{p_3, \alpha}{\mathfrak{p}}\right)\\ &= \left(\frac{u_2, \alpha}{\mathfrak{p}}\right)\left(\frac{\varpi, \alpha}{\mathfrak{p}}\right)\\ &= \left(\frac{\varpi, \alpha}{\mathfrak{p}}\right) \qquad (u_2, \alpha \in U_{\mathfrak{p}})\\ &= \frac{(\varpi, k_{\mathfrak{p}}(\sqrt{\alpha})/k_{\mathfrak{p}})\sqrt{\alpha}}{\sqrt{\alpha}}\,.\end{aligned}$$

Since $\mathfrak{p}$ is decomposed in $k(\sqrt{\alpha})/k$ by $[p_1, p_2, p_3] = 1$ and $(\varpi, k_{\mathfrak{p}}(\sqrt{\alpha})/k_{\mathfrak{p}})$ is the Frobenius automorphism over $\mathfrak{p}$ in $k(\sqrt{\alpha})/k$, $(\varpi, k_{\mathfrak{p}}(\sqrt{\alpha})/k_{\mathfrak{p}})(\sqrt{\alpha}) = \sqrt{\alpha}$.

(Case 4) $\mathfrak{p} = \infty$: Since $p_3 > 0$, $\left(\frac{\alpha, p_3}{\infty}\right) = 1$.

(Case 5) $\mathfrak{p} \mid 2$: If $\mathfrak{p} = (2)$, the above cases and the product formula for the Hilbert symbol yields $\left(\frac{\alpha, p_3}{\mathfrak{p}}\right) = 1$. If $(2) = \mathfrak{p}\cdot\mathfrak{p}'$ ($\mathfrak{p} \neq \mathfrak{p}'$), $k_{\mathfrak{p}} = k_{\mathfrak{p}'} = \boldsymbol{Q}_2$ and so we have

$$\left(\frac{\alpha, p_3}{\mathfrak{p}}\right) = \left(\frac{\alpha, p_3}{\mathfrak{p}'}\right) = (-1)^{\frac{p_3-1}{2}\cdot\frac{\alpha-1}{2}} = 1\,. \qquad \square$$

PROPOSITION 3.1.3. *Assume that the class number of k is 1. Then there are $X, Y, Z \in \mathcal{O}_k$ satisfying the following conditions*:

(1) $X^2 - p_3Y^2 - \alpha Z^2 = 0$,

(2) g.c.d$(X, Y, Z) = 1$.

PROOF. By Lemma 3.1.2, we have $\alpha \in N_{k_{\mathfrak{p}}(\sqrt{p_3})/k_{\mathfrak{p}}}(k_{\mathfrak{p}}(\sqrt{p_3})^\times)$ for any prime $\mathfrak{p}$ of k and so there are $X_{\mathfrak{p}}, Y_{\mathfrak{p}} \in k_{\mathfrak{p}}$ such that $X_{\mathfrak{p}}^2 - p_3Y_{\mathfrak{p}}^2 = \alpha$. By the Hasse principal, there are $\tilde{X}, \tilde{Y} \in k$ such that $\tilde{X}^2 - p_3\tilde{Y}^2 = \alpha$ from which the condition (1) holds by writing $\tilde{X} = \frac{X}{Z}, \tilde{Y} = \frac{Y}{Z}$ with $X, Y, Z \in \mathcal{O}_k$. Since $\mathcal{O}_k$ is the principal ideal domain by the assumption, we may choose $X, Y, Z \in \mathcal{O}_k$ so that the condition (2) is satisfied. $\square$

For $k_{13} = \boldsymbol{Q}(\sqrt{p_1}, \sqrt{p_3})$, let U be the unit group of $\mathcal{O}_{k_{13}}/(4)$ and $U(2)$ the 2-Sylow subgroup of U. Similarly, let $k'_{13} := \boldsymbol{Q}(\sqrt{p_1}, \sqrt{\alpha})$ and define $U' := (\mathcal{O}_{k'_{13}}/(4))^\times$ and $U'(2)$ to be the 2-Sylow subgroup of U'.

LEMMA 3.1.4. *The group $U(2)$ is given by*

$$U(2) = \langle -1\rangle \times \langle\sqrt{p_1}\rangle \times \langle\sqrt{p_3}\rangle \times \left\langle\frac{3 + \sqrt{p_1} + \sqrt{p_3} + \sqrt{p_1p_3}}{2}\right\rangle$$

$$\simeq \boldsymbol{Z}/2\boldsymbol{Z} \times \boldsymbol{Z}/2\boldsymbol{Z} \times \boldsymbol{Z}/2\boldsymbol{Z} \times \boldsymbol{Z}/2\boldsymbol{Z}\,.$$

Similarly, $U'(2)$ is given by

$$U'(2) = \langle -1 \rangle \times \langle \sqrt{p_1} \rangle \times \langle \sqrt{\alpha} \rangle \times \left\langle \frac{3 + \sqrt{p_1} + \sqrt{\alpha} + \sqrt{p_1\alpha}}{2} \right\rangle$$
$$\simeq \boldsymbol{Z}/2\boldsymbol{Z} \times \boldsymbol{Z}/2\boldsymbol{Z} \times \boldsymbol{Z}/2\boldsymbol{Z} \times \boldsymbol{Z}/2\boldsymbol{Z}\,.$$

PROOF. Since 2 is unramified in the extension $k_{13}/\boldsymbol{Q}$, we have the decomposition $(2) = \mathfrak{c}_1 \cdots \mathfrak{c}_r$. Therefore the order of U is given by

$$\Pi_{i=1}^r N\mathfrak{c}_i(N\mathfrak{c}_i - 1) = N((2))\Pi_{i=1}^r (N\mathfrak{c}_i - 1) = 16m$$

and so U has the order $16m$, where m is an odd integer. Let $A := \{\pm 1 \mod (4), \pm\sqrt{p_1} \mod (4), \pm\sqrt{p_3} \mod (4), \pm\sqrt{p_1p_3} \mod (4)\}$. Since $p_i \equiv 1 \mod 4$, each element of A has the order 2 and so $A \subset U(2)$. We show that the order of A is 8. Suppose $\sqrt{p_1} \equiv \sqrt{p_3} \mod (4)$ for example. Then $\sqrt{p_1} - \sqrt{p_3} = 4\beta$ for some $\beta \in \mathcal{O}_{k_{13}}$. Taking the norm $N_{k_{13}/\boldsymbol{Q}}$, we obtain

$$\frac{-p_1 - p_3}{16} + \frac{\sqrt{p_1p_3}}{8} \in \mathcal{O}_{\boldsymbol{Q}(\sqrt{p_1p_3})} = \left\{ \frac{a + b\sqrt{p_1p_3}}{2} \,\middle|\, a, b \in \boldsymbol{Z}, a \equiv b \mod 2 \right\},$$

which is a contradiction. Similary, using the structure of $\mathcal{O}_{k_1}$ and $\mathcal{O}_{k_3}$, we can check that any two elements in A are distinct. Hence we see that $U(2) = A \cup A \cdot \{(3+\sqrt{p_1}+\sqrt{p_3}+\sqrt{p_1p_3})/2 \mod (4)\}$. Replacing p_3 by α, the assertion for $U'(2)$ can be shown similarly. □

