ON A CERTAIN NUMERICAL INVARIANT OF LINK TYPES

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1. **Introduction.** Let l be an (oriented) tame link(1) of multiplicity μ in a 3-sphere S^3, and let L be a *diagram*, i.e. the image under a regular projection of S^3 into S^2. To L we can associate an integral square matrix M, called the *matrix* of a link l ([13], [14], also see §3). The quadratic form $f(x_1, x_2, \cdots, x_n)$ associated to the symmetric matrix $M + M^t$ induces some invariants of the original link type, where M^t denotes the transposed matrix of M [19], [22].

In this paper, we shall especially consider the *signature* $\sigma(f)$ of the quadratic form $f(x_1, \cdots, x_n)$ associated to $M + M^t$. The invariance of $\sigma(f)$ was proved by Trotter for $\mu = 1$ [22]. For the general case, it is not so difficult to prove that it is also an invariant of the link type (Theorem 3.1).

In §3, we shall define the matrix of a link for the convenience of the reader. In §5 we shall show that for any alternating link the signature is calculated immediately from the matrix. Then, from the properties of the matrix of a special alternating knot, it follows:

THEOREM 5.5. *Any special alternating knot is not amphicheiral unless it is a trivial knot.*

§§6–8 are concerned with *slice knots* or *links*, which were first defined by Fox and Milnor [4]. In determining the signature of a link(2) of this kind, the *nullity* of the matrix plays an important role. The nullity is determined easily from the *Alexander matrix* [3, II] rather than the matrix of a link. This is the reason to establish some relations between them in §4. In §8, the concept of a slice knot is generalized to the case of links, and a slice link in various senses will be defined. Cf. [5]. In particular, we obtain the following necessary condition for a knot to be a slice knot.

THEOREM 8.3. *If a knot is a slice knot, then the signature is always zero.*

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(1) A link l of multiplicity μ consists of μ ordered, oriented circles $l_1, \cdots, l_\mu$ imbedded in the 3-sphere S^3. Two links l and l' are of the *same type* or *isotopic* if $\mu = \mu'$ and there exists an orientation preserving homeomorphism f of S^3 onto itself such that $f \mid l_i = l_i'$ and $f \mid l_i$ is also orientation preserving, $i = 1, 2, \cdots, \mu$. A *knot* is a link of multiplicity $\mu = 1$. For any link, we select a "point of infinity" $\infty \in S^3 - l$ and consider $S^3 - \infty$ as a Cartesian product $R^1 \times R^1 \times R^1$.

(2) By the *signature* of a link l is meant the signature of the quadratic form $f(x_1, \cdots, x_n)$ associated to the matrix $M + M^t$, where M is the matrix of l. See (3.10) in §3.

Since the signature of the product knot (in the sense of [17]) is the sum of that of each component (Corollary 7.4), this theorem implies almost immediately

THEOREM 8.9. *The granny knot is not a slice knot*(3).

This theorem was proved recently by J. J. Andrews by means of the Minkowski unit

It is well known [18], [21] that given a link l there exists an orientable tame connected surface in S^3 spanning l. Therefore we may consider a locally flat connected surface in the *upper half space* of S^4 spanning l. The minimum genus h^* of these surfaces is a link invariant. §9 is devoted to prove the following:

THEOREM 9.1. *The absolute value of the signature of a link of multiplicity* μ *is not greater than* $2h^* + \mu - 1$.

Finally it would be interesting to investigate the relationship between the signature and other invariants of a knot. This last section will be concerned with the *unknotting number* of a knot. The unknotting number is a knot invariant which was first introduced by Wendt in 1937 [23]. Up to the present, only a few results have been found [10], [23]. In §10 we shall prove

THEOREM 10.1. *The absolute value of the signature of a knot is not greater than twice the unknotting number.*

This theorem assures the existence of a prime knot whose unknotting number is a given natural number n. For example, the torus knot of type $(2n + 1, 2)$ has the unknotting number n.

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2. **Equivalence of matrices.** We begin with definitions of equivalence of two matrices.

DEFINITION 2.1. Two $n \times n$ integral matrices are said to be s-*equivalent* if one is transformed into the other by a finite sequence of certain operations $\Lambda_i^{\pm 1}$ $(i = 1,2,3)$ defined as follows:

$$\Lambda_1\colon A \to TAT^t, \text{ with } T \text{ integral and unimodular,}$$

$$\Lambda_2\colon A \to \left[\begin{array}{c|cc} A & 0 & 0 \\ \hline 0 \cdots 0 & 0 & 1 \\ q_1 \cdots q_n & 0 & 0 \end{array}\right], \quad q_i \text{ being integers,}$$

$$\Lambda_3\colon A \to \left[\begin{array}{c|c} & 0\ q_1 \\ A & \vdots\ \vdots \\ & 0\ q_n \\ \hline 0 & 0\ 0 \\ 0 & 1\ 0 \end{array}\right].$$

(3) The *granny* knot is the product of the trefoil knot with itself [4].

If two matrices to be considered are symmetric, then they are said to be S-*equivalent* if one is transformed into the other by a finite sequence of operations $\Lambda_1^{\pm 1}$, $\Lambda_2'^{\pm 1}$:

$$\Lambda_2' : A \to \left[\begin{array}{c|cc} A & 0 & \\ \hline 0 & 0 & 1 \\ & 1 & 0 \end{array}\right].$$

From the definition, it follows

(2.1) *If M and N are* s-*equivalent, then their symmetrized matrices, $M + M^t$ and $N + N^t$, are* S-*equivalent.*

Any integral symmetric matrix A can be expressed in a diagonal form by a unimodular matrix R of rational numbers, i.e.

$$RAR^t = \begin{bmatrix} a_1 & & & \\ & a_2 & & 0 \\ & & \ddots & \\ & 0 & & \ddots \\ & & & & a_n \end{bmatrix}, \quad a_i \text{ being rational numbers.}$$

Then we can define $\sigma(A)$, the *signature* of A, as follows:

DEFINITION 2.2.

$$\sigma(A) = \sum_{i=1}^{n} \operatorname{sign} a_i,$$

where $\operatorname{sign} a_i = a_i/|a_i|$ if $a_i \neq 0$, and $\operatorname{sign} 0 = 0$.

It is clear that

(2.2) *$\sigma(A)$ is an invariant of the* S-*equivalence class.*

As one practical method to calculate the signature of a given matrix, the following well-known theorem will frequently be used [9].

(2.3) *Let A be a symmetric matrix of rank n. Then there is a sequence $\Delta_0 = 1, \Delta_1, \Delta_2, \cdots, \Delta_n$ (called the σ-series), of principal minors of A satisfying the following conditions:*

(1) *Δ_i is an $i \times i$ principal minor of Δ_{i+1} $(i = 1, 2, \cdots, n-1)$.*

(2) *No two consecutive matrices Δ_i and Δ_{i+1} are both singular.*

The signature of A is, then, given by

$$\sigma(A) = \sum_{i=0}^{n-1} \operatorname{sign}(\det \Delta_i \cdot \det \Delta_{i+1}), \tag{2.4}$$

where $\det \Delta_0 = 1$.

3. **The matrix of a link.** Let l be a link in S^3 of multiplicity μ and let L be a diagram of l. Let p be a regular projection of S^3 into S^2. The orientation of

L is determined by that of l. L is a 1-dimensional complex in S^2. To define the matrix by means of L we consider the following two cases.

Case I. *L is connected.*

L consists of some number, n say, of vertices $c_1, c_2, \cdots, c_n$ and $2n$ edges $e_1, e_2, \cdots, e_{2n}$, that are oriented and closed. Then the set of all e_i is divided into some subsets $T_1, T_2, \cdots, T_q$ in such a way that

(3.1) (1) T_i is an m-circuit([4]).

(2) T_i is a cycle with respect to the orientation of L.

(3) The inverse image $p^{-1}(T_i) \cap l$ has m components. T_i will be called a *Seifert circuit*. On the other hand, L divides S^2 into $n+2$ regions $r_1, r_2, \cdots, r_{n+2}$, that are open and connected. The Seifert circuits are classified into the following two classes:

DEFINITION 3.1. T_i is of the *first type* if it bounds a region r_j. Otherwise T_i is of the *second type*.

If L contains no Seifert circuit of the second type, then L is called a *special diagram* of l. l is called a *special* alternating link if l possesses a diagram that is special and alternating.

Now let $C_1, C_2, \cdots, C_m$ be Seifert circuits of the second type in L. Since $|C_i|$ is a simple closed curve in S^2, it divides S^2 into simply connected domains $|C_i|^+$ and $|C_i|^-$. Let $D(\gamma_1, \cdots, \gamma_m)$ be the closure of $|C_1|^{\gamma_1} \cap \cdots \cap |C_m|^{\gamma_m}$, where γ_i denotes $+$ or $-$. Then it is easy to show that

(3.2) *Only $m+1$ of the sets $D(\gamma_1, \cdots, \gamma_m)$ are nonempty. Let them be $D_1, D_2, \cdots, D_{m+1}$, called the Seifert domains. The boundary $\dot{D}_i$ of D_i consists of some Seifert circuits of the second type, and D_i and D_j $(i \neq j)$ have at most one Seifert circuit of the second type in common.*

Further, $r_1, r_2, \cdots, r_{n+2}$ are classified into two classes, α and β, in such a way that

(3.3) r_i belongs to the *class β*, or r_i is a *β-region*, if $\dot{r}_i$ is a Seifert circuit of the first type. Otherwise, r_i belongs to the *class α*, or r_i is an *α-region*.

This classification possesses the following properties [13].

(3.4) (1) *No two β-regions are adjacent, i.e. the boundaries of two β-regions have no edge in common.*

(2) *If two α-regions are adjacent, then the common edges belong to some Seifert circuits of the second type.*

Now at each vertex c in L, at most four different regions meet, and at least two among these four regions are α-regions and contained in the same Seifert domain, D_i say. Then we say *c belongs* to D_i. It is clear that

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([4]) An *m-circuit* T is a 1-complex with $m \geqq 1$ edges whose underlying space, denoted by $|T|$, is a simple closed curve; a 1-complex P is a *cycle* with respect to an orientation if in the free abelian group generated by the vertices of P, $\sum_{e \in P} \{(\text{terminal end point of } e) - (\text{initial end point of } e)\} = 0$. See [1].

(3.5) *Every vertex belongs to one and only one Seifert domain.*

Let $L_i = D_i \cap L$ and consider $p^{-1}(L_i) = \tilde{l}_i$. $\tilde{l}_i$ consists of some number of arcs in l. Especially for any vertex c, $p^{-1}(c)$ consists of two points $c^\#$ and $c^\flat$ [5]. If a vertex c is in D_i but *does not belong* to D_i, then two points $c^\#$ and $c^\flat$ that are in the boundary of $\tilde{l}_i$ can be joined by a segment $c^\flat$ in S^3 such that $p(c^\# \cup c^\flat \cup c^\flat) = c$. Thus we obtain a link l_i from $\tilde{l}_i$ such that $p(l_i) = L_i$. It should be noted that L_i has no Seifert circuit of the second type, i.e. L_i is a special diagram of l_i. We shall denote it symbolically, disregarding order, by

$$l = l_1 * l_2 * \cdots * l_{m+1}.$$

Similarly, $L = L_1 * L_2 * \cdots * L_{m+1}$.

Now for each vertex c of L, we shall define three indices η, ε and d.

DEFINITION 3.2. (1) Let c belong to a Seifert domain, D_k say. Then $\eta(c)$ is defined as $+1$ or -1 according as the rotation to make the overpass through c coincide with the underpass in an α-region contained in D_k is clockwise or counterclockwise.

(2) For any region r_i, if c is not in $\dot{r}_i$, then $d_{r_i}(c) = 0$. Otherwise $d_{r_i}(c)$ is defined as follows. Let r_j be the region that is opposite to r_i with respect to c.

(i) If $i = j$, then $d_{r_i}(c) = 0$.

(ii) If $i \neq j$, then $d_{r_i}(c) = 1$ or 0 according as r_i is on the left with respect to the direction of the underpass at c or not.

Suppose r_i, and hence r_j, is contained in D_k. Then $\varepsilon_{r_i}(c)$ is defined as follows.

(3) (i) If c is not in $\dot{r}_i$, or does not belong to D_k or if $i = j$, then $\varepsilon_{r_i}(c) = 0$.

(ii) Otherwise, $\varepsilon_{r_i}(c) = 2d_{r_i}(c) - 1$.

By means of these indices, the *matrix M of l with respect to L* is defined as follows:

DEFINITION 3.3. M is a block matrix:

$$(1) \qquad M = (M_{ij})_{i,j=1,2,\cdots,m+1},$$

$$M_{ii} = (a_{pq}^{(i)})_{p,q=1,2,\cdots,w_i},$$

where w_i denotes the number of α-regions contained in D_i, and

$$-a_{pq}^{(i)} = \sum \eta(c)\, d_{X_p}(c),$$

where the summation extends over all vertices that are in the intersection of the boundaries of two different α-regions X_p and X_q, both contained in D_i, and

[5] The one of $c^\#$ and $c^\flat$ with the larger z-coordinate is the *overcrossing* and the other is the *undercrossing*. The small segment of l containing the overcrossing or undercrossing will be called the *overpass* or *underpass*.

$$a_{pp}^{(i)} = -\sum_{q=1;p\neq q}^{w_i} a_{pq}^{(i)},$$

(2) $$M_{ij} = (b_{rs}^{(ij)})_{r=1,\cdots,w_i;s=1,\cdots w_j} \qquad (i \neq j),$$

$$-b_{rs}^{(ij)} = \sum \eta(c) d_{X_r}(c) \varepsilon_{X_s}(c),$$

where the summation extends over all vertices that are in the intersection of the boundaries of two α-regions X_r and X_s, contained in D_i and D_j respectively.

