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ON A CERTAIN PROBLEM OF REALIZATION IN HOMOTOPY THEORY

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In this paper it is shown that given any group π and any subgroup G of the centre of π there exists a 0-connected CW-Complex X with $\pi_1(X) \cong \pi$ under an isomorphism carrying the Gottlieb subgroup $G_1(X)$ of $\pi_1(X)$ onto G.

Introduction. Let X be a 0-connected CW-Complex and $x_0 \in X$. In [2] Gottlieb defined a certain subgroup $G(X, x_0)$ of the fundamental group $\pi_1(X, x_0)$ of X at x_0 and studied some of its properties. Earlier [4] Jiang Bo Ju defined a subgroup G_f of $\pi_1(X, f(x_0))$ corresponding to any map $f: X \to X$. The group G_f when $f = Id_X$ turns out to be precisely $G(X, x_0)$ studied by Gottlieb. These groups play a role in Nielsen-Wecken theory of fixed point classes and were investigated by R. F. Brown, W. J. Barnier, etc. In [3] Gottlieb defined the higher dimensional analogues $G_n(X, x_0) \subset \pi_n(X, x_0)$ of $G(X, x_0)$ and studied their properties. For any path σ joining x_0 to x_1 in X the isomorphism $\sigma_\#: \pi_n(X, x_1) \to \pi_n(X, x_0)$ carries $G_n(X, x_1)$ onto $G_n(X, x_0)$ for all $n \geqq 1$. Thus one can talk of the nth Gottlieb group $G_n(X)$ of X without reference to a base point. In [2] it is shown that always $G_1(X)$ is a subgroup of the centre of $\pi_1(X)$ and that if X is a $K(\pi, 1)$ CW-Complex $G_1(X)$ is precisely the centre of π.

Given any sequence of groups $(\pi_k)_{k \geqq 1}$ with π_k aberian for $k \geqq 2$ it is known that there exists a 0-connected CW-Complex X with $\pi_k(X) \cong \pi_k$ for all k. A natural question that suggests itself is the following:

Given a sequence of groups $(\pi_k)_{k \geqq 1}$ with π_k abelian for $k \geqq 2$ and subgroups G_k of π_k under what conditions does there exist a 0-connected CW-Complex with $\pi_k(X) \cong \pi_k$ under isomorphisms carrying $G_k(X)$ onto G_k?

Though we do not attempt to solve this general problem in this paper, we prove the following

THEOREM. *Given any group π and a subgroup G of the centre of π there exists a 0-connected CW-Complex X with $\pi_1(X) \cong \pi$ under an isomorphism carrying $G_1(X)$ onto G.*

Finally I wish to thank Professor W. Browder for some very profitable discussions I had with him in connection with this problem.

1. Discrete group actions. This section deals with some results that we need regarding the action of the fundamental group on the higher homotopy groups of a space. As we could not find any explicit reference we felt we should include them here. But before dealing with these results we recall briefly how the action of the fundamental group $\pi_1(X, x_0)$ on the homotopy group $\pi_n(X, x_0)$ is defined.

Let $\sigma: I \to X$ be any path in X with $\alpha(0) = x_0$ and $\alpha(1) = x_1$. Let $c \in \pi_n(X, x_1)$ be represented by $f: (S^n, *) \to (X, x_1)$ where $*$ denotes the base point in S^n. Let $h_\sigma: * \times I \cup S^n \times 1 \to X$ be given by $h_\sigma(*, t) = \sigma(t)$ for all $t \in I$ and $h_\sigma(z, 1) = f(z)$ for all $z \in S^n$. Then the isomorphism $\sigma_\sharp: \pi_n(X, x_1) \to \pi_n(X, x_0)$ carries c into the element of $\pi_n(X, x_0)$ represented by $g: (S^n, *) \to (X, x_0)$ where $g(z) = F(z, 0)$ with $F: S^n \times I \to X$ any extension of h_σ. It is known that $\sigma_\sharp$ depends only on the homotopy class $[\sigma]$ of the path σ and that if σ, τ are paths in X with $\sigma(0) = x_0$, $\sigma(1) = x_1 = \tau(0)$ and $\tau(1) = x_2$ then $\sigma \cdot \tau_\sharp: \pi_n(X, x_2) \to \pi_n(X, x_0)$ is the same as the composite $\sigma_\sharp \circ \tau_\sharp$. In particular the assignment $a \cdot c = \sigma_\sharp(c)$ for every $a \in \pi_1(X, x_0)$ and $c \in \pi_n(X, x_0)$ where σ is any loop at x_0 representing a, gives rise to an action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$.

Let π be a group and X, Y spaces on which π acts on the left. As usual we define an action of π on $X \times Y$ by $a(x, y) = (ax, ay)$ for any $a \in \pi$ and $(x, y) \in X \times Y$. The quotient space of $X \times Y$ under this action of π will be denoted by $X \times_\pi Y$. We recall the following.