LEMMA 3.1.5. *Assume $p_1 \equiv 5 \pmod 8$. Then there is a unit $\varepsilon \in \mathcal{O}_k^\times$ of the form $\varepsilon = s + t\sqrt{p_1}, s, t \in \boldsymbol{Z}, s \equiv 0, t \equiv 1 \pmod 2$. Such a unit ε satisfies $\varepsilon \equiv \pm\sqrt{p_1} \mod (4)$ in $U(2)$ and $U'(2)$.*

PROOF. Since $p_1 \equiv 1 \pmod 4$, the fundamental unit $\varepsilon_1 = \frac{s_1 + t_1\sqrt{p_1}}{2}$ ($s_1 \equiv t_1 \pmod 2$) of k satisfies $N_{k/\boldsymbol{Q}}(\varepsilon_1) = -1$. If $s_1 \equiv t_1 \equiv 0 \pmod 2$, we let $\varepsilon := \varepsilon_1 = s + t\sqrt{p_1}$, $s := s_1/2, t := t_1/2 \in \boldsymbol{Z}$, where we have $s \equiv 0, t \equiv 1 \pmod 2$, since $s^2 - p_1t^2 = -1$. Since $\varepsilon = s + t\sqrt{p_1} = s + s\sqrt{p_1} + (t - s)\sqrt{p_1}$ and $s + s\sqrt{p_1} \in 4\mathcal{O}_{k_{13}}$, $\varepsilon \equiv \pm\sqrt{p_1} \mod (4)$. Suppose $s_1 \equiv t_1 \equiv 1 \pmod 2$. Since $p_1 \equiv 5 \pmod 8$, we have $s_1^2 + 3p_1t_1^2 \equiv 3s_1^2 + p_1t_1^2 \equiv 0 \pmod 8$ and so

$$\varepsilon_1^3 = \frac{s_1(s_1^2 + 3p_1t_1^2) + t_1(3s_1^2 + p_1t_1^2)\sqrt{p_1}}{8} = s + t\sqrt{p_1}\,,$$

where $s = s_1(s_1^2 + 3p_1t_1^2)/8$, $t = t_1(3s_1^2 + t_1^2)/8 \in \boldsymbol{Z}$. Since $N_{k/\boldsymbol{Q}}(\varepsilon_1^3) = -1$, $\varepsilon = \varepsilon_1^3$ satisfies the desired conditions. □

The following theorem may be regarded as an analogue of Lemma 1.1.2.

THEOREM 3.1.6. *Assume that the class number of k is 1 and $p_1 \equiv 5 \pmod 8$. Then there are $X, Y, Z \in \mathcal{O}_k$ satisfying the following conditions*:

(1) $X^2 - p_3Y^2 - \alpha Z^2 = 0$,

(2) g.c.d$(X, Y, Z) = 1$, $(Z, 2) = 1$ (resp. g.c.d$(X, Y, Z) = 1$, $(Y, 2) = 1$),

(3) *There is* $\lambda \in \mathcal{O}_{k_{13}}$ (resp. $\lambda \in \mathcal{O}_{k'_{13}}$) *such that* $\lambda^2 \equiv X+Y\sqrt{p_3} \mod (4)$ (resp. $\lambda^2 \equiv X+Z\sqrt{\alpha} \mod (4)$).

PROOF. By Proposition 3.1.3, there are $X, Y, Z \in \mathcal{O}_k$ satisfying (1) and (2).

Case $(Z,2)=1$: Let $\theta := X+Y\sqrt{p_3}$ and $\overline{\theta} := \theta \mod (4)$. Then we easily see $\theta \in \mathcal{O}_{k_{13}}$ and $\overline{\theta} \in U$ since $(Z,2)=1$. Let n be the order of $\overline{\theta}$ in U.

(i) Suppose $n \not\equiv 0 \pmod 2$. Then it is easy to see that there is $\lambda \in \mathcal{O}_{k_{13}}$ such that $\lambda^2 \equiv \theta \mod (4)$.

(ii) Suppose $n \equiv 0 \pmod 2$. By Lemma 3.1.4, $\frac{n}{2} \not\equiv 0 \pmod 2$ and $\overline{\theta}^{\frac{n}{2}} \in U(2)$. Write $\theta^{\frac{n}{2}} = b_1 + b_2\sqrt{p_1} + b_3\sqrt{p_3} + b_4\sqrt{p_1p_3}$, $b_i \in \boldsymbol{Q}$. Since $N_{k_{13}/k}(\theta) = X^2 - p_3Y^2 = \alpha Z^2$, $N_{k_{13}/k}(\theta^{\frac{n}{2}}) = (\alpha Z^2)^{\frac{n}{2}}$. Since $\alpha = x + y\sqrt{p_1} = x - y + 2y \cdot \frac{1+\sqrt{p_1}}{2} \equiv 1 \mod (4)$, $(\alpha Z^2)^{\frac{n}{2}} \equiv (Z^{\frac{n}{2}})^2 \equiv 1 \mod (4)$. Therefore we have

(3.1.6.1)

$$(b_1 + b_2\sqrt{p_1} + b_3\sqrt{p_3} + b_4\sqrt{p_1p_3}) \cdot (b_1 + b_2\sqrt{p_1} - b_3\sqrt{p_3} - b_4\sqrt{p_1p_3}) \equiv 1 \mod (4)\,.$$

We claim that $\theta^{\frac{n}{2}} \equiv -1$ or $\pm\sqrt{p_1} \mod (4)$. Suppose this is not the case. Then, by Lemma 3.1.4, $\theta^{\frac{n}{2}} \equiv \pm\sqrt{p_3}$, $\pm\sqrt{p_1p_3}$ or $a \cdot (3+\sqrt{p_1}+\sqrt{p_3}+\sqrt{p_1p_3})/2$ $(a \in A) \mod (4)$ and so the coefficients of $\sqrt{p_3}$ or $\sqrt{p_1p_3}$ are not 0. Since any element of $U(2)$ has order 2, we have

$$(b_1 + b_2\sqrt{p_1} + b_3\sqrt{p_3} + b_4\sqrt{p_1p_3}) \cdot (b_1 + b_2\sqrt{p_1} - b_3\sqrt{p_3} - b_4\sqrt{p_1p_3}) \not\equiv 1 \mod (4)\,,$$

which contradicts to (3.1.6.1). Therefore, by Lemma 3.1.5, there is $\varepsilon \in \mathcal{O}_k^\times$ such that $\varepsilon\theta^{\frac{n}{2}} \equiv 1 \mod (4)$ and $\varepsilon^2 \equiv 1 \mod (4)$. Replacing (X,Y,Z) by $(\varepsilon X, \varepsilon Y, \varepsilon Z)$, (1), (2) holds obviously, and (3) is also satisfied because $\varepsilon\theta \mod (4)$ has the order $\frac{n}{2} \not\equiv 0 \pmod 2$.