From the definition we see immediately

(3.6) *Each row and column in M_{ii} corresponds to each α-region contained in D_i.*

(3.7) *At least one of M_{ij} or M_{ji} is a zero matrix for $i \neq j$ and*

$$\sum_{r=1}^{w_i} b_{rs}^{(ij)} = \sum_{s=1}^{w_j} b_{rs}^{(ij)} = 0.$$

Case II. *L consists of p ($\geqq 2$) connected components $L^{(1)}, \cdots, L^{(p)}$.*

The *matrix of l with respect to L* is defined as follows.

$$M = \begin{bmatrix} M^{(1)} & & & 0 \\ & M^{(2)} & & \\ & & \ddots & \\ 0 & & & M^{(p)} \\ & & & & M^{(p+1)} \end{bmatrix},$$

where $M^{(i)}$ $(i = 1, \cdots, p)$ denotes the matrix of $l^{(i)} = p^{-1}(L^{(i)})$ with respect to $L^{(i)}$, and $M^{(p+1)}$ denotes a $p \times p$ zero matrix.

Here we shall introduce some notations which will be used in the future.

Let N be an $m \times n$ matrix. Then by

$$N\begin{pmatrix} p_1 & \cdots & p_r \\ q_1 & \cdots & q_s \end{pmatrix}$$

is denoted the $r \times s$ matrix consisting of p_1th row, $\cdots$, p_rth row and q_1th column, $\cdots$, q_sth column, of N. In particular, by $N(p_1 \cdots p_r)$ is denoted the $r \times r$ matrix

$$N\begin{pmatrix} p_1 & \cdots & p_r \\ p_1 & \cdots & p_r \end{pmatrix}.$$

Further, by

$$\tilde{N}\begin{pmatrix} p_1 & \cdots & p_r \\ q_1 & \cdots & q_s \end{pmatrix}$$

is denoted the $(m-r) \times (n-s)$ matrix obtained from N by striking out p_1th row, $\cdots$, p_rth row and q_1th column, $\cdots$, q_sth column, and in particular,

$$\tilde{N}(p_1 \cdots p_r) = \tilde{N}\begin{pmatrix} p_1 & \cdots & p_r \\ p_1 & \cdots & p_r \end{pmatrix}.$$

Now let $M = (M_{ij})$ be the matrix of l with respect to L, and let M^* be the principal minor of M obtained by striking out the row and column containing a diagonal element $a^{(i)}_{r_i,r_i}$ in each M_{ii} $(i = 1,2,\cdots,m+1)$. In other words, we can choose $m+1$ positive integers $q_1,\cdots,q_{m+1}$ such that $M^* = \tilde{M}(q_1 \cdots q_{m+1})$. M^* will be called the *L-principal minor* of the type $(q_1,\cdots,q_{m+1})$ of M.

If L consists of p connected components, then M^* is defined as follows:

$$M^* = \begin{pmatrix} M^{(1)*} & & & & 0 \\ & M^{(2)*} & & & \\ & & \ddots & & \\ & & & M^{(p)*} & \\ 0 & & & & M^{(p+1)*} \end{pmatrix},$$

where $M^{(i)*}$ $(1 \leqq i \leqq p)$ denotes the L-principal minor of $M^{(i)}$ with respect to $L^{(i)}$, and $M^{(p+1)*}$ denotes the $(p-1) \times (p-1)$ zero matrix.

From the definitions we see immediately that

(3.8) $\det M^*$ *is independent of the choice of diagonal elements* $a^{(i)}_{r_i,r_i}$.

More precisely,

(3.9) *The* s*-equivalence class of* M^* *is independent of the choice of* $a^{(i)}_{r_i,r_i}$.

Moreover, we have the following

THEOREM 3.1. *The* s*-equivalence class of* M^* *is an invariant of link types, and so is the* S*-equivalence class of* $M^* + M^{*t}$.

This theorem can be proved as follows. If two link diagrams represent two links of the same type, then one is deformed into the other by finite sequences of certain deformations $\Omega_i^{\pm 1}$ $(i = 1,2,3)$ defined in [16, p. 7]. Consequently to prove the theorem, it is sufficient to show that no deformation $\Omega_i^{\pm 1}$ alters the s-equivalence class of the matrix. Proof can be given in various stages. Since the treatment on each stage is an elementary matter, we omit details.

Thus since the s-equivalence class of M^* depends only on the link type, we shall call it the *L-principal minor of* l. From Definition 2.2 and (2.2) we see

(3.10) *The signature* $\sigma(M^* + M^{*t})$ *is an invariant of the link type of* l *which will be called the signature of* l *and denoted by* $\sigma(l)$.

(3.11) *The nullity*(6), $n(M^* + M^{*t})$, *of* $M^* + M^{*t}$ *is also an invariant of the link type of* l; $n(M^* + M^{*t}) + 1$ *will be called the nullity of* l *and denoted by* $n(l)$.

(6) By the *nullity* $n(A)$ of a matrix A is meant the number of columns minus the rank of A.

4. **Alexander matrix.** Let G be the *group of a link* l of multiplicity μ, i.e. $G = \pi_1(S^3 - l)$. G is a quotient group of a free group X of finite rank by a normal subgroup. The associated homomorphism of X onto G is denoted by $\phi: X \to G$. The natural homomorphism from G onto its commutator quotient group G/G' is denoted by ψ. G/G' is a free abelian group of rank μ generated by $t_1, \cdots, t_\mu$, which are specified as follows. t_i is the element of G (mod G') which is represented by an oriented loop w in $S^3 - l$ such that the linking number of w with the jth component of l is equal to $+1$ or 0 according as $j = i$ or not. Further, the natural homomorphism from G/G' onto an infinite cyclic group Z generated by t is denoted by v, i.e. $v(t_i) = t$ for $i = 1, 2, \cdots, \mu$. These three homomorphisms ϕ, ψ and v are uniquely extended to ring homomorphisms between the integral group rings JX, JG, $J(G/G')$ and JZ. These ring homomorphisms will be denoted by the same letters.

In the preceding section, we knew that the S-equivalence class of an integral symmetric matrix $M^* + M^{*t}$ is an invariant of a link type. Moreover, the equivalence class of a matrix $M - tM^t$ over JZ, in the sense of Fox [3, II, p. 199], is an invariant of link type. This follows immediately from the proposition:

(4.1) *If two integral matrices M and N are* s*-equivalent, then $M - tM^t$ and $N - tN^t$ are equivalent over JZ.*

This result follows almost at once from the definition of equivalence and s-equivalence.

THEOREM 4.1. *Let l be a link. Then there exists a link l' isotopic to l such that the group G of l' possesses a presentation $G = (x_1, \cdots, x_n : r_1, \cdots, r_n)$ satisfying*

$$\left\| \theta\left(\frac{\partial r_i}{\partial x_j}\right) \right\| = M - tM^t, \tag{4.2}$$

where M denotes the matrix of l' with respect to its diagram L' and $\theta = v\psi\phi$.

Proof. It is not difficult to show that any link can be deformed isotopically to a link l' that has a connected and special diagram L'.

Now as is shown in (3.3) we can divide all regions of S^2 that are divided by L', into two classes, α and β. Let n and m be the number of α- and β-regions. Then we associate the generators x_i and y_j of a free group X of rank $n + m$ to α-regions and β-regions respectively. Let $c_1, \cdots, c_q$ be double points of L' on the boundary of a β-region B_j. Then $d_{B_j}(c_i)$ depends only on B_j. Thus we can classify all β-regions into two classes $\mathfrak{B}_0$ and $\mathfrak{B}_1$ as follows:

(4.3) *If $d_{B_j}(c_i) = 0$ then B_j belongs to $\mathfrak{B}_0$. Otherwise B_j belongs to $\mathfrak{B}_1$.*

Now take a β-region, B_1 say, belonging to $\mathfrak{B}_1$, and fix it. Next to every double point c of L', we associate an element $w(c)$ of X as follows: Suppose that the four regions g_i, g_j, g_k and g_l meet at c in such a way that we pass through these regions in the cyclic order just named as we go around the point c counterclock-

wise. Among these regions, just two nonconsecutive regions, g_i and g_k say, are β-regions. Suppose g_i belongs to $\mathfrak{B}_0$. Then $w(c)$ is defined as $y_i x_j^{-1} y_k x_l^{-1}$ or $y_i x_l^{-1} y_k x_j^{-1}$ according as $\eta(c) = 1$ or -1. Since L has $n + m - 2$ double points, we have $n + m - 2$ elements $w_1, \cdots, w_{n+m-2}$ of X. Then G has a presentation

$$G = (x_1, \cdots, x_n, y_1, \cdots, y_m: \ w_1, \cdots, w_{n+m-2}, \ y_1). \tag{4.4}$$

This is known as the *Dehn presentation* of the group of a link. It is easy to show [12] that

$$\theta(x_i) = t, \text{ for } i = 1, \cdots, n, \text{ and} \tag{4.5}$$

$$\theta(y_j) = t^2 \text{ or } 1 \text{ according as } B_j \text{ belongs to } \mathfrak{B}_0 \text{ or not.}$$

The next step is to eliminate all y_j. Let H be the *graph*(7) of L and let H^* be the *dual graph*. Take a maximal tree T in H and fix it. Each vertex of T is the center b_j of B_j and each edge contains one and only one double point of L'. For any vertex b_j, there is a unique path P_{1j} from b_1, the center of B_1, to b_j. By means of this path P_{1j}, a given y_j can be written in the form: $y_j = u_j y_1 u_j'$, where u_j is an element of the subgroup X_1 of X generated by $x_1, \cdots, x_n$. Since $y_1 = 1$ in G, we can eliminate all y_j and m relators from (4.4), and we obtain the following presentation:

$$G = (x_1, \cdots, x_n: \ R_1, \cdots, R_{n-1}), \tag{4.6}$$

where the R_i are nontrivial relators that are obtained from some w_k by replacing y_j by $u_j u_j'$.

Now consider a maximal tree of H^* which is disjoint to T. Such a tree exists and it is determined uniquely. We denote it by T^*. T^* is a *dual tree* of T. Vertices of T^* are centers of α-regions. Since R_i corresponds to a double point, c_i say, that is not contained in T, and since these double points are contained in T^*, we see each R_i one-one corresponds to each edge, f_i say, of T^*. Let $f_1, \cdots, f_p$ be edges incident to a vertex v_k of T^*. Since f_j contains only one double point, c_j say, we can assign a relator, R_j say, to f_j. Let $\mathfrak{S}_i$ be the set of all elements of the free group X_1 of the form: $R_{i_1}^{\delta_1} R_{i_2}^{\delta_2} \cdots R_{i_p}^{\delta_p}$, where $(i_1 \ i_2 \cdots i_p)$ is a permutation of $(1\,2 \cdots p)$ and $\delta_j = \varepsilon_{w_k}(c_j)\eta(c_j)$, W_k denoting the α-region whose center is v_k. Select an element S_i from each $\mathfrak{S}_i$ as its representative. Then $\{S_1, \cdots, S_n\}$ constitutes a complete system of defining relations of G, for T^* is a maximal tree. In other words, we obtain the following presentation of G:

$$G = (x_1, \cdots, x_n: \ S_1, \cdots, S_n). \tag{4.7}$$

It is not so difficult to show from (4.5) that (4.7) is a presentation of the required type. q.e.d.

(7) Take a point in each α- or β-region and fix it. It is called the *center* of a region. Then by the *graph* (or the *dual graph*) of L is meant the 1-complex obtained by connecting the centers of α-regions (or of β-regions) with the double points lying on their boundaries by simple arcs.

From this theorem, we see

(4.8) *The nullity of the reduced Alexander matrix*(8) *at* $t=-1$ *is equal to the nullity of its link.*

Moreover, denoting the reduced Alexander polynomial of l', hence of l, by $\Delta(t)$, we have from Theorem 4.1,

$$\varepsilon\Delta(t) = t^{\lambda}\det(M^* - tM^{*t}), \tag{4.9}$$

where λ *is an integer so that the least degree of a term of* $\Delta(t)$ *is zero, and* $\varepsilon = \pm 1$.

In particular, if L is a diagram of an alternating link, then we can prove that $\lambda = 0$ [13]. That is to say,

$$\varepsilon\Delta(t) = \det(M^* - tM^{*t}), \text{ or, } \textit{equivalently}, \det M^* \neq 0. \tag{4.10}$$

In the following to fix $\Delta(t)$, we may assume that $\Delta(1) \geqq 0$. Therefore, since $\det(M^* - M^{*t}) \geqq 0$, always we may assume in (4.9) and (4.10) that $\varepsilon = +1$.

5. **Signature of alternating links.** A link l is said to be *splittable* if there is a 2-sphere $S^2 \subset S^3 - l$ such that both components of $S^3 - S^2$ contain points of l. More precisely, we say that l is *split* into n components $l_1, \cdots, l_n$, or n links $l_1, \cdots, l_n$ are *separated* from one another, if there are $n-1$ disjoint 2-spheres $S_1, \cdots, S_{n-1} \subset S^3 - l$ such that each component of $S^3 - \bigcup_{i=1}^{n-1} S_i$ contains one component l_i of l. l is denoted, then, by $l = l_1 \circ \cdots \circ l_n$. Similarly, a diagram L of l is said to be *separated* if L consists of n, say, disjoint 1-complexes in S^2.

A link to be considered in this section is assumed to be *nonsplittable* unless otherwise mentioned.

LEMMA 5.1. *If* l *is an alternating link, then* $\Delta(-1) \neq 0$ *and* $\operatorname{sign}\Delta(0) = \operatorname{sign}\Delta(-1)$.

Proof. Since l is an alternating link, $\Delta(t) \neq 0$ [1]. Thus we can write $\Delta(t) = c_0 + c_1 t + \cdots + c_d t^d$, $d \geqq 1$, where $(-1)^{i+j} c_i c_j \geqq 0$, for any i, j [1], [13]. Since $c_0 \neq 0$, we have $c_0\Delta(-1) = c_0 \sum_{i=0}^{d}(-1)^i c_i = c_0^2 + \sum_{i=1}^{d}(-1)^i c_0 c_i \geqq c_0^2 > 0$. Therefore we see that $\Delta(-1) \neq 0$ and $\operatorname{sign} c_0 = \operatorname{sign}\Delta(-1)$. q.e.d.

LEMMA 5.2. *If* l *is a special alternating link, then* $|\sigma(l)|$ *is equal to the degree of* $\Delta(t)$.

Proof. Let L be a diagram of l, which is special alternating. Then the $n \times n$

(8) For any presentation $G = (x_1, \cdots, x_n;\ r_1, \cdots, r_m)$ of the group of a link l, its $n \times m$ Jacobian matrix at $\psi\phi$, $A^{\phi\psi} = \|\psi\phi(\partial r_i/\partial x_j)\|$ is called the *Alexander matrix* of l [3, II]. By the dth *elementary ideal*, $\varepsilon_d(A^{\phi\psi})$ $(d \geqq 1)$, of $A^{\phi\psi}$ is meant the ideal generated by the minor determinants of A of order $n-d$. Since ε_d is an invariant of a link type l, it may be called the elementary ideal of l. In particular, $\varepsilon_d(A^{\theta})$ will be called the dth *reduced* elementary ideal, denoted by $\tilde{\varepsilon}_d(l)$. $\tilde{\varepsilon}_1(l)$ is a principal ideal and its generator $\Delta(t)$ is called the *reduced Alexander polynomial* [1].

L-principal minor M^* of the matrix M of l with respect to L has the following properties [13]:

(5.1) (1) No elements on the diagonal of M^* are zero and they are of the same sign ε.

(2) All elements except those in the diagonal of M^* are of the same sign $-\varepsilon$ or 0.

(3) The sign of the sum of all elements of each row and column is ε or 0.

(4) For any m, $1 \leqq m \leqq n$, there exists the number q such that the sign of the sum of the first m elements on qth row and qth column in M^* is ε.

Then we can show that $\det M^* > 0$ [13]. Since $\varepsilon M^* + \varepsilon M^{*t}$ and all its principal minors have the properties (5.1)(1)–(4), we see that $\det[(\varepsilon M^* + \varepsilon M^{*t})(12\cdots i)] > 0$, $1 \leqq i \leqq n$. Thus to calculate the signature σ of the matrix $\varepsilon M^* + \varepsilon M^{*t}$, we can select a σ-series of $\varepsilon M^* + \varepsilon M^{*t}$ as $\Delta_i = (\varepsilon M^* + \varepsilon M^{*t})(12\cdots i), 1 \leqq i \leqq n$. Therefore, $|\sigma(l)| = |\sigma(\varepsilon M^* + \varepsilon M^{*t})| = |\sum_{i=0}^{n-1} \operatorname{sign}(\det \Delta_i \cdot \det \Delta_{i+1})| = n$. Since n is the rank of M^*, i.e., the degree of $\Delta(t)$, Lemma 5.2 is proved.

Lemma 5.2 is extended to any alternating link as follows.

THEOREM 5.3. *Let* $l = l_1 * l_2 * \cdots * l_{m+1}$. *Then*

$$\sigma(l) = \sum_{i=1}^{m+1} \sum_{p=1}^{w_i - 1} \operatorname{sign} a_{p,p}^{(i)}.$$

This formula simplifies the calculation of $\sigma(l)$ for alternating links.

Proof. Let W_i and W_j be two different α-regions in a Seifert domain D_k. W_i and W_j are said to be *connectible* if there are two points P_i and P_j on $\dot{W}_i$ and $\dot{W}_j$ respectively and if these two points can be joined by a simple polygonal arc α_{ij} such that

(5.2) α_{ij} is contained in a β-region B except for P_i and P_j, i.e., α_{ij} does not intersect L except at P_i and P_j.

α_{ij} will be called a *joining arc* between the connectible regions W_i and W_j. Now let us replace α_{ij} by a sufficiently narrow (untwisted) band β such that two arcs, β_1 and β_3 say, among the four arcs β_1, β_2, β_3 and β_4 contained in $\dot{\beta}$, are in W_i and W_j respectively, and that β_2 and β_4 are in B. Since B is a β-region, β can be oriented in such a way that the orientation induced on $\dot{\beta}$ from β is consistent with that of β_1 and β_3. Thus we obtain an oriented link $l' = (l - (\beta_1 \cup \beta_3)) \cup (\beta_2 \cup \beta_4)$. The link type of l' is independent of the choice of α_{ij} satisfying (5.2). l' will be called the link obtained by *connecting* W_i and W_j. Since a diagram L' of l' is of the form $L' = (L - (\beta_1' \cup \beta_3')) \cup (\beta_2' \cup \beta_4')$, $p(\beta_i) = \beta_i'$, the number of the α-regions in $S^2 - L'$ is one less than in $S^2 - L$. And the L-principal minor N^* of l' is given as follows.

(5.3) If M^* is the L-principal minor of l of type $(q_1, \cdots, q_{m+1})$, then $N^* = \tilde{M}(q_1 \cdots q_{k-1} q_{k_i} q_{k_j} q_{k+1} \cdots q_{m+1})$, where q_{k_i} and q_{k_j} are the order of row and column corresponding to W_i and W_j in M.

We are now in position to prove the theorem. We may assume without loss of generality that the α-regions $W_1, \cdots, W_{w_i}$ in D_i are ordered in such a way that

(5.4) W_{j+1} is connectible with at least one of $W_1, \cdots, W_i$ for $j = 1, \cdots, w_i - 1$.

It is clear that a diagram of the link obtained from an alternating link by connecting two α-regions is also alternating. Let us define $f(r) = \sum_{i=1}^{r}(w_i - 1)$, $r = 1, \cdots, m+1$ and $f(0) = 0$. Set $F_s = M^*(f(m+1) - s + 1, \cdots, f(m+1))$ for $s = 1, 2, \cdots, f(m+1)$. For any s, $1 \leqq s \leqq f(m+1)$, there exists a unique integer r such that $f(r) < f(m+1) - s + 1 \leqq f(r+1)$, $0 \leqq r \leqq m+1$. Then $\det(F_s - tF_s^t)$ will be considered as the reduced Alexander polynomial, denoted by $\Delta_s(t)$, of the alternating link $l_{(s)}$. $l_{(s)}$ is the link whose diagram $L_{(s)}$ is obtained from L by considering each of r Seifert domains $D_1, \cdots, D_r$ as one β-region and by connecting [9] $f(m+1) - s - f(r) + 1$ connectible α-regions $W_1, \cdots, W_{f(m+1)-s-f(r)+1}$. Therefore from (4.10) and Corollary 1.40 in [13], we have

$$\begin{aligned} \operatorname{sign}(\det F_s) &= \operatorname{sign}\Delta_{(s)}(0) \\ &= \operatorname{sign}\left[\det \tilde{M}_{r+1,r+1}(1\,2 \cdots \eta\, w_{r+1}) \prod_{u=r+2}^{m+1} \det \tilde{M}_{uu}(w_u)\right], \end{aligned} \tag{5.5}$$

where $\eta = f(m+1) - s - f(r) + 1$.

Since the sign of any diagonal element in M_{ii} is constant and since it depends only on i, we can denote it by ε_i. We have then

$$\begin{aligned} &\varepsilon_{r+1}^{w_{r+1}-\eta-1} \cdots \varepsilon_{m+1}^{w_{m+1}-1}\ \Delta_{(s)}(0) \\ &= \det[\varepsilon_{r+1}\tilde{M}_{r+1,r+1}(1\,2 \cdots \eta\, w_{r+1})] \prod_{u=r+2}^{m+1} \det[\varepsilon_u \tilde{M}_{uu}(w_u)]. \end{aligned} \tag{5.6}$$

Since each factor in the right-hand side in (5.6) is positive (cf. Lemma 5.2), it follows

$$\begin{aligned} \operatorname{sign}(\det F_s) &= \operatorname{sign}\Delta_{(s)}(0) \\ &= \varepsilon_{r+1}^{w_{r+1}-\eta-1} \cdots \varepsilon_{m+1}^{w_{m+1}-1} \\ &= \operatorname{sign}(a_{\eta+1,\eta+1}^{(r+1)} \cdots a_{w_{r+1}-1,w_{r+1}-1}^{(r+1)} a_{11}^{(r+2)} \cdots a_{w_{m+1}-1,w_{m+1}-1}^{(m+1)}). \end{aligned} \tag{5.7}$$

Since for any s, $1 \leqq s \leqq f(m+1)$, $\det(F_s + F_s^t) \neq 0$ from Lemma 5.1, we can select a series of principal minors $F_s + F_s^t$ of $M^* + M^{*t}$ as its σ-series. Then from (2.4) and (5.7), it follows

---

[9] It is clear that the link type of a link obtained by connecting some number of connectible α-regions is independent of the order of connection.

$$\sigma(M^* + M^{*t}) = \sum_{s=1}^{f(m+1)-1} \operatorname{sign}[(\det(F_s + F_s^t)\cdot\det(F_{s+1} + F_{s+1}^t)]$$

$$= \sum_s \operatorname{sign}(\det F_s \cdot \det F_{s+1})$$

$$= \sum_s \operatorname{sign}(a_{\eta+1,\eta+1}^{(r+1)}\cdots)(a_{\eta+2,\eta+2}^{(r+1)}\cdots)$$

$$= \sum_s \operatorname{sign} a_{\eta+1,\eta+1}^{(r+1)}. \qquad \text{q.e.d.}$$

REMARK. Theorem 5.3 is valid for a splittable alternating link.

From Lemma 5.2 and Theorem 5.3, it follows

THEOREM 5.4. *Let l be an alternating link and let $l = l_1 * \cdots * l_{m+1}$. Then*

$$\sigma(l) = \sigma(l_1) + \cdots + \sigma(l_{m+1}).$$

This theorem is not necessarily true for nonalternating links.

Now let l' be the reflected inverse, the so-called *mirror imaged link*, of a link l, and let M and M' be the matrices of l and l'. Then it is clear from the definition that $M'^* + M'^{*t}$ is S-equivalent to $-(M^* + M^{*t})$, and hence $\sigma(l') = -\sigma(l)$. Thus we have

(5.8) *If l is amphicheiral, i.e. if l and l' are of the same type, then $\sigma(l) = 0$.*

From Lemma 5.2 and (5.8), it follows

THEOREM 5.5. *If l is a nonsplittable special alternating link, then l is not amphicheiral unless l is a trivial knot.*

This was a conjecture of the author [15].

Finally we prove the following

THEOREM 5.6. *Let k be any knot. If $|\sigma(k)| \equiv 2m \pmod 4$ ([10]), then $|\Delta(-1)| \equiv (-1)^m \pmod 4$.*

Proof. As we noted at the end of §4, the Alexander polynomial $\Delta(t)$ of k is given by

$$\Delta(t) = t^{\lambda}\det(M^* - tM^{*t}),$$

where λ is an integer and $\Delta(0) \neq 0$. Therefore $|\Delta(-1)| = |\det(M^* + M^{*t})|$. Let $\det(M^* - tM^{*t}) = c_0 + c_1 t + \cdots + c_d t^d$, where d is even, $c_i = c_{d-i}$ $(0 \leqq i \leqq d)$ and c_0 may be zero. Since $|\sigma(M^* + M^{*t})| \equiv 2m \pmod 4$, from the definition there is a unimodular matrix R of rational numbers such that

([10]) For any knot k, $\sigma(k)$ is always even.

$$(5.9)\qquad R(M^* + M^{*t})R^t = \begin{bmatrix} a_1 & & & & & & & \\ & \ddots & & & & & & \\ & & a_n & & & & 0 & \\ & & & -a_1' & & & & \\ & & & & \ddots & & & \\ & & & & & -a_n' & & \\ & & 0 & & & & \varepsilon b_1 & \\ & & & & & & & \ddots \\ & & & & & & & & \varepsilon b_{4q+2m} \end{bmatrix},$$

where a_i, a_j' and $b_k > 0$([11]), $\varepsilon = \pm 1$ and $d = 2n + 4q + 2m$. Thus we see that $\det(M^* + M^{*t}) = (-1)^n \varepsilon^{4q+2m} a_1 \cdots a_n a_1' \cdots a_n' b_1 \cdots b_{4q+2m} = (-1)^n a_1 \cdots b_{4q+2m}$. Therefore we see

$$(5.10)\qquad \operatorname{sign}[\det(M^* + M^{*t})] = (-1)^n.$$

On the other hand, since $\det(M^* + M^{*t}) = \sum_{j=0}^{d}(-1)^j c_j$ and $\det(M^* - M^{*t}) = \sum_{j=0}^{d} c_j = 1$, it follows that

$$\begin{aligned} \det(M^* + M^{*t}) &= 2\sum_{j=0}^{(d/2)-1}(-1)^j c_j + (-1)^{d/2} c_{n+2q+m-1} \\ &= 2\sum_{j=0}^{(d/2)-1}(-1)^j c_j + (-1)^{n+m}\left[1 - 2\sum_{j=0}^{(d/2)-1} c_j\right] \qquad (5.11) \\ &\equiv (-1)^{n+m} \pmod 4. \end{aligned}$$

Therefore we obtain from (5.10) and (5.11),

$$\begin{aligned} |\Delta(-1)| &= |\det(M^* + M^{*t})| \\ &= \operatorname{sign}[\det(M^* + M^{*t})] \cdot \det(M^* + M^{*t}) \\ &\equiv (-1)^n (-1)^{n+m} \pmod 4 \\ &\equiv (-1)^m \pmod 4. \qquad \text{q.e.d.} \end{aligned}$$

This theorem implies that if $|\sigma(k)| \equiv 2 \pmod 4$ then the square free part of $|\Delta(-1)|$ must contain a prime number p of the form $4s + 3$. Therefore, in this case the nonamphicheirality of a knot is shown by means of the Minkowski unit C_p of the quadratic form [7], [16]. However if $\sigma(k) \equiv 0 \pmod 4$, the Minkowski unit may be powerless. The knots 9_{36} and 9_{43} are in this case ([12]) [16].

([11]) For, $\det(M^* + M^{*t}) \neq 0$.

([12]) The signatures of these knots are ± 4.