Case $(Y,2)=1$: Let $\theta' := X + Z\sqrt{\alpha}$. Replacing θ by θ' and p_3 by α, the above proof works well by using Lemma 3.1.4. □

Let $\boldsymbol{a} = (X,Y,Z)$ be a triple of integers in $\mathcal{O}_k$ satisfying (1), (2), (3) in Theorem 3.1.6 and fix it once and for all. We let

$$\begin{cases} \theta := X + Y\sqrt{p_3} & \text{if } (Z,2)=1\,, \\ \theta' := X + Z\sqrt{\alpha} & \text{if } (Y,2)=1\,, \end{cases}$$

and set

$$\begin{cases} \theta_1 := \theta, \\ \theta_2 := X - Y\sqrt{p_3}\,, \\ \theta_3 := \overline{X} + \overline{Y}\sqrt{p_3}\,, \\ \theta_4 := \overline{X} - \overline{Y}\sqrt{p_3}\,, \end{cases} \qquad \begin{cases} \theta'_1 = \theta'\,, \\ \theta'_2 = X - Z\sqrt{\alpha}\,, \\ \theta'_3 = \overline{X} + \overline{Z}\sqrt{\overline{\alpha}}\,, \\ \theta'_4 = \overline{X} - \overline{Z}\sqrt{\overline{\alpha}}\,, \end{cases}$$

where $\overline{X}$, $\overline{Y}$ and $\overline{\alpha}$ are conjugates of X, Y and α over $\boldsymbol{Q}$ respectively.

DEFINITION 3.1.7. We then define the number field K by

$$K = K_{\boldsymbol{a}} = \begin{cases} \boldsymbol{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3}, \sqrt{\theta_1\theta_2}, \sqrt{\theta_1\theta_3}, \sqrt{\theta_1}) & \text{if } (Z,2)=1\,, \\ \boldsymbol{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3}, \sqrt{\theta'_1\theta'_2}, \sqrt{\theta'_1\theta'_3}, \sqrt{\theta'_1}) & \text{if } (Y,2)=1\,. \end{cases}$$

For the latter use, we set, for the case of $(Y, 2) = 1$,

$$\begin{cases} \eta_1 := \left(\sqrt{\theta_1'} + \sqrt{\theta_2'}\right)^2 = 2X + 2Y\sqrt{p_3}\,, \\ \eta_2 := \left(\sqrt{\theta_1'} - \sqrt{\theta_2'}\right)^2 = 2X - 2Y\sqrt{p_3}\,, \\ \eta_3 := \left(\sqrt{\theta_3'} + \sqrt{\theta_4'}\right)^2 = 2\overline{X} + 2\overline{Y}\sqrt{p_3}\,, \\ \eta_4 := \left(\sqrt{\theta_3'} - \sqrt{\theta_4'}\right)^2 = 2\overline{X} - 2\overline{Y}\sqrt{p_3}\,. \end{cases}$$

THEOREM 3.1.8. (1) *We have*

$$K = \begin{cases} \boldsymbol{Q}(\sqrt{\theta_1}, \sqrt{\theta_2}, \sqrt{\theta_3}, \sqrt{\theta_4}) & \text{if } (Z, 2) = 1, \\ \boldsymbol{Q}\left(\sqrt{\theta_1'}, \sqrt{\theta_2'}, \sqrt{\theta_3'}, \sqrt{\theta_4'}\right) = \boldsymbol{Q}(\sqrt{\eta_1}, \sqrt{\eta_2}, \sqrt{\eta_3}, \sqrt{\eta_4}) & \text{if } (Y, 2) = 1\,. \end{cases}$$

(2) *The extension $K/\boldsymbol{Q}$ is a Galois extension whose Galois group is isomorphic to $N_4(\boldsymbol{F}_2)$.*

PROOF. (1) Case $(Z, 2) = 1$: It is easy to see $\sqrt{\theta_2}, \sqrt{\theta_3} \in K$. Noting that

$$\begin{aligned} \theta_1\theta_2\theta_3\theta_4 &= N_{k_{13}/\boldsymbol{Q}}(\theta_1) \\ &= N_{k/\boldsymbol{Q}}(N_{k_{13}/k}(\theta_1)) \\ &= N_{k/\boldsymbol{Q}}(\alpha Z^2) \\ &= p_2 h^2 \quad (h \in \boldsymbol{Z})\,, \end{aligned} \tag{3.1.8.1}$$

we have $\sqrt{\theta_4} \in K$ and hence $\boldsymbol{Q}(\sqrt{\theta_1}, \sqrt{\theta_2}, \sqrt{\theta_3}, \sqrt{\theta_4}) \subset K$. Next we show the converse inclusion. Write $\theta_1 = a_1 + a_2\sqrt{p_1} + a_3\sqrt{p_3} + a_4\sqrt{p_1 p_3}$ ($a_i \in \boldsymbol{Q}$). By considering the prime factorization of the ideal (αZ^2) in k_1, we find $\alpha Z^2 \notin \boldsymbol{Z}$. Then, by the equality $\theta_1\theta_2 = \alpha Z^2$, we find that the number of i ($1 \le i \le 4$) with $a_i = 0$ is at most one. Since $\theta_1 + \theta_2 = 2(a_1 + a_2\sqrt{p_1})$, $\theta_1 + \theta_3 = 2(a_1 + a_3\sqrt{p_3})$ and $\theta_1 + \theta_4 = 2(a_1 + a_4\sqrt{p_1 p_3})$, $\sqrt{p_1}, \sqrt{p_3} \in \boldsymbol{Q}(\sqrt{\theta_1}, \sqrt{\theta_2}, \sqrt{\theta_3}, \sqrt{\theta_4})$. By (3.1.8.1), we get $K \subset \boldsymbol{Q}(\sqrt{\theta_1}, \sqrt{\theta_2}, \sqrt{\theta_3}, \sqrt{\theta_4})$.

Case $(Y, 2) = 1$: First, let us show $\boldsymbol{Q}(\sqrt{\theta_1'}, \sqrt{\theta_2'}, \sqrt{\theta_3'}, \sqrt{\theta_4'}) = \boldsymbol{Q}(\sqrt{\eta_1}, \sqrt{\eta_2}, \sqrt{\eta_3}, \sqrt{\eta_4})$. By the definition of η_i's, obviously the inclusion $\supset$ holds. Since $\sqrt{\eta_1} + \sqrt{\eta_2} = 2\sqrt{\theta_1'}$, $\sqrt{\eta_1} - \sqrt{\eta_2} = 2\sqrt{\theta_2'}$, $\sqrt{\eta_3} + \sqrt{\eta_4} = 2\sqrt{\theta_3'}$, $\sqrt{\eta_3} - \sqrt{\eta_4} = 2\sqrt{\theta_4'}$, we obtain the converse inclusion $\subset$.

Next, we show $K = \boldsymbol{Q}(\sqrt{\theta_1'}, \sqrt{\theta_2'}, \sqrt{\theta_3'}, \sqrt{\theta_4'})$. It is easy to see $\sqrt{\theta_2'}, \sqrt{\theta_3'} \in K$. Since $\theta_1'\theta_2' = X^2 - \alpha Z^2 = p_3 Y^2$, we have $\theta_3'\theta_4' = \overline{X}^2 - \overline{\alpha}\overline{Z}^2 = p_3\overline{Y}^2$. So, $\theta_1'\theta_2'\theta_3'\theta_4' = p_3^2(Y\overline{Y})^2 \in \boldsymbol{Q}$ and $\sqrt{\theta_4'} \in K$. For the converse inclusion, it suffices to show $K \subset \boldsymbol{Q}(\sqrt{\eta_1}, \sqrt{\eta_2}, \sqrt{\eta_3}, \sqrt{\eta_4})$. By considering the prime factorization of the ideal $(\alpha(2Z)^2)$ in k_1, we find $\alpha(2Z)^2 \notin \boldsymbol{Z}$. By $N_{k_{13}/\boldsymbol{Q}}(\eta_1) = 4p_2h^2$ and the argument similar to the case of $(Z, 2) = 1$, we have $\sqrt{p_i} \in \boldsymbol{Q}(\sqrt{\eta_1}, \sqrt{\eta_2}, \sqrt{\eta_3}, \sqrt{\eta_4})$ ($i = 1, 2, 3$).