6. **Nullity of links of some kind.** Let l be a splittable link: $l = l_1 \circ l_2$, l_i being links. Take a small arc α_i on each l_i and join α_1 and α_2 in S^3 by a band β that does not intersect l except at α_1 and α_2. We can give a direction to β so that $(l_1 - \alpha_1) \cup (l_2 - \alpha_2) \cup (\beta - \alpha_1 - \alpha_2) = l'$ is an oriented link. l' will be called the link obtained by *joining* l_1, l_2 at α_1, α_2 by a band β and denoted by $l' = l_1 \oplus l_2$. Although the type of $l_1 \oplus l_2$ is not determined by that of each l_i, we shall see later that some invariants are determined by that of each l_i. It is clear that

(6.1) *The multiplicity of $l_1 \oplus l_2$ is less than that of $l_1 \circ l_2$ by one.*

The product $l_1 \# l_2$ of l_1 and l_2 in the sense of [8] and [17] is a kind of $l_1 \oplus l_2$. This *joining operation* will be extended to the multiple case as follows. Let $l = l_1 \circ l_2 \circ \cdots \circ l_n$, $n \geqq 2$. Then $l_1 \oplus l_2 \oplus \cdots \oplus l_n$ is a link obtained from l by joining consecutively l_i and l_{i+1} $(i = 1, \cdots, n-1)$, at α_i and α'_{i+1} by a band β_i, where α_i and α'_i are disjoint arcs on l_i, and β_i meets l only in α_i and α'_{i+1}. In this section a link of this kind will chiefly be considered.

First we shall prove the following

LEMMA 6.1. *The nullity of a link l is not greater than its multiplicity μ.*

Proof. It is well known [18], [21] that l can be spanned by a connected orientable surface F in S^3. Let the genus of F be h. Then the reduced Alexander matrix $A(t)$ of l is equivalent to a $(2h + \mu - 1) \times (2h + \mu)$ matrix [21]

$$\begin{pmatrix} A_{11}(t) & A_{12}(t) & 0 \\ A_{21}(t) & A_{22}(t) & 0 \end{pmatrix},$$

where $A_{11}(t)$ is a $2h \times 2h$ matrix. Since $\det A_{11}(1) = 1$ [21], $\det A_{11}(-1) \neq 0$, which implies rank $A(-1) \geqq 2h$. Thus we see $n(l) = n(A(-1)) \leqq 2h + \mu - 2h = \mu$. q.e.d.

The following lemma is fundamental in the following sections.

LEMMA 6.2. *Let l be a link consisting of μ links $l_1, \cdots, l_\mu$ of the form: $l_j = l_{j,0} \oplus l_{j,1} \oplus \cdots \oplus l_{j,m_j}$, where $l_{j,0}$ and $l_{j,k}$ $(k>0)$ denote links and trivial knots respectively, and these $l_{j,k}$ $(k>0)$ are separated from one another (m_j may be zero). Then*

(1) $n(l) = n(l_{1,0} \cup \cdots \cup l_{\mu,0})$.

(2) *If $n(l) = d$, the dth reduced elementary ideal $\tilde{\varepsilon}_d(l)$ of l is of the form: $\tilde{\varepsilon}_d(l) = \tilde{\varepsilon}_d(l_{1,0} \cup \cdots \cup l_{\mu,0}) f(t) f(t^{-1})$, where $f(t)$ denotes an integral polynomial on t and $f(1) = \pm 1$.*

Proof. Let G be the *group* of l, i.e. $G = \pi_1(S^3 - l)$. We shall first give a presentation of G. We may assume without loss of generality that the diagrams $L_{j,k}$ $(k>0)$ of $l_{j,k}$ are circles in S^2 and that the diagrams $L_{p,q}$ of $l_{p,q}$ are separated from one another in such a way that one of two components into which S^2 is divided by $L_{j,k}$ $(k > 0)$ contains no diagrams $L_{p,q}$.

Now starting from the well-known *Wirtinger* presentation of G, after applying Tietze transformations, we have finally the following presentation (cf. [21]).

$$
\begin{aligned}
G = (&a_{i,j,k},\ 1 \leqq i \leqq \mu,\ 0 \leqq j \leqq m_i,\ 1 \leqq k \leqq \lambda_{i,j},\\
&B_{i,j,k},\ 1 \leqq i \leqq \mu,\ 0 \leqq j \leqq m_i - 1,\ 1 \leqq k \leqq \nu_{i,j}:\\
&R_{i,0,k},\ 1 \leqq i \leqq \mu,\ 1 \leqq k \leqq \lambda_{i,0},\\
&S_{i,j,k},\ 1 \leqq i \leqq \mu,\ 1 \leqq j \leqq m_i,\ 1 \leqq k \leqq \lambda_{i,j},\\
&T_{i,j,k},\ 1 \leqq i \leqq \mu,\ 0 \leqq j \leqq m_i - 1,\ 1 \leqq k \leqq \nu_{i,j} - 1,\\
&U_{i,j},\ 1 \leqq i \leqq \mu,\ 0 \leqq j \leqq m_i - 1)
\end{aligned}
\tag{6.2}
$$

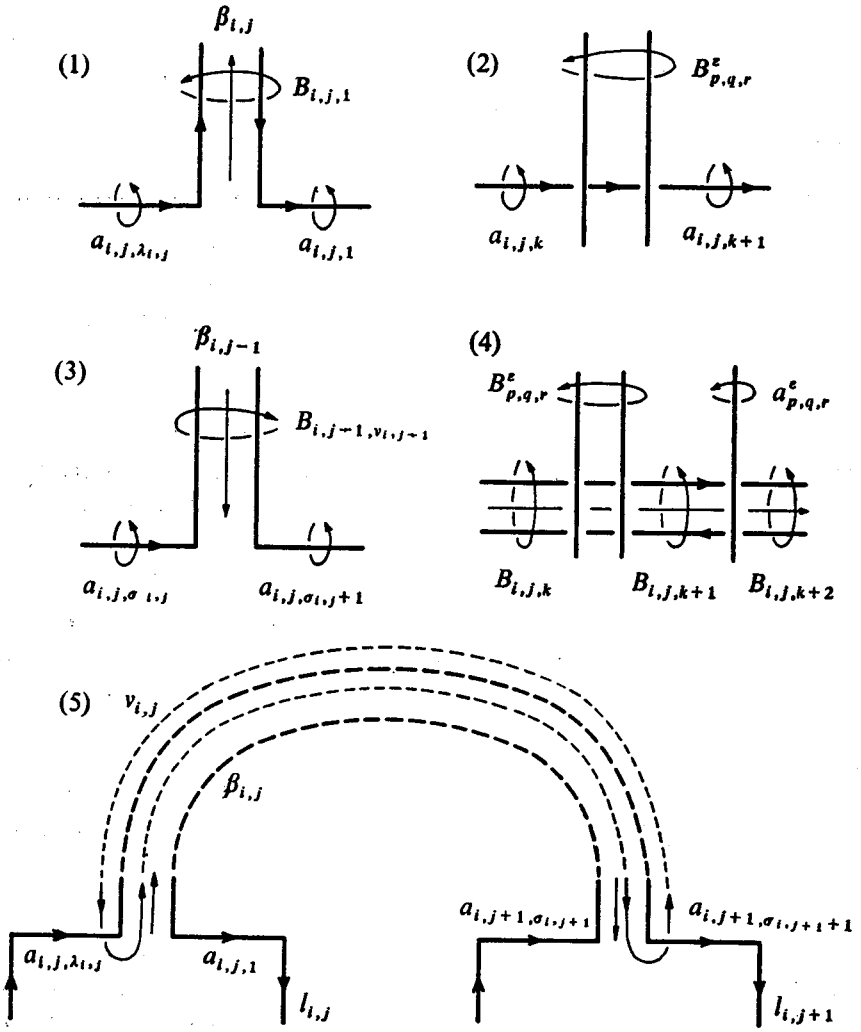

FIGURE 1

(1) $a_{i,j,k}$ are elements of G represented by loops going once around arcs of $l_{i,j}$ in the positive direction.

(2) $B_{i,j,k}$ are elements of G represented by loops going once around bands $\beta_{i,j}$ positively([13]) that connects $l_{i,j}$ with $l_{i,j+1}$. See Figure 1.

(3) Relators are of the forms ($\varepsilon = \pm 1$):

(i) For $k < \lambda_{i,0}$,

$$R_{i,0,k} = a_{i,0,k}X^{\varepsilon}_{p,q,r}a^{-1}_{i,0,k+1}X^{-\varepsilon}_{p,q,r},$$

and

$$R_{i,0,\lambda_{i,0}} = a_{i,0,\lambda_{i,0}}a^{-1}_{i,0,1}B^{-1}_{i,0,1},$$

where $X_{p,q,r} = a_{p,0,r}$ or $B_{p,q,r}$. See Figure 1(1)(2).

(ii) For $k \neq \sigma_{i,j}$, $\lambda_{i,j}$,

$$S_{i,j,k} = a_{i,j,k}B^{\varepsilon}_{p,q,r}a^{-1}_{i,j,k+1}B^{-\varepsilon}_{p,q,r},$$

and

$$S_{i,j,\sigma_{i,j}} = a_{i,j,\sigma_{i,j}}a^{-1}_{i,j,\sigma_{i,j}+1}B_{i,j-1,\nu_{i,j-1}},$$

$$S_{i,j,\lambda_{i,j}} = a_{i,j,\lambda_{i,j}}a^{-1}_{i,j,1}B^{-1}_{i,j,1}.$$

See Figure 1(1)–(3).

(iii) $$T_{i,j,k} = B_{i,j,k}X^{\varepsilon}_{p,q,r}B^{-1}_{i,j,k+1}X^{-\varepsilon}_{p,q,r},$$

where $X_{p,q,r} = a_{p,q,r}$ or $B_{p,q,r}$. See Figure 1(4).

(iv) $$U_{i,j} = u_{i,j}a_{i,j,\lambda_{i,j}}u^{-1}_{i,j}a^{-1}_{i,j+1,\sigma_{i,j+1}+1},$$

where $U_{i,j}$ are relators corresponding to loops $v_{i,j}$ in Figure 1(5).

Let $A(t)$ be the reduced Alexander matrix corresponding to the presentation (6.2) of the group G of link l [3, II].

Now we shall study the contribution made to $A(t)$ by the various type of relators. From now on, all the derivatives will be considered to be evaluated in $JZ(t)$, and only the nonzero contributions will be described [3, I]. Remark that $\theta(a_{i,j,k}) = t$ and $\theta(B_{i,j,k}) = 1$.

(6.3) (1) For $k \neq \lambda_{i,0}$,

$$\frac{\partial R_{i,0,k}}{\partial a_{i,0,k}} = 1, \qquad \frac{\partial R_{i,0,k}}{\partial a_{i,0,k+1}} = -t^{\delta},$$

$\delta = 0$ or ± 1 according as $X_{p,q,r} = B_{p,q,r}$ or $a_{p,q,r}$.

$$\frac{\partial R_{i,0,k}}{\partial X_{p,q,r}} = \varepsilon(t-1) \text{ or } t^{\delta}-1$$

according as $\delta = 0$ or not.

([13]) $\beta_{i,j}$ is oriented in the direction from $l_{i,j}$ to $l_{i,j+1}$.

For $k=\lambda_{i,0}$,

$$\frac{\partial R_{i,0,k}}{\partial a_{i,0,k}}=1,\quad \frac{\partial R_{i,0,k}}{\partial a_{i,0,1}}=-1,\quad \frac{\partial R_{i,0,k}}{\partial B_{i,0,1}}=-1.$$

(2)

$$\frac{\partial S_{i,j,k}}{\partial a_{i,j,k}}=1,\quad \frac{\partial S_{i,j,k}}{\partial a_{i,j,k+1}}=-1.$$

For $k\neq\sigma_{i,j},\ \lambda_{i,j}$,

$$\frac{\partial S_{i,j,k}}{\partial B_{p,q,r}}=\varepsilon(t-1),$$

(3)

$$\frac{\partial S_{i,j,\sigma_{i,j}}}{\partial B_{i,j-1,\nu_{i,j-1}}}=1,\qquad \frac{\partial S_{i,j,\lambda_{i,j}}}{\partial B_{i,j,1}}=-1.$$

$$\frac{\partial T_{i,j,k}}{\partial B_{i,j,k}}=t^{\delta},$$

$\delta=0$ or ± 1 according as $X_{p,q,r}=B_{p,q,r}$ or $a_{p,q,r}$.

$$\frac{\partial T_{i,j,k}}{\partial B_{i,j,k+1}}=-1.$$

(4)

$$\frac{\partial U_{i,j}}{\partial a_{i,j\,\lambda_{i,j}}}=\theta(u_{i,j})+(1-t)\frac{\partial u_{i,j}}{\partial a_{i,j,\lambda_{i,j}}}.$$

$$\frac{\partial U_{i,j}}{\partial a_{i,j+1,\sigma_{i,j+1}+1}}=-1+(1-t)\frac{\partial u_{i,j}}{\partial a_{i,j+1,\sigma_{i,j+1}+1}}.$$

If $x\neq a_{i,j,\lambda_{i,j}}$ or $a_{i,j+1,\sigma_{i,j+1}+1}$, then

$$\frac{\partial U_{i,j}}{\partial x}=(1-t)\frac{\partial u_{i,j}}{\partial x}.$$

For convenience sake, we introduce the following notations. For $1\leqq i,\alpha\leqq\mu$,

$$R_a(i,0:\alpha,\beta)=\left\|\frac{\partial R_{i,0,k}}{\partial a_{\alpha,\beta,\gamma}}\right\|_{1\leqq k\leqq\lambda_{i,0},\ 1\leqq\gamma\leqq\lambda_{\alpha,\beta}},$$

for $0\leqq\beta\leqq m_\alpha$.

$$R_B(i,0:\alpha,\beta)=\left\|\frac{\partial R_{i,0,k}}{\partial B_{\alpha,\beta,\gamma}}\right\|_{1\leqq k\leqq\lambda_{i,0},\ 1\leqq\gamma\leqq\nu_{\alpha,\beta}},$$

for $0\leqq\beta\leqq m_\alpha-1$.

$$S_a(i,j:\alpha,\beta)=\left\|\frac{\partial S_{i,j,k}}{\partial a_{\alpha,\beta,\gamma}}\right\|_{1\leqq k\leqq\lambda_{i,j},\ 1\leqq\gamma\leqq\lambda_{\alpha,\beta}},$$

for $1\leqq j\leqq m_i,\ 0\leqq\beta\leqq m_\alpha$.

$$S_B(i,j:\alpha,\beta) = \left\| \frac{\partial S_{i,j,k}}{\partial B_{\alpha,\beta,\gamma}} \right\|_{1\leqq k\leqq \lambda_{i,j},\,1\leqq\gamma\leqq\nu_{\alpha,\beta}},$$

for $1\leqq j\leqq m_i,\ 0\leqq\beta\leqq m_\alpha-1$.

$$T_a(i,j:\alpha,\beta) = \left\| \frac{\partial T_{i,j,k}}{\partial a_{\alpha,\beta,\gamma}} \right\|_{1\leqq k\leqq \nu_{i,j}-1,\,1\leqq\gamma\leqq\lambda_{\alpha,\beta}},$$

for $0\leqq j\leqq m_i-1,\ 0\leqq\beta\leqq m_\alpha$.

$$T_B(i,j:\alpha,\beta) = \left\| \frac{\partial T_{i,j,k}}{\partial B_{\alpha,\beta,\gamma}} \right\|_{1\leqq k\leqq \nu_{i,j}-1,\,1\leqq\gamma\leqq\nu_{\alpha,\beta}},$$

for $1\leqq j\leqq m_i-1,\ 0\leqq\beta\leqq m_\alpha-1$.

$$U_a(i,0:\alpha,\beta) = \left\| \frac{\partial U_{i,j}}{\partial a_{\alpha,\beta,\gamma}} \right\|_{1\leqq j\leqq m_i-1,\,1\leqq\gamma\leqq\lambda_{\alpha,\beta}},$$

for $1\leqq\beta\leqq m_\alpha$.

$$U_B(i,0:\alpha,\beta) = \left\| \frac{\partial U_{i,j}}{\partial B_{\alpha,\beta,\gamma}} \right\|_{1\leqq j\leqq m_i-1,\,1\leqq\gamma\leqq\nu_{\alpha,\beta}},$$

for $1\leqq\beta\leqq m_\alpha-1$.

Now, from (6.3) we see easily that

(6.4) (1) $R_a(i,0:\alpha,\beta)=0$ *if* $\beta\neq 0$, *and the matrix*

$$\begin{bmatrix} R_a(1,0:1,0) & \cdots & R_a(1,0:\mu,0) \\ \vdots & & \vdots \\ R_a(\mu,0:1,0) & & R_a(\mu,0:\mu,0) \end{bmatrix}$$

is the reduced Alexander matrix of $l_{1,0}\cup\cdots\cup l_{\mu,0}$.

(2) $S_a(i,j:\alpha,\beta)=0$ *if* $(i,j)\neq(\alpha,\beta)$, *and* $S_a(i,j:i,j)$ *are* $(\lambda_{i,j}\times\lambda_{i,j})$ *matrices of the forms*:

$$\begin{bmatrix} 1 & -1 & & & & 0 \\ 0 & 1 & \ddots & & & \\ \vdots & & \ddots & \ddots & & \\ \vdots & & & \ddots & \ddots & \\ \vdots & & & & \ddots & -1 \\ -1 & 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}.$$

(3) $T_a(i,j:\alpha,\beta)=0$ *for any* (i,j) *and* (α,β).

(4) $T_B(i,j:\alpha,\beta)=0$ *if* $(i,j)\neq(\alpha,\beta)$ *and* $T_B(i,j:i,j)$ *are* $(\nu_{i,j}-1)\times\nu_{i,j}$ *matrices of the forms*:

$$\begin{bmatrix} t^{\delta_{i,j,1}} & -1 & & & \\ & t^{\delta_{i,j,2}} & -1 & & 0 \\ & & \ddots & \ddots & \\ 0 & & & \ddots & \\ & & & & t^{\delta_{i,j,\xi}} \quad -1 \end{bmatrix}$$

where $\xi = v_{i,j} - 1$ *and* $\delta_{i,j,k} = 0$ *or* ± 1.

Thus $A(t)$ is of the form

$$\begin{bmatrix} R_a & R_B \\ S_a & S_B \\ 0 & T_B \\ U_a & U_B \end{bmatrix}.$$

Now consider the matrix $S_a(i,j:i,j)$. Add the first column to the second and then the second to the third and so on, and then add all rows to the last. Thus $S_a(i,j:i,j)$ is transformed into the $(\lambda_{i,j} - 1) \times (\lambda_{i,j} - 1)$ identity matrix bordered by one zero column and row. By applying such a transformation to all $S_a(i,j:i,j)$, $A(t)$ is transformed into the matrix that is equivalent in $JZ(t)$ to the following

$$A'(t) = \begin{bmatrix} R_a(i,0:\alpha,0) & 0 & R_B(i,0:\alpha,\beta) \\ 0 & 0 & S'_B(i,j:\alpha,\beta) \\ 0 & 0 & T_B(i,j:\alpha,\beta) \\ U_a(i,0:\alpha,0) & U'_a(i,0:\alpha,\beta) & U_B(i,0:\alpha,\beta) \end{bmatrix},$$

where S' and U' are defined as follows:

(6.5) $S'_B(i,j:\alpha,\beta)$ *are* $1 \times v_{\alpha,\beta}$ *matrices of the forms*:

$$S'_B(i,j:\alpha,\beta) = \left(\sum_{k=1}^{\lambda_{i,j}} \frac{\partial S_{i,j,k}}{\partial B_{\alpha,\beta,1}} \cdots \sum_{k=1}^{\lambda_{i,j}} \frac{\partial S_{i,j,k}}{\partial B_{\alpha,\beta,v_{\alpha,\beta}}} \right).$$

$U'_a(i,0:\alpha,\beta)$ *are* $m_i \times 1$ *matrices of the forms*:

$$U'_a(i,0:\alpha,\beta) = \begin{bmatrix} \sum_{\gamma=1}^{\lambda_{\alpha,\beta}} \frac{\partial U_{i,0}}{\partial a_{\alpha,\beta,\gamma}} \\ \vdots \\ \sum_{\gamma=1}^{\lambda_{\alpha,\beta}} \frac{\partial U_{i,m_i-1}}{\partial a_{\alpha,\beta,\gamma}} \end{bmatrix}.$$

We remark that the sub-matrix $\| U'_a(i,0:\alpha,\beta) \|_{1 \leqq i,\alpha \leqq \mu, 1 \leqq \beta \leqq m_\alpha}$ is the square matrix of degree $\sum_{i=1}^{\mu} m_i$.

Next consider the matrix $T_B(i,j:i,j)$. Add the first column that is multiplied by $t^{-\delta_{i,j,1}}$ to the second and then the second multiplied by $t^{-\delta_{i,j,2}}$ to the third and so on. Thus $T_B(i,j:i,j)$ is transformed into the $(v_{i,j}-1) \times (v_{i,j}-1)$ identity matrix with one zero column. By applying such a transformation to all T_B, $A(t)$ is transformed into the matrix that is equivalent to the following

$$A''(t) = \begin{bmatrix} R_a(i,0:\alpha,0) & 0 & R'_B(i,0:\alpha,\beta) \\ 0 & 0 & S''_B(i,j:\alpha,\beta) \\ U_a(i,0:\alpha,0) & U'_a(i,0:\alpha,\beta) & U'_B(i,0:\alpha,\beta) \end{bmatrix},$$

where $S''_B(i,j:\alpha,\beta)$ is a 1×1 matrix of the form:

$$S''_B(i,j:\alpha,\beta) = \left(\sum_{\gamma=1}^{\nu_{\alpha,\beta}} t^{\eta_\gamma} \sum_{k=1}^{\lambda_{i,j}} \frac{\partial S_{i,j,k}}{\partial B_{\alpha,\beta,\gamma}} \right). \tag{6.6}$$

Let us denote

$$U'_a(i:\alpha) = \| U'_a(i,0:\alpha,\beta) \|_{1 \leqq \beta \leqq m_\alpha}$$

and

$$S''_B(i:\alpha) = \| S''_B(i,j:\alpha,\beta) \|_{1 \leqq j \leqq m_i, 0 \leqq \beta \leqq m_\alpha - 1}.$$

Both matrices are $m_i \times m_\alpha$ matrices. $\| S''_B(i:\alpha) \|$ is a square matrix of degree $\sum_{i=1}^{\mu} m_i$.

Now to prove Lemma 6.2, we shall prove the following:

$$\overline{U'_a(i:\alpha)^t} = -S''_B(\alpha:i), \tag{6.7}$$

where the bar over a symbol denotes the so-called conjugation [6] *of a matrix* M *in* $JZ(t)$ *that replaces* t *in* M *by* t^{-1}.

Proof of (6.7). Let us denote

$$U'_a(i:\alpha) = \| x_{p,q} \|_{1 \leqq p \leqq m_i,\ 1 \leqq q \leqq m_\alpha}$$

and

$$S''_B(\alpha:i) = \| y_{r,s} \|_{1 \leqq r \leqq m \quad 1 \leqq s \leqq m_i}.$$

Then we must prove that $\bar{x}_{p,q} = -y_{q,p}$.

From (6.5) and (6.6) it follows that

(6.8) (1) For $i \neq \alpha$, or $i = \alpha$ and $|q-p| > 1$,

$$x_{p,q} = \sum_{\gamma=1}^{\lambda_\alpha} \frac{\partial U_{i,p-1}}{\partial a_{\alpha,q,\gamma}} = (1-t) \sum_{\gamma=1}^{\lambda_{\alpha,q}} \frac{\partial u_{i,p-1}}{\partial a_{\alpha,q,\gamma}},$$

and

$$y_{q,p} = \sum_{\gamma=1}^{\nu_{i,p-1}} t^{\eta_\gamma} \sum_{k=1}^{\lambda_{\alpha,q}} \frac{\partial S_{\alpha,q,k}}{\partial B_{i,p-1,\gamma}} = (t-1) \sum_{\gamma=1}^{\nu_{i,p-1}} t^{\eta_\gamma} y(\alpha, q : i, p-1, \gamma),$$

where $\eta_\gamma = -\sum_{\xi=\gamma}^{\nu_{i,p-1}} \delta_{\alpha,p-1,\xi}$ and

$$y(\alpha, q : i, p-1, \gamma) = \frac{1}{t-1} \sum_{k=1}^{\lambda_{\alpha,q}} \frac{\partial S_{\alpha,q,k}}{\partial B_{i,p-1,\gamma}}.$$

(2) For $i = \alpha$ and $|q-p| \leqq 1$,

$$x_{p,p-1} = \sum_{\gamma=1}^{\lambda_{i,p-1}} \frac{\partial U_{i,p-1}}{\partial a_{i,p-1,\gamma}} = \theta(u_{i,p-1}) + (1-t) \sum_{\gamma=1}^{\lambda_{i,p-1}} \frac{\partial u_{i,p-1}}{\partial a_{i,p-1,\gamma}},$$

$$x_{p,p} = \sum_{\gamma=1}^{\lambda_{i,p}} \frac{\partial U_{i,p-1}}{\partial a_{i,p,\gamma}} = -1 + (1-t) \sum_{\gamma=1}^{\lambda_{i,p}} \frac{\partial u_{i,p-1}}{\partial a_{i,p,\gamma}},$$

$$y_{p,p} = \sum_{\gamma=1}^{\nu_{i,p-1}} t^{\eta_\gamma} \sum_{k=1}^{\lambda_{i,p}} \frac{\partial S_{i,p,k}}{\partial B_{i,p-1,\gamma}} = t^{\delta_{i,p-1,\nu_{i,p-1}}} + (t-1)\phi_{p,p}(t),$$

$$y_{p,p+1} = \sum_{\gamma=1}^{\nu_{i,p}} t^{\eta_\gamma} \sum_{k=1}^{\lambda_{i,p}} \frac{\partial S_{i,p,k}}{\partial B_{i,p,\gamma}} = -t^{\delta_{i,p-1,\nu_{i,p}}} + (t-1)\phi_{p,p+1}(t).$$

First we consider the case where $i \neq \alpha$ or $i = \alpha$ and $|p-q| > 1$. Since

$$\tilde{x}_{p,q} = (1-t^{-1}) \sum_\gamma \overline{\frac{\partial u_{i,p-1}}{\partial a_{\alpha,q,\gamma}}} = t^{-1}(t-1) \sum_\gamma \overline{\frac{\partial u_{i,p-1}}{\partial a_{\alpha,q,\gamma}}},$$

to prove (6.7) it is sufficient to show that

$$-t^{-1} \sum_\gamma \overline{\frac{\partial u_{i,p-1}}{\partial a_{\alpha,q,\gamma}}} = \sum_\gamma t^{\eta_\gamma} y(\alpha, q : i, p-1, \gamma). \tag{6.9}$$

Let us denote

$$\frac{\partial u_{i,p-1}}{\partial a_{\alpha,q,\gamma}} = \varepsilon_1 t^{\zeta_1} + \cdots + \varepsilon_d t^{\zeta_d} \tag{6.10}$$

and

$$\sum_\gamma t^{\eta_\gamma} y(\alpha, q : i, p-1, \gamma) = \delta_1 t^{\chi_1} + \cdots + \delta_f t^{\chi_f}. \tag{6.11}$$

Then from (6.3)(2)(4) and (6.8)(1), $d, \varepsilon_j, \zeta_j, f, \delta_k$ and χ_k are interpreted as follows.

(6.12)(1) d is the *absolute sum* ([14]) of the number of times that the band

([14]) Let P and N be the number of times that a band β crosses (under or over) some $L_{i,j}$ from left to right and from right to left respectively. Then $P+N$ and $P-N$ will be called the ***absolute*** sum and ***algebraic*** sum of the number of times that β crosses (under or over) $L_{i,j}$.

$\beta_{i,p-1}$ crosses under $L_{\alpha,q}$. That is to say, each term of the left hand side of (6.9) appears each time $\beta_{i,p-1}$ crosses under $L_{\alpha,q}$.

(2) $\varepsilon_j = +1$ or -1 according as $\beta_{i,p-1}$ crosses under $L_{\alpha,q}$ from right to left or from left to right.

(3) Let us suppose that $\varepsilon_j t^{\zeta_j}$ corresponds to a crossing of $\beta_{i,p-1}$ and $L_{\alpha,q}$. Then $-\zeta_j$ is the *algebraic sum*([14]) of the number of times that the sub-band β' of $\beta_{i,p-1}$ bounded by this crossing and the arc $\alpha'_{i,p}$, at where $\beta_{i,p-1}$ is attached to $l_{i,p}$([13]), crosses under any $L_{t,s}$ plus $(\varepsilon_j - 1)/2$.