(2) Case $(Z, 2) = 1$: First, $K/\boldsymbol{Q}$ is a Galois extension, because K is the splitting field of $\Pi_{i=1}^4(T^2 - \theta_i) = \Pi_{\sigma\in\mathrm{Gal}(k_{13}/\boldsymbol{Q})}(T^2 - \sigma(\theta_1)) \in \boldsymbol{Z}[T]$. Next, let $k_{123} = \boldsymbol{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3})$, $K_1 := k_{123}(\sqrt{\theta_1\theta_2})$ and $K_2 := K_1(\sqrt{\theta_1\theta_3})$. Since $\theta_3\theta_4 = \overline{\theta_1\theta_2}$ and

$\sqrt{\theta_3\theta_4} = h\sqrt{p_2}/\sqrt{\theta_1\theta_2} \in K_1$, K_1/k_{123} is a Galois extension. Let us show $[K_1 : k_{123}] = 2$. Define $\sigma \in \mathrm{Gal}(k_{123}/\boldsymbol{Q})$ by

$$\sigma : (\sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3}) \mapsto (-\sqrt{p_1}, -\sqrt{p_2}, \sqrt{p_3})\,.$$

Let $\tilde{\sigma} \in \mathrm{Gal}(K_1/\boldsymbol{Q})$ be an extension of σ. Then we have

$$(\tilde{\sigma}(\sqrt{\theta_1\theta_2}))^2 = \tilde{\sigma}(\theta_1\theta_2) = \theta_3\theta_4$$

and so $\tilde{\sigma}(\sqrt{\theta_1\theta_2}) = \pm\sqrt{\theta_3\theta_4}$. Therefore we have

$$\tilde{\sigma}^2(\sqrt{\theta_1\theta_2}) = \tilde{\sigma}(\pm\sqrt{\theta_3\theta_4}) = \tilde{\sigma}(\pm h\sqrt{p_2}/\sqrt{\theta_1\theta_2}) = -\sqrt{\theta_1\theta_2}\,.$$

Since $\tilde{\sigma}^2 \,|_{k_{123}} = \mathrm{id}$, $\sqrt{\theta_1\theta_2} \notin k_{123}$ and hence $[K_1 : k_{123}] = 2$. Similarly we can show that K_2/K_1 is a Galois extension and $[K_2 : K_1] = [K : K_2] = 2$. Hence we have $[K : \boldsymbol{Q}] = [K : K_2][K_2 : K_1][K_1 : k_{123}][k_{123} : \boldsymbol{Q}] = 64$.

Case $(Y, 2) = 1$: $K/\boldsymbol{Q}$ is a Galois extension, because K is the splitting field of $\Pi_{i=1}^4(T^2 - \eta_i) = \Pi_{\sigma\in\mathrm{Gal}(k_{13}/\boldsymbol{Q})}(T^2 - \sigma(\eta_1)) \in \boldsymbol{Z}[T]$. Let $K_1' := k_{123}(\sqrt{\eta_1\eta_2})$ and $E_2' := k_{123}(\sqrt{\eta_1\eta_3})$. By the argumet similar to the case $(Z, 2) = 1$, we have $[K : \boldsymbol{Q}] = [K : K_2'][K_2' : K_1'][K_1' : k_{123}][k_{123} : \boldsymbol{Q}] = 64$.

Finally, by the computer calculation using GAP, we have the following presentation of the group $N_4(\boldsymbol{F}_2)$:

$$N_4(\boldsymbol{F}_2) = \left\langle g_1, g_2, g_3 \;\middle|\; \begin{array}{l} g_1^2 = g_2^2 = g_3^2 = (g_1g_3)^2 = 1 \\ (g_1g_2)^4 = (g_2g_3)^4 = (g_1g_2g_3)^4 = 1 \\ ((g_1g_2g_3g_2)^2g_3)^2 = 1 \end{array} \right\rangle,$$

where g_1, g_2 and g_3 are words representing the following matrices respectively:

$$g_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Case $(Z, 2) = 1$: We define $\tau_1, \tau_2, \tau_3 \in \mathrm{Gal}(K/\boldsymbol{Q})$ by

$$\begin{aligned}
\tau_1 :\ & (\sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3}, \sqrt{\theta_1\theta_2}, \sqrt{\theta_1\theta_3}, \sqrt{\theta_1}, \sqrt{\theta_2}, \sqrt{\theta_3}, \sqrt{\theta_4}) \\
& \mapsto (-\sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3}, \sqrt{\theta_3\theta_4}, \sqrt{\theta_1\theta_3}, \sqrt{\theta_3}, \sqrt{\theta_4}, \sqrt{\theta_1}, \sqrt{\theta_2}) \\
\tau_2 :\ & (\sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3}, \sqrt{\theta_1\theta_2}, \sqrt{\theta_1\theta_3}, \sqrt{\theta_1}, \sqrt{\theta_2}, \sqrt{\theta_3}, \sqrt{\theta_4}) \\
& \mapsto (\sqrt{p_1}, -\sqrt{p_2}, \sqrt{p_3}, -\sqrt{\theta_1\theta_2}, -\sqrt{\theta_1\theta_3}, -\sqrt{\theta_1}, \sqrt{\theta_2}, \sqrt{\theta_3}, \sqrt{\theta_4}) \\
\tau_3 :\ & (\sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3}, \sqrt{\theta_1\theta_2}, \sqrt{\theta_1\theta_3}, \sqrt{\theta_1}, \sqrt{\theta_2}, \sqrt{\theta_3}, \sqrt{\theta_4}) \\
& \mapsto (\sqrt{p_1}, \sqrt{p_2}, -\sqrt{p_3}, \sqrt{\theta_1\theta_2}, \sqrt{\theta_2\theta_4}, \sqrt{\theta_2}, \sqrt{\theta_1}, \sqrt{\theta_4}, \sqrt{\theta_3})\,.
\end{aligned}$$

Then we can easily check $\tau_1^2 = \tau_2^2 = \tau_3^2 = (\tau_1\tau_3)^2 = \mathrm{id}$, $(\tau_1\tau_2)^4 = (\tau_2\tau_3)^4 = (\tau_1\tau_2\tau_3)^4 = \mathrm{id}$, $((\tau_1\tau_2\tau_3\tau_2)^2\tau_3)^2 = \mathrm{id}$. Thus the correspondence $\tau_i \mapsto g_i$ $(i = 1, 2, 3)$ gives an isomorphism $\mathrm{Gal}(K/\boldsymbol{Q}) \simeq N_4(\boldsymbol{F}_2)$.