(6.13)(1) f is the absolute sum of the number of times that the sub-band $\beta_{i,p-1,\gamma}$ of $\beta_{i,p-1}$ corresponding to an element $B_{i,p-1,\gamma}$ of G (see Figure 1(4)) crosses over $L_{\alpha,q}$. That is to say, each term of the right hand side of (6.9) appears each time $\beta_{i,p-1,\gamma}$ crosses over $L_{\alpha,q}$.

(2) δ_k is $+1$ or -1 according as $\beta_{i,p-1,\gamma}$ crosses over $L_{\alpha,q}$ from right to left or from left to right.

(3) Suppose that $\delta_k t^{\chi_k}$ corresponds to a crossing of $\beta_{i,p-1,\gamma}$ and $L_{\alpha,q}$. Then χ_k is the algebraic sum of the number of times that the sub-band β'' of $\beta_{i,p-1}$ bounded by this crossing and $\alpha'_{i,p}$ crosses under any $L_{t,s}$ plus $(\omega_k - 1)/2$, where ω_k is $+1$ or -1 according as $L_{t,s}$ crosses between $\beta_{i,p-1,\gamma}$ and $\beta_{i,p-1,\gamma+1}$ from right to left or from left to right, and $\omega_k = +1$ if another band passes through between them.

Let us denote by $P^{i,j}_{\alpha,\beta}$ (or $Q^{i,j}_{\alpha,\beta}$) the algebraic sum of the number of times that a band $\beta_{i,j}$ crosses over (or under) an $L_{\alpha,\beta}$ ($\beta > 0$). Then from the assumption on a diagram of l it is easy to show that

(6.14) $P^{i,j}_{\alpha,\beta} = -Q^{i,j}_{\alpha,\beta}$ *for* $i \neq \alpha$, *or* $i = \alpha$ *and* $\beta \neq j \pm 1$.

Further, l can be deformed isotopically so that (6.14) holds for all (i,j) and (α,β).

We are now in position to prove (6.9). From (6.12) and (6.13) it follows that if the band $\beta_{i,p-1}$ successively([15]) crosses over (or under) $L_{\alpha,q}$, then two terms corresponding to these crossings cancel each other. Suppose that $\beta_{i,p-1}$ *alternatingly* crosses $L_{\alpha,q}$, and that $\beta_{i,p-1}$ first crosses under $L_{\alpha,q}$ and then crosses over $L_{\alpha,q}$. Let εt^{ζ} and δt^{χ} be the terms corresponding to these crossings which are terms in (6.10) and (6.11) respectively. Then it is easy to show that $\varepsilon = -\delta$ and $\chi = -\zeta - 1$. Thus these two terms cancel each other in the equation (6.9). In this way, two terms in (6.9) cancel each other. This proves (6.9).

In the case where $i = \alpha$ and $p = q \pm 1$, from (6.8)(2), we can prove (6.7) by almost the same way. Thus proof of (6.7) is completed.

Now we consider the matrices $U'_a = \| U'_a(i:\alpha) \|_{1 \leq i,\alpha \leq \mu}$, and $S''_B = \| S''_B(i:\alpha) \|_{1 \leq i,\alpha \leq \mu}$. From (6.3)(4) and (6.8) it follows that $[U'_\alpha(i:\alpha)]_{t=1} = 0$ if $i \neq \alpha$ and

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([15]) $\beta_{i,p-1}$ may cross over or under other bands.

$$[U'_a(i:i)]_{t=1} = \begin{bmatrix} -1 & & & & \\ 1 & -1 & 0 & & \\ & \cdot & \cdot & & \\ & 0 & \cdot & \cdot & \\ & & & 1 & -1 \end{bmatrix}.$$

Thus we have $\det(U'_a)_{t=1} = \pm 1$. Hence U'_a and S''_B are nonsingular. Denoting $\det U'_a = f(t)$, from (6.7) and the form of $A''(t)$, we obtain Lemma 6.2.

The following lemma is proved by almost the same method as is used in Lemma 6.2.

LEMMA 6.3. $n(l_1 \oplus l_2) = n(l_1) + n(l_2) - 1$. *Moreover, if* $n(l_1) = a$, *and* $n(l_2) = b$, *then*

$$\tilde{\varepsilon}_{a+b-1}(l_1 \oplus l_2) = \tilde{\varepsilon}_a(l_1)\tilde{\varepsilon}_b(l_2)f(t)f(t^{-1}),$$

where $f(1) = \pm 1$.

This lemma was proved in [20] for the special case where l_1 is a knot and l_2 is a trivial knot.

LEMMA 6.4. $n(l_1 \circ l_2) = n(l_1) + n(l_2)$.

Proof is easy from the definition.

7. **Signature of links of some kind.** Let l be a link consisting of μ knots $k_1, \cdots, k_\mu$. By joining two knots, k_i and k_j say, by a band β, we have a link l' consisting of $\mu - 1$ knots $k_1, \cdots, k_{i-1}, k', k_{i+1}, \cdots, k_{j-1}, k_{j+1}, \cdots, k_\mu$. Then the signatures of l and l' are related to their nullity as follows.

LEMMA 7.1. (1) $|n(l') - n(l)| \leqq 1$.
(2) *If* $|n(l') - n(l)| = 1$, *then* $\sigma(l) = \sigma(l')$, *and otherwise* $|\sigma(l) - \sigma(l')| = 1$.

Proof. Let L' be a diagram of l', i.e. an image of a regular projection p of l'. The image $p(\beta)$ of a band β is divided into a number of α- and β-regions. We can take a projection p such that L' is connected and that there exists a β-region B in $p(\beta)$ as is shown in Figure 2. Two α-regions W_1 and W_2 adjacent to a β-region B belong to the same Seifert domain and they are connectible. It is clear that a

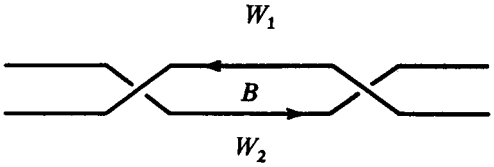

FIGURE 2

diagram L of l can be represented by that obtained from L' by connecting W_1 and W_2. Therefore, by the definition, the matrices M and M' of l and l' are given by (cf. (5.3))

$$M = \left(\begin{array}{c|c} a_{11} & M_{12} \\ \hline M_{21} & M_{22} \end{array}\right), \qquad M' = \left[\begin{array}{cc|c} b_{11} & b_{12} & \\ b_{21} & b_{22} & M'_{12} \\ \hline \multicolumn{2}{c|}{M'_{21}} & M'_{22} \end{array}\right],$$

where $a_{11} = b_{11} + b_{12} + b_{21} + b_{22}$ and $M_{22} = M'_{22}$. The first and second rows and columns of M' correspond to W_1 and W_2 respectively. Let us take the L-principal minor M^* and M'^* of M and M' as follows:

$$M^* = \tilde{M}(1\, q_2 \cdots q_{m+1}) = \tilde{M}_{22}(q_2 - 1 \cdots q_{m+1} - 1)$$

and

$$M'^* = \tilde{M}'(1\, q_2 + 1 \cdots q_{m+1} + 1).$$

Consider the symmetrized matrices N and N' of M^* and M'^*. N is a principal minor of N'. To determine the signatures of l and l', our consideration will be divided into several cases.

Case I. $\det N \neq 0$.

Let $r = \operatorname{rank} N$. Hence $n(N) = 0$. Let $\Delta_0, \Delta_1, \cdots, \Delta_r = N$ be a σ-series of N. Since N is a principal minor of N', $\operatorname{rank} N' \geqq \operatorname{rank} N = r$ and $\operatorname{rank} N' \leqq r + 1$.

(i) Suppose $\operatorname{rank} N' = r$, i.e. $n(N') = 1$. Then $\det N' = 0$. Thus we may select a σ-series of N' as that of N. We have, then, $n(N') - n(N) = 1$ and $\sigma(N') = \sigma(N)$.

(ii) Suppose $\operatorname{rank} N' = r + 1$, i.e. $n(N') = 0$. Since $\det N' \neq 0$, we selec[t] a σ-series $\{\Delta'_i\}$ of N' as follows:

$$\Delta'_i = \Delta_i, \quad i = 0, 1, \cdots, r, \quad \text{and } \Delta'_{r+1} = N'.$$

Then we see that

$$\sigma(N') = \sigma(N) + \operatorname{sign}(\det \Delta_r \cdot \det N') = \sigma(N) \pm 1.$$

Thus in this case the lemma is true.

Case II. $\det N = 0$.

Let $s = \operatorname{rank} N$ and let $n(N) = q$. There is a unimodular integral matrix T such that

$$TNT^t = \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix},$$

where P is an $s \times s$ matrix and nonsingular. Thus we see

$$\begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix} N' \begin{pmatrix} 1 & 0 \\ 0 & T^t \end{pmatrix} = \left[\begin{array}{c|cc} 2b_{22} & P_1 & P_2 \\ \hline P_1^t & P & 0 \\ \\ P_2^t & 0 & 0 \end{array}\right],$$

where $P_1 = (b_1 \cdots b_s)$ and $P_2 = (b_{s+1} \cdots b_{s+q})$ are $1 \times s$ and $1 \times q$ matrices respectively. Since P is nonsingular, it follows that $s \leqq \operatorname{rank} N' \leqq s+2$. Let $\Delta_0, \Delta_1, \cdots, \Delta_s = P$ be a σ-series of P. We select a σ-series $\{\Delta_i'\}$ of N' as follows:

(i) If $\operatorname{rank} N' = s+2$, there is a nonzero element, b_{s+1} say, in P_2. Δ_i' is defined, then, as follows:

$$\Delta_i' = \Delta_i, \qquad i = 0, 1, \cdots, s,$$

and

$$\Delta_{s+1}' = D_{s+1} = \left(\begin{array}{c|c} 2b_{22} & P_1 \\ \hline P_1^t & P \end{array}\right)$$

and

$$\Delta_{s+2}' = \left[\begin{array}{c|cc} 2b_{22} & P_1 & b_{s+1} \\ \hline P_1^t & P & 0 \\ b_{s+1} & 0 & 0 \end{array}\right].$$

Then we see

$$\sigma(N') = \sigma(N) + \operatorname{sign}(\det \Delta_s' \cdot \det \Delta_{s+1}') + \operatorname{sign}(\det \Delta_{s+1}' \cdot \det \Delta_{s+2}'). \tag{7.1}$$

Since $\operatorname{sign}(\det \Delta_s') = -\operatorname{sign}(\det \Delta_{s+2}')$, the last two terms in (7.1) cancel each other. Hence $\sigma(N') = \sigma(N)$. In this case $n(N') = s+q+1-(s+2) = q-1$ and $n(N) = q$. Thus $n(N) - n(N') = 1$.

(ii) If $\operatorname{rank} N' = s+1$, then $P_2 = 0$ and D_{s+1} is nonsingular. Thus $\{\Delta_i'\}$ can be defined as follows: $\Delta_i' = \Delta_i$, $i = 0, 1, \cdots, s$, and $\Delta_{s+1}' = D_{s+1}$. It is clear that $|\sigma(N) - \sigma(N')| = 1$ and $n(N) = n(N')$.

(iii) If $\operatorname{rank} N' = s$, then $P_2 = 0$ and D_{s+1} is singular. Thus Δ_i' is defined as Δ_i. Then $\sigma(N) = \sigma(N')$ and $n(N') - n(N) = 1$.

Thus the lemma is completely proved.

From the definition of the matrix of a link, it follows immediately that

LEMMA 7.2. $\sigma(l_1 \circ l_2) = \sigma(l_1) + \sigma(l_2)$.

LEMMA 7.3. $\sigma(l_1 \oplus l_2) = \sigma(l_1) + \sigma(l_2)$.

Proof. Apply Lemma 7.1 to the case $l = l_1 \circ l_2$ and $l' = l_1 \oplus l_2$. Since, from Lemmas 6.3 and 6.4, $n(l_1 \circ l_2) = n(l_1) + n(l_2)$ and $n(l_1 \oplus l_2) = n(l_1) + n(l_2) - 1$, we see from Lemma 7.2 that $\sigma(l_1 \oplus l_2) = \sigma(l_1 \circ l_2) = \sigma(l_1) + \sigma(l_2)$. q.e.d.

COROLLARY 7.4. $\sigma(l_1 \# l_2) = \sigma(l_1) + \sigma(l_2)$.

8. **Slice links.** Let $H_{(a,b)}$, $H_{[a,b)}$, $H_{(a,b]}$ and $H_{[a,b]}$ $(-\infty \leqq a, b \leqq \infty)$ be subspaces of the 4-sphere S^4 defined as follows:

$$H_{(a,b)} = \{x = (x_1, x_2, x_3, x_4) \mid a < x_4 < b\},$$
$$H_{[a,b)} = \{x \mid a \leqq x_4 < b\},$$
$$H_{(a,b]} = \{x \mid a < x_4 \leqq b\},$$
$$H_{[a,b]} = \{x \mid a \leqq x_4 \leqq b\}.$$

In particular, H_a means a hyperplane $H_{[a,a]}$.

Consider a polyhedral locally flat connected orientable surface([16]) F in general position in S^4 and cut it by the family of hyperplanes $H_t, -\infty < t < \infty$. We can assume without loss of generality that there are only finite number of t-values that are singular. A singular hyperplane may intersect F in an isolated point, which may be either a maximum or minimum for the height, called an *extreme point*, or it may intersect F in a graph with just one node, which is of order four. These nodes are called *saddle points*. Extreme points or saddle points will be called *singular points*.

Let a link l be the boundary of F. Then F is said to be in *normal* position in S^4 if F is placed in such a way that

(8.1)(1) $F \subset H_{[-1,n]}$ for a sufficiently large $n > 0$ and $l \subset F \cap H_0$.

(2) All minimal points lie on H_{-1}.

(3) All maximal points lie on H_n.

(4) All saddle points are in $H_{(0,n)}$ and they are ordered in order of their heights.

It is obvious that given a surface F we can deform it into a surface that is in normal position.