Case $(Y, 2) = 1$: We note $K = \boldsymbol{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3}, \sqrt{\eta_1\eta_2}, \sqrt{\eta_1\eta_3}, \sqrt{\eta_1})$, because $\sqrt{\eta_3\eta_4} = 4h\sqrt{p_2}/\sqrt{\eta_1\eta_2} \in K_1$. Then the assertion can be shown in a way similar to the case $(Z, 2) = 1$, by replacing θ_i with η_i. □

Next, let us study the ramification in our extension $K/\boldsymbol{Q}$. First, we recall the following well-known fact on the ramification in a Kummer extension.

LEMMA 3.1.9 ([B, Lemma 6]). *Let l be a prime number and E a number field containing a primitive l-th root of unity. Let $E(\sqrt[l]{a})$ $(a \in \mathcal{O}_E)$ be a Kummer extension over E of degree l. Suppose $(a) = \mathfrak{q}^m \mathfrak{a}$ where $\mathfrak{q}$ is a prime ideal in E which does not divide l, $(\mathfrak{q}, \mathfrak{a}) = 1$ and $l \mid m$. Then $\mathfrak{q}$ is unramified in $E(\sqrt[l]{a})/E$.*

THEOREM 3.1.10. *All prime numbers ramified in the extension $K/\boldsymbol{Q}$ are p_1, p_2 and p_3 with ramification index 2.*

PROOF. Case $(Z, 2) = 1$: Let us study the ramification in the extension $k_{13}(\sqrt{\theta_1})/k_{13}$. Since $(T - \frac{\lambda+\sqrt{\theta_1}}{2})(T - \frac{\lambda-\sqrt{\theta_1}}{2}) = (T - \frac{\lambda}{2})^2 - (\frac{\sqrt{\theta_1}}{2})^2 = T^2 - \lambda T + \frac{\lambda^2}{4} - \frac{\theta_1}{4}$ with λ, $\frac{\lambda^2-\theta_1}{4} \in \mathcal{O}_{k_{13}}$, we find $\frac{\lambda+\sqrt{\theta_1}}{2} \in \mathcal{O}_{k_{13}(\sqrt{\theta_1})}$. Since the relative discriminant of $\frac{\lambda+\sqrt{\theta_1}}{2}$ in $k_{13}(\sqrt{\theta_1})/k_{13}$ is given by

$$\begin{vmatrix} 1 & \frac{\lambda+\sqrt{\theta_1}}{2} \\ 1 & \frac{\lambda-\sqrt{\theta_1}}{2} \end{vmatrix}^2 = \left(\frac{\lambda - \sqrt{\theta_1}}{2} - \frac{\lambda + \sqrt{\theta_1}}{2}\right)^2 = \theta_1\,,$$

we find that any prime factor of 2 is unramified in $k_{13}(\sqrt{\theta_1})/k_{13}$.

Next, let us look closely at the prime factorization of the ideal (θ_1) in k_{13}. We let

$$(\theta_1) = \mathfrak{Q}_1^{e_1}\mathfrak{Q}_2^{e_2}\cdots\mathfrak{Q}_r^{e_r}$$

be the prime factorization of (θ_1) and let $\mathfrak{q}_i = \mathfrak{Q}_i \cap k$. Since $N_{k_{13}/k}(\theta) = X^2 - p_3Y^2 = \alpha Z^2$, we have

$$N_{k_{13}/k}((\theta_1)) = (\alpha Z^2) = \mathfrak{p}_2^m\mathfrak{a}^2\,, \tag{3.1.10.1}$$

where $\mathfrak{a} := (Z)$ is an ideal in k. Now the prime factorization of $\mathfrak{q}_i$ in k_{13}/k has the following three cases:

(i) $\mathfrak{q}_i = \mathfrak{Q}_i^2 \quad N_{k_{13}/k}(\mathfrak{Q}_i) = \mathfrak{q}_i$,

(ii) $\mathfrak{q}_i = \mathfrak{Q}_i \quad N_{k_{13}/k}(\mathfrak{Q}_i) = \mathfrak{q}_i^2$,

(iii) $\mathfrak{q}_i = \mathfrak{Q}_i\mathfrak{Q}_i' \quad N_{k_{13}/k}(\mathfrak{Q}_i) = \mathfrak{q}_i,\ N_{k_{13}/k}(\mathfrak{Q}_i') = \mathfrak{q}_i$

Case (i): If e_i is odd, it contradicts to (3.1.10.1). Hence e_i is even.

Case (ii): Since $\theta_1 \in \mathfrak{Q}_i$ and $\mathfrak{q}_i = \mathfrak{Q}_i$, $\theta_2 = a_1 + a_2\sqrt{p_1} - a_3\sqrt{p_3} - a_4\sqrt{p_1p_3} = X - Y\sqrt{p_3} \in \mathfrak{Q}_i$. Since $\mathfrak{p}_2$ is decomposed in k_{13}/k, we see, by (3.1.10.1), $Z \in \mathfrak{Q}_i$. Further, since $\mathfrak{Q}_i$ is not a prime factor of 2 by $(Z, 2) = 1$ and $2X = \theta_1 + \theta_2 \in \mathfrak{Q}_i$, $2Y\sqrt{p_3} = \theta_1 - \theta_2 \in \mathfrak{Q}_i$ and $X, Y, Z \in k$, we have $X, Y, Z \in \mathfrak{q}_i$, which contradicts to g.c.d$(X, Y, Z) = 1$.

Case (iii): Suppose $\mathfrak{P}$ and $\mathfrak{P}'$ are prime factors of $\mathfrak{p}_2$. Since the exponent m in (3.1.10.1) is odd, one of $\mathfrak{P}$ and $\mathfrak{P}'$ appears odd times in the prime factorization of (θ_1). Let $\mathfrak{P}$ be that one. When $\mathfrak{Q}_i \neq \mathfrak{P}$, assume e_i is odd. By (3.1.10.1), $\mathfrak{Q}_i'$ also appears odd times in the prime factorization of (θ_1). Therefore we have $\theta_1 \in \mathfrak{Q}_i\mathfrak{Q}_i' = \mathfrak{q}_i$ and $\theta_2 \in \mathfrak{q}_i$, and so $2X = \theta_1 + \theta_2 \in \mathfrak{Q}_i$, $2Y\sqrt{p_3} = \theta_1 - \theta_2 \in \mathfrak{Q}_i$. This deduces $X, Y, Z \in \mathfrak{q}_i$, which contradicts to g.c.d $(X, Y, Z) = 1$. Thus e_i must be even.

Getting all together, we find that (θ_1) has the form $\mathfrak{P}^{m_1}\mathfrak{A}^2$ (m_1: odd). Then, by Lemma 3.1.9, ramified finite primes in $k_{13}(\sqrt{\theta_1})/k_{13}$ must be lying over p_2. Similarly, we see that ramified finite primes in $k_{13}(\sqrt{\theta_i})/k_{13}$ ($i = 2, 3, 4$) are all lying over p_2. This shows that any ramified finite prime in the extension $K = k_{13}(\sqrt{\theta_1}, \sqrt{\theta_2}, \sqrt{\theta_3}, \sqrt{\theta_4})/k_{13}$ is lying over p_2. Since $k_{13}/\boldsymbol{Q}$ is unramified outside p_1, p_3, we conclude that all ramified prime numbers in $K/\boldsymbol{Q}$ are p_1, p_2 and p_3.

Finally, we show that the ramification indices of p_i's in $K/\boldsymbol{Q}$ are all 2. We easily see that this is true for p_1 and p_3, because the ramification indices of p_1 and p_2 in $k_{13}/\boldsymbol{Q}$ are 2 and any prime factor of p_1 or p_3 is unramified in K/k_{13}. So it suffices to show our assertion for p_2. Let $\mathfrak{p}_{2i}$ be a prime factor in k_{13} of p_2 which is ramified in $k_{13}(\sqrt{\theta_1})/k_{13}$. Since we have $\mathfrak{p}_{2i} = \mathfrak{Q}_i^2$ in $k_{13}(\sqrt{\theta_1})$, by considering the prime factorization of the ideal (θ_i) in $k_{13}(\sqrt{\theta_1})$, we see by Lemma 3.1.9 that $\mathfrak{Q}_i$ is unramified in $k_{13}(\sqrt{\theta_1}, \sqrt{\theta_i})$. Therefore any prime factor of p_2 ramified in $k_{13}(\sqrt{\theta_1})/k_{13}$ is unramified in $k_{13}(\sqrt{\theta_1}, \sqrt{\theta_2}, \sqrt{\theta_3}, \sqrt{\theta_4})/k_{13}(\sqrt{\theta_1})$. Thus the ramification index of p_2 is 2.

Case $(Y, 2) = 1$: As in the case of $(Z, 2) = 1$, we consider the prime factorization of (θ_1') in k_{13}'. Then, by a similar argument, we find that (θ_1') has the ideal decomposition of the form $\mathfrak{Q}'\mathfrak{B}^2$ where any prime factor of $\mathfrak{Q}'$ is lying over p_3. This shows by Lemma 3.1.9 that any ramified finite prime in $k_{13}'(\sqrt{\theta_1'})/k_{13}'$ is lying over p_3. Similarly, we see that finite ramified primes in $k_{13}'(\sqrt{\theta_2'})/k_{13}'$, $\overline{k_{13}'}(\sqrt{\theta_3'})/\overline{k_{13}'}$ and $\overline{k_{13}'}(\sqrt{\theta_4'})/\overline{k_{13}'}$ are all lying over p_3. Hence all ramified prime numbers in $K/\boldsymbol{Q}$ are p_1, p_2 and p_3. The assertion on the ramification indices of p_i's can also be shown by an argument similar to the case of $(Z, 2) = 1$. □

THEOREM 3.1.11. *We have*

$$K = \begin{cases} k_{\{p_1,p_2\}}k_{\{p_2,p_3\}}(\sqrt{\theta_1}) & \text{if } (Z,2)=1\,, \\ k_{\{p_1,p_2\}}k_{\{p_3,p_2\}}(\sqrt{\theta_1'}) & \text{if } (Y,2)=1\,. \end{cases}$$

PROOF. Case $(Z, 2) = 1$: First we have

$$\begin{aligned} \boldsymbol{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{\theta_1\theta_2}) &= \boldsymbol{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{\alpha Z^2}) \\ &= \boldsymbol{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{\alpha}) \\ &= k_{\{p_1,p_2\}}\,. \end{aligned}$$

Next, it is easy to see that $\boldsymbol{Q}(\sqrt{p_2}, \sqrt{p_3}, \sqrt{\theta_1\theta_3})$ is a dihedral extension over $\boldsymbol{Q}$ of degree 8. Since all prime numbers ramified in $\boldsymbol{Q}(\sqrt{p_2}, \sqrt{p_3}, \sqrt{\theta_1\theta_3})/\boldsymbol{Q}$ are p_2 and p_3 with ramification index 2 by Theorem 3.1.10, we have

$$\boldsymbol{Q}(\sqrt{p_2}, \sqrt{p_3}, \sqrt{\theta_1\theta_3}) = k_{\{p_3,p_2\}}$$

by Theorem 1.1.7. Hence we have

$$K = k_{\{p_1,p_2\}}k_{\{p_3,p_2\}}(\sqrt{\theta_1})\,.$$

Case $(Y,2)=1$: Noting that $\eta_1 = 2X + 2Y\sqrt{p_3}$, $\eta_2 = 2X - 2Y\sqrt{p_3}$ and $\eta_3 = 2\overline{X} + 2\overline{Y}\sqrt{p_3}$, we have

$$\begin{aligned} \boldsymbol{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{\eta_1\eta_2}) &= \boldsymbol{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{4\alpha Z^2}) \\ &= \boldsymbol{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{\alpha}) \\ &= k_{\{p_1,p_2\}}\,. \end{aligned}$$

By the same argument as in the case of $(Z,2)=1$ replacing θ_i with η_i, we have $\boldsymbol{Q}(\sqrt{p_2}, \sqrt{p_3}, \sqrt{\eta_1\eta_3}) = k_{\{p_3,p_2\}}$. Hence we have, by Theorem 3.1.8,

$$\begin{aligned} K &= \boldsymbol{Q}(\sqrt{\eta_1}, \sqrt{\eta_2}, \sqrt{\eta_3}, \sqrt{\eta_4}) \\ &= \boldsymbol{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3}, \sqrt{\eta_1\eta_2}, \sqrt{\eta_1\eta_3}, \sqrt{\eta_1}) \\ &= k_{\{p_1,p_2\}}k_{\{p_3,p_2\}}(\sqrt{\theta_1'})\,. \end{aligned}$$

□

3.2. The 4-th multiple residue symbol. Let p_1, p_2, p_3 and p_4 be four prime numbers satisfying

$$\text{(3.2.1)} \qquad \begin{cases} p_1 \equiv 5 \pmod 8,\ \ p_i \equiv 1 \pmod 4\ (i=2,3,4)\,, \\ \left(\dfrac{p_i}{p_j}\right) = 1\ (1 \le i \ne j \le 4),\ [p_i, p_j, p_k] = 1\ (i,j,k:\ \text{distinct})\,, \end{cases}$$

and we assume that the class number of $k_1 = \boldsymbol{Q}(\sqrt{p_1})$ is 1.

Let K be the field defined in Definition 3.1.7.

DEFINITION 3.2.2. We define the 4-*th multiple residue symbol* $[p_1, p_2, p_3, p_4]$ by

$$[p_1, p_2, p_3, p_4] = \begin{cases} 1 & \text{if } p_4 \text{ is completely decomposed in } K/\boldsymbol{Q}\,, \\ -1 & \text{otherwise.} \end{cases}$$

We let

$$L := \begin{cases} \boldsymbol{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3}, \sqrt{\theta_1\theta_2}, \sqrt{\theta_1\theta_3}) & \text{if } (Z,2)=1\,, \\ \boldsymbol{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3}, \sqrt{\eta_1\eta_2}, \sqrt{\eta_1\eta_3}) & \text{if } (Y,2)=1\,. \end{cases}$$

Case $(Z,2)=1$: Let $\tau_1, \tau_2, \tau_3 \in \mathrm{Gal}(K/\boldsymbol{Q})$ be as in the proof of Theorem 3.1.8 and we let

$$\xi_1 := \sqrt{\theta_1\theta_2} + \sqrt{\theta_3\theta_4}\,, \quad \xi_2 := \sqrt{\theta_1\theta_3} + \sqrt{\theta_2\theta_4}\,, \quad \xi_3 := \sqrt{\theta_1} + \sqrt{\theta_2} + \sqrt{\theta_3} + \sqrt{\theta_4}\,.$$

Then, the subfields of $K/\boldsymbol{Q}$ which corresponds by Galois theory to the subgroups generated by τ_1, τ_2, τ_3 and $(\tau_1\tau_2\tau_3\tau_2)^2$ are $\boldsymbol{Q}(\sqrt{p_2}, \sqrt{p_3}, \xi_1, \sqrt{\theta_1\theta_3}, \xi_3)$, $\boldsymbol{Q}(\sqrt{p_1}, \sqrt{p_3}, \sqrt{\theta_2}, \sqrt{\theta_3}, \sqrt{\theta_4})$, $\boldsymbol{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{\theta_1\theta_2}, \xi_2, \xi_3)$ and F, respectively. By the assumption (3.2.1), p_4 is completely decomposed in the extension $F/\boldsymbol{Q}$.