Now a knot k is called a *slice knot* if k is a cross section of a locally flat 2-*sphere* F in S^4 with H_t for some $t, -\infty < t < \infty$ [4]. The object of this section is to give a necessary condition for a given knot to be a slice knot by means of the signature of the knot. Up to the present the following conditions are known ([17]) [4].

(8.2) *If k is a slice knot, then its Alexander polynomial is of the form $f(t)f(t^{-1})$ and its Minkowski units are always $+1$.*

Let k be a slice knot, i.e. there exists a 2-sphere F such that $F \cap H_t = k$ for some t. Put F in normal position. Let $k_t = F \cap H_t$. From (8.1)(1), k is a component of a link $k_0 = F \cap H_0$. Exactly, k_0 is of the form: $k_0 = k \circ \iota_1 \circ \cdots \circ \iota_m$, where ι_i denotes a trivial knot and m is the number of the minimal points. It is easy to see that if $H_{(a,b)}$ has no saddle point of F, then any two links k_t and $k_{t'}$, $a < t, t' < b$, are of the same link type. If H_t has a saddle point, then the *saddle point transformation* gives rise to the following transformation on two links $k_{t-\varepsilon}$ and $k_{t+\varepsilon}$ for a sufficiently small number $\varepsilon > 0$:

([16]) By a *surface* is meant a tamely imbedded 2-manifold with or without boundary.

([17]) Recently the second condition was obtained by J. J. Andrews. See *The Minkowski unit of slice knots*, Abstract 599–20, Notices Amer. Math. Soc. **10** (1963), p. 253.

$$k' \circ k'' \to k' \oplus k''$$

or conversely. Moreover, it is clear that $k_{n-\varepsilon}$ is of the form: $k_{n-\varepsilon} = \iota_1 \circ \cdots \circ \iota_r$, r being the number of the maximal points. From these observations, we have the following

LEMMA 8.1. *If a knot k is a slice knot, then a knot of the type $k \oplus \iota \oplus \cdots \oplus \iota$ is isotopic to the knot of the type $\iota \oplus \cdots \oplus \iota$, for some number of trivial knot ι's.*

The definition and this lemma can be extended to links in the following form [5].

DEFINITION 8.1. (1) A link l of multiplicity μ is called a *slice link in the strong sense* if l is a cross section of a union of μ disjoint locally flat 2-spheres in S^4.

(2) A link l is called a *slice link (in the ordinary sense)* if l is a cross section of a locally flat 2-sphere in S^4.

(3) A link l is called a *slice link in the weak sense* if l bounds a locally flat (not necessarily connected) surface of genus 0 in $H_{[t,\infty)}$, $-\infty < t < \infty$.

LEMMA 8.2. (1) *If a link $l = k_1 \cup \cdots \cup k_\mu$ is a slice link in the strong sense, then the link l' consisting of knots of the type $k_i \oplus \iota \oplus \cdots \oplus \iota$ is isotopic to the link of multiplicity μ of the type $(\iota \oplus \cdots \oplus \iota) \circ \cdots \circ (\iota \oplus \cdots \oplus \iota)$, where all k_i and ι's are separated from one another.*

(2) *If a link l is a slice link, then the link l'' obtained from l', of the type considered in* (1), *by joining suitably its components by bands is isotopic to a link of multiplicity μ' $(\leqq \mu)$ of the type $(\iota \oplus \cdots \oplus \iota) \circ \cdots \circ (\iota \oplus \cdots \oplus \iota)$.*

(3) *If a link l is a slice link in the weak sense, then l'' constructed in* (2) *above is isotopic to a knot of the type $\iota \oplus \iota \oplus \cdots \oplus \iota$.*

Obviously, if a link is a slice link in the strong sense, it is a slice link in the ordinary sense, and further it is also a slice link in the weak sense. The converse is true in the restricted cases as follows [5].

(8.3) *If k is a slice knot in the weak sense, then k is a slice knot in the strong sense. If l is a slice link of multiplicity 2 in the ordinary sense, then it is a slice link in the strong sense.*

We are now in position to give some necessary conditions for a given link to be a slice link in any sense.

THEOREM 8.3. *If a knot k is a slice knot, then $\sigma(k) = 0$.*

This is a corollary of the following

THEOREM 8.4. *If l is a slice link of multiplicity μ in the strong sense, then $n(l) = \mu$ and $\sigma(l) = 0$. Moreover, $\tilde{\varepsilon}_\mu(l)$ is a principal ideal that is generated by an element of the form $f(t)f(t^{-1})$, $f(1) = \pm 1$.*

Proof. For the link l' of multiplicity μ constructed in Lemma 8.2(1), we see

$$\begin{aligned}\sigma(l') &= \sigma[(k_1 \oplus \iota_{1,1} \oplus \cdots \oplus \iota_{1,m_1}) \cup \cdots \cup (k_\mu \oplus \iota_{\mu,1} \oplus \cdots \oplus \iota_{\mu,m})] \\ &= \sigma[(\iota \oplus \cdots \oplus \iota) \circ \cdots \circ (\iota \oplus \cdots \oplus \iota)] = 0.\end{aligned} \tag{8.4}$$

Now from Lemma 6.2, we have $n(l') = n(l)$. Since l' is isotopic to a link of multiplicity μ, whose components are separated from one another, $n(l') = \mu$. Therefore $n(l) = \mu$.

Next to prove $\sigma(l) = 0$, it is sufficient to show

$$\sigma(l) = \sigma(l'). \tag{8.5}$$

Consider $l'' = (k_1 \cup \cdots \cup k_\mu) \circ \iota_{1,1} \circ \cdots \circ \iota_{\mu,m_\mu}$ of the multiplicity $\mu + \sum_{j=1}^{\mu} m_j$. l'' is obtained from l' by removing all bands connecting k_i with $\iota_{i,1}$, and $\iota_{i,j}$ with $\iota_{i\,j+1}$. Since $n(l'') = n(k_1 \cup \cdots \cup k_\mu) + \sum_{j=1}^{\mu} m_j = \mu + \sum_{j=1}^{\mu} m_j$ and hence $n(l'')$ has the greatest value, we see from Lemma 7.1(1) that the nullity must increase by one each time a band is removed. Thus from Lemma 7.1(2), we see that the signature is unchanged. Therefore $\sigma(l'') = \sigma(l')$. On the other hand, $\sigma(l'') = \sigma(l) + \sigma(\iota_{1,1}) + \cdots + \sigma(\iota_{\mu,m_\mu}) = \sigma(l)$. Thus we have $\sigma(l) = 0$. That $\tilde{\varepsilon}_\mu(l) = (f(t)f(t^{-1}))$ is clear from Lemma 6.2(2). This completes the proof of Theorem 8.4.

THEOREM 8.5. *If l is a slice link of multiplicity μ in the weak sense, then* $|\sigma(l)| \leqq \mu - 1$.

Proof. Since l' constructed in Lemma 8.2(3) is isotopic to a knot of the type $\iota \oplus \cdots \oplus \iota$, we see $\sigma(l') = 0$ and $n(l') = 1$. Consider a link $l'' = (k_1 \oplus \iota \oplus \cdots \oplus \iota) \cup \cdots \cup (k_\mu \oplus \iota \oplus \cdots \oplus \iota)$. l' is obtained from l'' by joining two components consecutively by a band. Since $n(l'') = n(k_1 \cup \cdots \cup k_\mu) \leqq \mu$ from Lemma 6.2, we have $|\sigma(l)| \leqq \mu - 1$ from Lemma 7.1 and from the fact that $\sigma(l') = 0$. q.e.d.

THEOREM 8.6. *Let l be a link of multiplicity μ consisting of ν links $l_1, \cdots, l_\nu$. Suppose that every l_i bounds a 2-cell C_i in $H_{[t,\infty)}$ for a fixed t, $-\infty < t < \infty$, and that the C_i are disjoint from each other. Then* $|\sigma(l)| \leqq \mu - \nu$.

Proof. As we see in Lemma 8.2(3), a link of the type $l_i \oplus \iota \oplus \cdots \oplus \iota$ is isotopic to a link of the type $\iota \oplus \cdots \oplus \iota$ and these links are separated. Therefore from Theorems 8.4. and 8.5, since $n(l) \geqq \nu$ and $|\sigma(l_i)| \leqq \mu_i - 1$, μ_i denoting the multiplicity of l_i, we have

$$|\sigma(l)| \leqq \sum_{i=1}^{\nu} (\mu_i - 1) = \sum_{i=1}^{\nu} \mu_i - \nu = \mu - \nu. \tag{8.6}$$

q.e.d.

From this theorem we have

COROLLARY 8.7. *If l is a slice link of multiplicity μ, then* $|\sigma(l)| \leqq \frac{1}{2}(\mu - 1)$.

Proof. Since l is a cross section of a 2-sphere F in S^4, l bounds ν and ν', say, disjoint 2-cells in $H_{[0,\infty)}$ and $H_{(-\infty,0]}$ respectively. Therefore from Theorem 8.6,

we see that $|\sigma(l)| \leqq \mu - \nu$ and $|\sigma(l)| \leqq \mu - \nu'$. Since F is a 2-sphere, $\nu + \nu' = \mu + 1$. Thus $2|\sigma(l)| \leqq 2\mu - (\nu + \nu') = \mu - 1$. q.e.d.

This corollary is strengthened as follows.

THEOREM 8.8. *If l is a slice link, then $\sigma(l) = 0$.*

Proof will be given in the next section.

Finally consider a product of k with itself. From Corollary 7.4, we see $\sigma(k \# k) = 2\sigma(k)$. Thus if k is a knot such that $\sigma(k) \neq 0$, then $\sigma(k \# k) \neq 0$. Therefore Theorem 8.3 implies that $k \# k$ is not a slice knot. In particular, since for the trefoil knot k, $\sigma(k) = \pm 2$, $\sigma(k \# k) = \pm 4$. Thus we have

THEOREM 8.9. *A granny knot is not a slice knot.*

REMARK. It is well known that the product of k with its mirror image is always a slice knot [4].

9. **Genus of a link.** Let l be an oriented link of multiplicity μ in S^3. l always bounds an orientable connected tame surface F in S^3 [18],[21]. The minimal genus of these surfaces is called the *genus* of a link l, denoted by $h(l)$. Then as is well known, the following inequality holds [18],[21].

(9.1) *The degree of the reduced Alexander polynomial of l is not greater than $2h + \mu - 1$.*

Now since l bounds a surface F in S^3, l always bounds an orientable locally flat connected surface F in the *upper half space* $H_{[0,\infty)}$ in S^4. The minimal genus of these surfaces is an invariant of the link types. It is denoted by $h^*(l)$. Clearly for any link

$$0 \leqq h^*(l) \leqq h(l). \tag{9.2}$$

Moreover it is also clear

(9.3) *A knot k is a slice knot if and only if $h^*(k) = 0$.*

Thus in (9.2) the inequalities cannot be replaced by equalities. For example, for the stevedore's knot k [4], $h^*(k) = 0$ but $h(k) = 1$.

The object of this section is to show a relation similar to (9.1). That is,

THEOREM 9.1. *If a link l bounds ν disjoint locally flat orientable surfaces F_i in $H_{[0,\infty)}$, then*

$$|\sigma(l)| \leqq 2 \sum_{i=1}^{\nu} (\textit{genus of } F_i) + \mu - \nu. \tag{9.4}$$

Consequently the following inequality holds:

$$|\sigma(l)| \leqq 2h^*(l) + \mu - 1. \tag{9.5}$$

Suppose that a link l of multiplicity μ bounds a connected surface F. Let p and q denote the number of *extreme* points and *saddle* points in F. Then we shall show first that

LEMMA 9.2. *$2g = q - p + 2 - \mu$, where g denotes the genus of F.*

Proof. Proof will be completed by calculating the Euler characteristic $\chi(F)$ [2, p. 463, footnote (1)]. We may assume that F is in normal position. Consider the 1-complex K obtained by intersecting F with every singular hyperplane. Subdivide F by introducing the vertices and edges of K. Then any nonsingular hyperplane H_t meets an equal number of faces and edges of the subdivided surface F. Since F is bounded by μ disjoint simple closed curves, $\chi(F) = 2 - \mu - 2g$ of F is equal to the Euler characteristic $\chi(K)$ of K. On the other hand, by the assumption, each saddle point is a common point of only two simple closed curves through it. Thus we see $\chi(K) = (p + q + \mu) - (2q + \mu) = p - q$. Hence we obtain the required formula.

Let F be an orientable surface that consists of m connected components F_i. Then by the genus, $g(F)$, of F is meant the sum of the genera of F_i. Moreover, $v(F)$ denotes the number of connected components of F and $\mu(l)$ the multiplicity of a link l. Then from Lemma 9.2, we have immediately

(9.6) *If l bounds an orientable surface F, then*

$$2g(F) = q(F) - p(F) + 2v(F) - \mu(l),$$

where $p(F)$ and $q(F)$ denote the sum of the number of extreme points and saddle points, respectively, of each connected component of F.

We are now in position to prove Theorem 9.1. Proof will be done by induction on the genus of a surface F spanning l.

First of all, in the case where l bounds a surface of genus 0, the theorem is true from Theorem 8.6. Suppose that the theorem is true for the genus of a surface $< g$, and that l bounds a surface F of genus $g(F) = g$. We may suppose F is in normal position. Let $F_t = F \cap H_{[t,n]}$. F_t is a surface spanning a link (or a linear graph). Then it is clear

(9.7) *If $t < t'$, then $g(F_{t'}) \leqq g(F_t) \leqq g(F_{-1}) = g$ and $v \leqq v(F_t) \leqq v(F_{t'})$.*

Now the genus of F_t may be changed by one when the hyperplane H_t passes through a saddle point. So we divide the set of saddle points in F into two classes $\mathfrak{A}$ and $\mathfrak{B}$ as follows.

(9.8) A saddle point s_i in H_{t_i} belongs to $\mathfrak{A}$ if $g(F_{t_i+\varepsilon}) < g(F_{t_i-\varepsilon})$. Otherwise s_i belongs to $\mathfrak{B}$.