Case $(Y,2)=1$: We let $\tau_1, \tau_2, \tau_3 \in \mathrm{Gal}(K/\boldsymbol{Q})$ and ξ_1, ξ_2, ξ_3 be defined by replacing θ_i in the case $(Z,2)=1$ with η_i $(1 \le i \le 4)$. Then, as in the case $(Z,2)=1$ the subfields of $K/\boldsymbol{Q}$ which corresponds by Galois theory to the subgroups generated by τ_1, τ_2, τ_3 and $(\tau_1\tau_2\tau_3\tau_2)^2$ are $\boldsymbol{Q}(\sqrt{p_2}, \sqrt{p_3}, \xi_1, \sqrt{\eta_1\eta_3}, \xi_3)$, $\boldsymbol{Q}(\sqrt{p_1}, \sqrt{p_3}, \sqrt{\eta_2}, \sqrt{\eta_3}, \sqrt{\eta_4})$,

$\boldsymbol{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{\eta_1\eta_2}, \xi_2, \xi_3)$ and F, respectively. By the assumption (3.2.1), p_4 is completely decomposed in the extension $F/\boldsymbol{Q}$.

Let $\mathfrak{P}_4$ be a prime ideal in F lying over p_4 and let $\sigma_{\mathfrak{P}_4} = \left(\frac{K/F}{\mathfrak{P}_4}\right) \in \mathrm{Gal}(K/F)$ be the Frobenius automorphism of $\mathfrak{P}_4$. Note that $\mathfrak{P}_4$ is decomposed in K/F if and only if p_4 is completely decomposed in $K/\boldsymbol{Q}$. So we have, by Definition 3.2.2,

$$[p_1, p_2, p_3, p_4] = \begin{cases} 1 & \sigma_{\mathfrak{P}_4} = \mathrm{id}_K\,, \\ -1 & \sigma_{\mathfrak{P}_4} \neq \mathrm{id}_K\,. \end{cases} \tag{3.2.3}$$

Let $S := \{p_1, p_2, p_3, p_4\}$. Then, by Theorem 2.2.2, we have

$$\begin{aligned} G_S(2) &= \mathrm{Gal}(\boldsymbol{Q}_S(2)/\boldsymbol{Q}) \\ &= \langle x_1, x_2, x_3, x_4 \mid x_1^{p_1-1}[x_1, y_1] = \cdots = x_4^{p_4-1}[x_4, y_4] = 1\rangle\,. \end{aligned}$$

Let $\hat{F}$ be the free pro-2 group on x_1, x_2, x_3, x_4 and let $\pi : \hat{F}(2) \to G_S(2)$ be the natural homomorphism. Since $K \subset \boldsymbol{Q}_S(2)$ by Theorem 3.1.10, we have the natural homomorphism $\psi : G_S(2) \to \mathrm{Gal}(K/\boldsymbol{Q})$. Let $\varphi := \pi \circ \psi : \hat{F} \to \mathrm{Gal}(K/\boldsymbol{Q})$. We then see that

$$\varphi(x_1) = \tau_1\,, \quad \varphi(x_2) = \tau_2\,, \quad \varphi(x_3) = \tau_3, \quad \varphi(x_4) = 1\,.$$

Therefore the relations among τ_1, τ_2 and τ_3 are equivalent to the following relations:

$$\begin{gathered} \varphi(x_1)^2 = \varphi(x_2)^2 = \varphi(x_3)^2 = \varphi(x_1x_3)^2 = 1\,, \quad \varphi(x_4) = 1\,, \\ \varphi(x_1x_2)^4 = \varphi(x_2x_3)^4 = \varphi(x_1x_2x_3)^4 = \varphi((x_1x_2x_3x_2)^2x_3)^2 = 1\,. \end{gathered} \tag{3.2.4}$$

On the other hand, by the assumption (3.2.1), we have $\overline{\mu}_2(1234) = \mu_2(1234)$.

THEOREM 3.2.5. *We have*

$$[p_1, p_2, p_3, p_4] = (-1)^{\mu_2(1234)}\,.$$

PROOF. By (3.2.3), we have

$$\varphi(y_4) = \begin{cases} 1 & \text{if } [p_1, p_2, p_3, p_4] = 1\,, \\ (\tau_1\tau_2\tau_3\tau_2)^2 = \varphi((x_1x_2x_3x_2)^2) & \text{if } [p_1, p_2, p_3, p_4] = -1\,. \end{cases}$$

By (3.2.4), $\mathrm{Ker}(\varphi)$ is generated as a normal subgroup of $\hat{F}$ by

$$x_1^2, x_2^2, x_3^2, (x_1x_3)^2, x_4, (x_1x_2)^4, (x_2x_3)^4, (x_1x_2x_3)^4 \text{ and } ((x_1x_2x_3x_2)^2x_3)^2$$

and one has

$$
\begin{aligned}
&M_2((x_1)^2) = (1+X_1)^2 = 1 + X_1^2\,,\\
&M_2((x_2)^2) = (1+X_2)^2 = 1 + X_2^2\,,\\
&M_2((x_3)^2) = (1+X_3)^2 = 1 + X_3^2\,,\\
&M_2((x_1x_3)^2) = ((1+X_1)(1+X_3))^2 \equiv 1 \bmod \deg \ \geq 2\,,\\
&M_2((x_1x_2)^4) = ((1+X_1)(1+X_2))^4 \equiv 1 \bmod \deg \ \geq 4\,,\\
&M_2((x_2x_3)^4) = ((1+X_2)(1+X_3))^4 \equiv 1 \bmod \deg \ \geq 4\,,\\
&M_2((x_1x_2x_3)^4) = ((1+X_1)(1+X_2)(1+X_3))^4 \equiv 1 \bmod \deg \ \geq 4\,,\\
&M_2(((x_1x_2x_3x_2)^2x_3)^2)\\
&\equiv 1 + X_3^2 + X_1^2X_3 + X_1X_3^2 + X_1X_3^2 + X_3X_1^2 + X_3^2X_1 \bmod \deg \ \geq 4\,.
\end{aligned}
$$

Therefore $\mu_2((1);*), \mu_2((2);*), \mu_2((3);*), \mu_2((12);*), \mu_2((23);*), \mu_2((123);*)$ take their values 0 on $\mathrm{Ker}(\varphi)$. If $\varphi(y_4) = 1$, $\mu_2(1234) = \mu_2((123);y_4) = 0$ by $\varphi(y_4) \in \mathrm{Ker}(\varphi)$. If $\varphi(y_4) = (\tau_1\tau_2\tau_3\tau_2)^2 = \varphi((x_1x_2x_3x_2)^2)$, we can write $y_4 = (x_1x_2x_3x_2)^2R$, where $R \in \mathrm{Ker}(\varphi)$. Then comparing the coefficients of $X_1X_2X_3$ in the equality $M_2(y_4) = M_2((x_1x_2x_3x_2)^2)M_2(R)$, we have

$$
\begin{aligned}
\mu_2(1234) &= \mu_2((123);y_4)\\
&= \mu_2((123);(x_1x_2x_3x_2)^2) + \mu_2((12);(x_1x_2x_3x_2)^2)\mu_2((3);R)\\
&\quad + \mu_2((1);(x_1x_2x_3x_2)^2)\mu_2((23);R) + \mu_2((123);R)\\
&= 1\,.
\end{aligned}
$$

This yields our assertion. □

EXAMPLE 3.2.6. Let $(p_1, p_2, p_3, p_4) := (5, 8081, 101, 449)$. Then we have

$$
\begin{cases}
\theta_1 = 25 + 2\sqrt{5} + 2\sqrt{101}\,,\\
\theta_2 = 25 + 2\sqrt{5} - 2\sqrt{101}\,,\\
\theta_3 = 25 - 2\sqrt{5} + 2\sqrt{101}\,,\\
\theta_4 = 25 - 2\sqrt{5} - 2\sqrt{101}\,,
\end{cases}
\qquad
\begin{cases}
k_{\{p_1,p_2\}} = \boldsymbol{Q}(\sqrt{5}, \sqrt{8081}, \sqrt{241 + 100\sqrt{5}})\,,\\
k_{\{p_3,p_2\}} = \boldsymbol{Q}(\sqrt{8081}, \sqrt{101}, \sqrt{1009 + 100\sqrt{101}})\,,
\end{cases}
$$

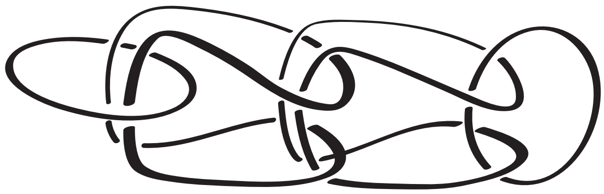
and

$$
K = k_{\{p_1,p_2\}} \cdot k_{\{p_3,p_2\}}(\sqrt{25 + 2\sqrt{5} + 2\sqrt{101}})\,.
$$

Then we have

$$
\begin{cases}
\left(\dfrac{p_i}{p_j}\right) = 1 \ \ (1 \leq i \neq j \leq 4)\,, \quad [p_i, p_j, p_k] = 1 \ \ (i, j, k : \text{ distinct})\,,\\
[p_1, p_2, p_3, p_4] = -1\,.
\end{cases}
$$

In view of Example 2.1.3, this 4-tuple prime numbers may be called *Milnor primes*.



Finally, two remarks are in order.

REMARK 3.2.7. (1) By Theorem 3.2.5, the shuffle relation for arithmetic Milnor invariants (Theorem 2.2.3 (3)) yields the following shuffle relation for the 4-th multiple residue symbol

$$\prod_{(ijk)\in \mathrm{PSH}(I,J)} [p_i, p_j, p_k, p_l] = 1\,,$$

where I, J are multi-indices with $|I| + |J| = 3$ and $\mathrm{PSH}(I, J)$ is the set of proper shuffles of I and J, and $1 \le l \le 4$. It is also expected that our 4-th multiple residue symbols satisfy the cyclic symmetry, although we are not able to prove it in the present paper. We hope to study the reciprocity law for the 4-th multiple residue symbol in the future.

(2) In this paper, we are concerned only with 2-extensions over $\boldsymbol{Q}$ as a generalization of Rédei's work. If a base number field k contains the group of l-th roots of unity μ_l for an odd prime number l and the maximal pro-l Galois group over k unramified outside a set of certain primes $S = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\} \cup \{\mathfrak{p}|\infty\}$ is a Koch type pro-l group, we can intoduce μ_l-valued multiple residue symbol $[\mathfrak{p}_1, \ldots, \mathfrak{p}_r]$ in a similar manner.

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REFERENCES

[A] F. AMANO, On Rédei's dihedral extension and triple reciprocity law, Proc. Japan Acad. Ser. A Math. Sci. 90 (2014), 1–5.

[B] B. J. BIRCH, Cyclotomic fields and Kummer extensions. Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965), pages 85–93. Thompson, Washington, D.C., 1967.

[CFL] K. T. CHEN, R. H. FOX AND R. C. LYNDON, Free differential calculus. IV. The quotient groups of the lower central series, Ann. of Math. (2) 68 (1958), 81–95.

[F] R. H. FOX, Free differential calculus. I: Derivation in the free group ring, Ann. of Math. 57 (1953), 547–560.

[I] Y. IHARA, On Galois representations arising from towers of coverings of $\mathbf{P}^1 \setminus \{0, 1, \infty\}$, Invent. Math. 86 (1986), no. 3, 427–459.

[K1] H. KOCH, Galois theory of p-extensions. With a foreword by I. R. Shafarevich. Translated from the 1970 German original by Franz Lemmermeyer. With a postscript by the author and Lemmermeyer. Springer Monographs Math. Springer-Verlag, Berlin, 2002.

[K2] H. KOCH, On p-extension with given ramification, Appendix in: K. Haberland, Galois cohomology of algebraic number fields, 89–126, VEB Deutscher Verlag der Wissenschaften, Berlin, 1978.

[Mi1] J. MILNOR, Link groups, Ann. of Math. 59 (1954), 177–195.

[Mi2] J. MILNOR, Isotopy of links, in Algebraic Geometry and Topology, A symposium in honor of S. Lefschetz (edited by R.H. Fox, D.C. Spencer and A.W. Tucker), 280–306, Princeton University Press, Princeton, N.J., 1957.

[Mo1] M. MORISHITA, Milnor's link invariants attached to certain Galois groups over **Q**, Proc. Japan Acad. Ser. A Math. Sci. 76 (2000), 18–21.

[Mo2] M. MORISHITA, On certain analogies between knots and primes, J. Reine Angew. Math. 550 (2002), 141–167.

[Mo3] M. MORISHITA, Milnor invariants and Massey products for prime numbers, Compos. Math. 140 (2004), 69–83.

[Mo4] M. MORISHITA, Knots and Primes—An introduction to arithmetic topology, Universitext, Springer, London, 2012.

[Mu] K. MURASUGI, Nilpotent coverings of links and Milnor's invariant, Low-dimensional topology (Chelwood Gate, 1982), 106–142, London Math. Soc. Lecture Note Ser., 95, Cambridge Univ. Press, Cambridge-New York, 1985.

[O] T. ODA, Note on meta-abelian quotients of pro-l free groups, preprint, 1985.

[R] L. RÉDEI, Ein neues zahlentheoretisches Symbol mit Anwendungen auf die Theorie der quadratischen Zahlkörper I, J. Reine Angew. Math. 180 (1939), 1–43.

[V1] D. VOGEL, On the Galois group of 2-extensions with restricted ramification, J. Reine Angew. Math. 581 (2005), 117–150.

[V2] D. VOGEL, A letter to M. Morishita, 2008, February.

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