Let s_α be the saddle point in $\mathfrak{A}$ with the lowest height and let s_α be in H_α. Clearly for a sufficiently small $\varepsilon > 0$, $g(F_{\alpha+\varepsilon}) = g(F) - 1$, where $F_{\alpha+\varepsilon} = F \cap H_{[\alpha+\varepsilon,n]}$. Consider two links $l_{\alpha-\varepsilon} = F \cap H_{\alpha-\varepsilon}$ and $l_{\alpha+\varepsilon} = F \cap H_{\alpha+\varepsilon}$. Then we shall show that

$$\mu(l_{\alpha+\varepsilon}) = \mu(l_{\alpha-\varepsilon}) + 1 \text{ and } v(F_{\alpha+\varepsilon}) = v(F_{\alpha-\varepsilon}). \tag{9.9}$$

Proof. Let p_t and q_t denote the number of extreme points and saddle points of F_t. We can see easily that $p_{\alpha-\varepsilon} = p_{\alpha+\varepsilon}$, $q_{\alpha-\varepsilon} = q_{\alpha+\varepsilon} + 1$, and from Lemma 9.2, the genera of $F_{\alpha-\varepsilon}$ and $F_{\alpha+\varepsilon}$ are given by

$$2g(F_{\alpha-\varepsilon}) = 2g(F) = q_{\alpha-\varepsilon} - p_{\alpha-\varepsilon} + 2v(F_{\alpha-\varepsilon}) - \mu(l_{\alpha-\varepsilon}),$$

and

$$2g(F_{\alpha+\varepsilon}) = 2\{g(F)-1\} = q_{\alpha-\varepsilon} - 1 - p_{\alpha-\varepsilon} + 2v(F_{\alpha+\varepsilon}) - \mu(l_{\alpha+\varepsilon}).$$

Therefore we see that

$$q_{\alpha-\varepsilon} - p_{\alpha-\varepsilon} + 2v(F_{\alpha-\varepsilon}) - \mu(l_{\alpha-\varepsilon}) = q_{\alpha-\varepsilon} - p_{\alpha-\varepsilon} + 2v(F_{\alpha+\varepsilon}) - \mu(l_{\alpha+\varepsilon}) + 1,$$

which reduces to

$$2\{v(F_{\alpha-\varepsilon}) - v(F_{\alpha+\varepsilon})\} = \mu(l_{\alpha-\varepsilon}) - \mu(l_{\alpha+\varepsilon}) + 1.$$

If $v(F_{\alpha-\varepsilon}) < v(F_{\alpha+\varepsilon})$, i.e. if $v(F_{\alpha-\varepsilon}) + 1 = v(F_{\alpha+\varepsilon})$, then $\mu(l_{\alpha-\varepsilon}) - \mu(l_{\alpha+\varepsilon}) = -3$, which is impossible. Since $v(F_{\alpha-\varepsilon}) \leqq v(F_{\alpha+\varepsilon})$ from (9.7), we have $v(F_{\alpha-\varepsilon}) = v(F_{\alpha+\varepsilon})$ and hence $\mu(l_{\alpha+\varepsilon}) = \mu(l_{\alpha-\varepsilon}) + 1$.

Now applying the assumption of induction to $F_{\alpha+\varepsilon}$, we have

$$\begin{aligned} |\sigma(l_{\alpha+\varepsilon})| &\leqq 2\{g(F) - 1\} + \mu(l_{\alpha+\varepsilon}) - v(F_{\alpha+\varepsilon}) \\ &= 2g(F) - 2 + \mu(l_{\alpha-\varepsilon}) + 1 - v(F_{\alpha-\varepsilon}) \\ &= 2g(F) + \mu(l_{\alpha-\varepsilon}) - v(F_{\alpha-\varepsilon}) - 1. \end{aligned} \tag{9.10}$$

On the other hand, since $|\sigma(l_{\alpha-\varepsilon}) - \sigma(l_{\alpha+\varepsilon})| \leqq 1$ from Lemma 7.1, we have

$$|\sigma(l_{\alpha-\varepsilon})| \leqq |\sigma(l_{\alpha+\varepsilon})| + 1 \leqq 2g(F) + \mu(l_{\alpha-\varepsilon}) - v(F_{\alpha-\varepsilon}). \tag{9.11}$$

Next we shall consider the changes of $\mu(l_t)$ and $v(F_t)$ when H_t passes through saddle points in $H_{(0,\alpha)}$, which belong to $\mathfrak{B}$. Let $s_\beta \in H_\beta$ be an arbitrary saddle point in F_β, $0 < \beta < \alpha$. Since s_β belongs to $\mathfrak{B}$, $g(F_{\beta-\varepsilon}) = g(F_{\beta+\varepsilon}) = g(F)$ for a sufficiently small $\varepsilon > 0$. Then we can prove the following (9.12) in the same way as used in proof of (9.9).

(9.12) (1) $\mu(l_{\beta+\varepsilon}) = \mu(l_{\beta-\varepsilon}) - 1$ *if and only if* $v(F_{\beta+\varepsilon}) = v(F_{\beta-\varepsilon})$.

(2) $\mu(l_{\beta+\varepsilon}) = \mu(l_{\beta-\varepsilon}) + 1$ *if and only if* $v(F_{\beta+\varepsilon}) = v(F_{\beta-\varepsilon}) + 1$.

From Lemma 7.1 and (9.12), we see

(9.13) (1) *If* $v(F_{\beta+\varepsilon}) = v(F_{\beta-\varepsilon})$, *then* $|\sigma(l_{\beta+\varepsilon}) - \sigma(l_{\beta-\varepsilon})| \leqq 1$.

(2) *If* $v(F_{\beta+\varepsilon}) = v(F_{\beta-\varepsilon}) + 1$, *then* $\sigma(l_{\beta+\varepsilon}) = \sigma(l_{\beta-\varepsilon})$.

Suppose that there are u saddle points of F in $H_{(0,\alpha)}$ such that the case (9.13) (1) occurs when H_t passes through each of them. Suppose that there are v saddle points of F such that the case (9.13)(2) occurs. Further, since l_0 is of the form $l \circ \iota_1 \circ \cdots \circ \iota_r$, r being the number of minimal points, we see that there are r saddle

points such that when H_t passes through each of these r points the following equalities hold:

$$(9.14) \qquad v(F_{\beta-\varepsilon}) = v(F_{\beta+\varepsilon}),\ \ \mu(l_{\beta+\varepsilon}) = \mu(l_{\beta-\varepsilon}) + 1, \text{ and } \sigma(l_{\beta+\varepsilon}) = \sigma(l_{\beta-\varepsilon}).$$

Thus from (9.14) we have finally

$$\begin{aligned} |\sigma(l)| &= |\sigma(l_0)| \leqq |\sigma(l_{\alpha-\varepsilon})| + u - r \\ &\leqq 2g(F) + \mu(l_{\alpha-\varepsilon}) - v(F_{\alpha-\varepsilon}) + u - r \\ &= 2g(F) + \{\mu(l_{\alpha-\varepsilon}) - v + u\} - \{v(F_{\alpha-\varepsilon}) - v\} - r \qquad (9.15) \\ &= 2g(F) + \mu(l_0) - v(F_0) - r \\ &= 2g(F) + \mu(l) - v(F), \end{aligned}$$

because $\mu(l) = \mu(l_0) - r$ and $v(F_0) = v(F)$. This proves (9.4).

Since v disjoint surfaces spanning a link l can be connected by joining them by cylinders without changing the genus, $h^*(l)$ always satisfies the inequality

$$|\sigma(l)| \leqq 2h^*(l) + \mu(l) - 1.$$

Thus Theorem 9.1 is completely proved.

Proof of Theorem 8.8. Since l is a slice link, there is a 2-sphere F such that $F \cap H_0 = l$. F is contained in $H_{[-n,n]}$ for a sufficiently large number $n > 0$. If H_t $(-n \leqq t \leqq n)$ contains neither saddle points nor extreme points, then each of the two surfaces $F_t^{\#} = F \cap H_{[t,n]}$ and $F_t^{\flat} = F \cap H_{[-n,t]}$ is of genus 0.

Now let s be a saddle point of F and let $s \in H_t$. Then we see that

(9.16) $\sigma(l_{t-\varepsilon}) = \sigma(l_{t+\varepsilon})$ *for a sufficiently small* $\varepsilon > 0$.

For, if $\mu(l_{t+\varepsilon}) = \mu(l_{t-\varepsilon}) + 1$, then from (9.12) (2) we obtain that $v(F^{\#}_{t+\varepsilon}) = v(F^{\#}_{t-\varepsilon}) + 1$. If $\mu(l_{t+\varepsilon}) = \mu(l_{t-\varepsilon}) - 1$, then similarly we obtain that $v((F^{\flat}_{t+\varepsilon}) = v(F^{\flat}_{t-\varepsilon}) - 1$. Thus $\sigma(l_{t+\varepsilon}) = \sigma(l_{t-\varepsilon})$ in both cases.

For an extreme point $m \in H_u$, clearly $\sigma(l_{u-\varepsilon}) = \sigma(l_{u+\varepsilon})$. Thus it follows that $\sigma(l) = \sigma(F \cap H_0) = \sigma(l_{\gamma-\varepsilon}) = \sigma(\iota \circ \cdots \circ \iota) = 0$, where γ denotes the largest value of the heights of extreme points. q.e.d.

For a special alternating knot k, since $2h(k)$ is equal to the degree of the Alexander polynomial of k [1], [12], from Lemma 5.2 and Theorem 9.1, we have

THEOREM 9.3. *For a special alternating knot* k,

$$h(k) = h^*(k).$$

10. **Unknotting number of knots.** Let k be a knot in S^3 and let K be a diagram of k. K has in general at least one double point. As is well known, any knot k can be deformed into a trivial knot by employing a finite number of *unknotting operations* Γ defined as follows [23].

(Γ) *Change an underpass into an overpass at a double point.*

The minimum number of unknotting operations required to deform a given knot into a trivial knot is called the *unknotting number* of k, denoted by $u(k)$. The object of this section is to show the following

THEOREM 10.1. *For any knot* k, $|\sigma(k)| \leqq 2u(k)$.

Proof. Let k_1 be the knot obtained by employing once an unknotting operation upon k. Then it is easy to see that the symmetrized matrices N and N_1 of k and k_1 with respect to their diagrams K and K_1 are given as follows.

$$(10.1) \quad N = \left[\begin{array}{cc|c} a & c & N_{12} \\ c & b & \\ \hline N_{12}^t & & N_{22} \end{array}\right] \text{ and } N_1 = \left[\begin{array}{cc|c} a-2 & c+2 & N_{12} \\ c+2 & b-2 & \\ \hline N_{12}^t & & N_{22} \end{array}\right].$$

The first and second rows and columns correspond to two α-regions W_1 and W_2 that meet at the double point, at which the unknotting operation is applied. Therefore, we may assume without loss of generality that L-principal minors of N and N_1 are given by

$$N^* = \begin{pmatrix} b & N_{12}^* \\ N_{12}^{*t} & N_{22}^* \end{pmatrix} \text{ and } N_1^* = \begin{pmatrix} b-2 & N_{12}^* \\ N_{12}^{*t} & N_{22}^* \end{pmatrix}.$$

Let m be the number of rows of N_{22}^*. To prove Theorem 10.1, it is sufficient to show that

$$(10.2) \qquad |\sigma(k) - \sigma(k_1)| \leqq 2.$$

Proof will be done by dividing it into two cases.

Case I. $\det N_{22}^* \neq 0$.

Then rank $N_{22}^* = m$. Let $\Delta_0, \Delta_1, \cdots, \Delta_m = N_{22}^*$ be the σ-series of N_{22}^*. Then since $\det N^* \neq 0$ and $\det N_1^* \neq 0$, the σ-series of N^* and N_1^* are given by

$$(10.3) \qquad \Delta_0, \Delta_1, \cdots, \Delta_m, N^* \text{ and } \Delta_0, \Delta_1, \cdots, \Delta_m, N_1^*.$$

Therefore it follows $|\sigma(k) - \sigma(k_1)| \leqq 2$.

Case II. $\det N_{22}^* = 0$.

The matrix N_{22}^* is the symmetric L-principal minor of the matrix of a link l obtained from k by connecting W_1 and W_2. Since the multiplicity of l is two, rank $N_{22}^* = m - 1$. Hence there is an integral unimodular matrix T such that

$$T N_{22}^* T^t = \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix},$$

where P is a nonsingular matrix of rank $m-1$. Consider two matrices Q and Q_1:

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix} N^* \begin{pmatrix} 1 & 0 \\ 0 & T^t \end{pmatrix} = \begin{bmatrix} b & P_1 & p_1 \\ P_1^t & P & 0 \\ p_1 & 0 & 0 \end{bmatrix}$$

and

$$Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix} N_1^* \begin{pmatrix} 1 & 0 \\ 0 & T^t \end{pmatrix} = \begin{bmatrix} b-2 & P_1' & p_1 \\ P_1'^t & P & 0 \\ p_1 & 0 & 0 \end{bmatrix}.$$

Then, denoting the σ-series of P by $\Delta_0, \Delta_1, \cdots, \Delta_{m-1} = P$, we can select the σ-series of Q and Q_1 as follows:

$$\Delta_0, \Delta_1, \cdots, \Delta_m = \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix}, \quad \Delta_{m+1} = Q, \text{ and}$$

(10.4)

$$\Delta_0, \Delta_1, \cdots, \Delta_m = \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix}, \quad \Delta_{m+1}' = Q_1 .$$

Then it is easy to show that

(10.5) $$\sigma(Q) = \sigma(P) \quad \text{and} \quad \sigma(Q_1) = \sigma(P).$$

Thus we see $\sigma(k) = \sigma(Q) = \sigma(Q_1) = \sigma(k_1)$. This completes the proof of Theorem 10.1.

The following theorem is proved by a method similar to that used in proof of Theorem 9.1.

THEOREM 10.2. *For any knot k, $h^*(k) \leqq u(k)$.*

From Theorems 9.3 and 10.2, we have

COROLLARY 10.3. *For any special alternating knot k, $|\sigma(k)| =$ degree of $\Delta(t) = 2h^*(k) = 2h(k) \leqq 2u(k)$.*

From this corollary, we obtain

(10.6) *The unknotting number of the torus knot of type $(2n+1, 2)$ is just $|n|$.*

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