DEFINITION 1.1. π is said to act properly discontinuously X if given $x \in X$ there exists an open set U of X with $x \in U$ and satisfying the following condition:

$$a, a' \in \ ; aU \cap a'U \neq \emptyset \Longrightarrow a = a' .$$

LEMMA. 1.2. *Suppose the action of π on X is properly discontinuous. Then for any action of π on Y the action of π on $X \times Y$ described above is properly discontinuous.*

Proof. Let $(x, y) \in X \times Y$. Let U be an open set in X with $z \in U$ and satisfying the requirement in Definition 1.1. Then $V = U \times Y$ is open in $X \times Y$; $(x, y) \in V$ and $aV = aU \times aY = aU \times Y$ for any $a \in \pi$. Hence $aV \cap a'V \neq \emptyset \Rightarrow aU \cap a'U \neq \emptyset \Rightarrow a = a'$.

For the rest of this section X will denote a 0-connected space admitting of a simply connected covering space $\widetilde{X} \xrightarrow{p} X$. The Deck transformation group of the covering $\widetilde{X} \xrightarrow{p} X$ will be denoted by $\mathscr{D}(X)$. Let $\tilde{x}_0$ be a chosen base point of $\widetilde{X}$ and $x_0 =$

$p(\tilde{x}_0)$. As is well-known $\mathscr{D}(X)$ is isomorphic to $\pi_1(X, x_0)$. A specific isomorphism

$$(1) \qquad \lambda_X: \mathscr{D}(X) \longrightarrow \pi_1(X, x_0)$$

is got as follows. For any $a \in \mathscr{D}(X)$ let $\tilde{\alpha}_a$ be any path in $\tilde{X}$ joining $\tilde{x}_0$ to $a\tilde{x}_0$. Then $p \circ \tilde{\alpha}_a = \alpha_a$ is a loop at x_0 in X and $\lambda_X(a) =$ the homotopy class $[\alpha_a]$ of the loop α_a. This isomorphism λ_X definitely depends on the choice of the base point $\tilde{x}_0$ in $\tilde{X}$.

Let us denote the group $\mathscr{D}(X)$ by π. Let Y be a 1-connected space and y_0 a base point in Y. Assume that the group π acts on the left on Y further satisfying the condition $ay_0 = y_0$ for all $a \in \pi$. By Lemma 1.2 the action of π on $\tilde{X} \times Y$ is properly discontinuous. Denote the quotient space $\tilde{X} \times_\pi Y$ by Q and the canonical quotient map $\tilde{X} \times Y \rightarrow Q$ by q. Since $\tilde{X} \times Y$ is simply connected and the action of π on $\tilde{X} \times Y$ is properly discontinuous it follows that $q: \tilde{X} \times Y \rightarrow Q$ is the universal covering of Q and π is the Deck transformation group of the covering $q: \tilde{X} \times Y \rightarrow Q$. Let $q(\tilde{x}_0, y_0) = u_0$. Using $(\tilde{x}_0, y_0)$ as the base point in $\tilde{X} \times Y$ as described in the previous paragraph we have an isomorphism

$$(2) \qquad \lambda_Q: \pi \longrightarrow \pi_1(Q, u_0)$$

Let now n be an integer $\geqq 2$ and let us assume $\pi_n(X) = 0$. Let $j: Y \rightarrow \tilde{X} \times Y$ be the inclusion given by $j(y) = (\tilde{x}_0, y)$. From the fact that $\pi_n(\tilde{X}) \cong \pi_n(X) = 0$ we see that

$$j_*: \pi_n(Y, y_0) \longrightarrow \pi_n(\tilde{X} \times Y, (\tilde{x}_0, y_0))$$

is an isomorphism. Since $q: \tilde{X} \times Y \rightarrow Q$ is the universal covering of Q the map $q_*: \pi_n(\tilde{X} \times Y, (\tilde{x}_0, y_0)) \rightarrow \pi_n(Q, u_0)$ is an isomorphism. Denoting the composite $q \circ j$ by q' we thus see that $q'_*: \pi_n(Y, y_0) \rightarrow \pi_n(Q, u_0)$ is an isomorphism.

For any $v \in \pi_1(Q, u_0)$ and $w \in \pi_n(Q, u_0)$ we denote by $v \cdot w$ the element in $\pi_n(Q, u_0)$ got by the action of v on w under the usual action of $\pi_1(Q, u_0)$ on $\pi_n(Q, u_0)$. Since $ay_0 = y_0$ for any $a \in \pi$ the action of π on Y gives rise to an action of π on $\pi_n(Y, y_0)$. For any $a \in \pi$ and $c \in \pi_n(Y, y_0)$ we denote the element of $\pi_n(Y, y_0)$ got under this action by $a * c$. The following proposition assets that under the isomorphisms $\lambda_Q: \pi \rightarrow \pi_1(Q, u_0)$ and $q'_*: \pi_n(Y, y_0) \rightarrow \pi_n(Q, u_0)$ these two actions correspond. More precisely we have

PROPOSITION 1.3. *$q'_*(a * c) = \lambda_Q(a) \cdot q'_*(c)$ for all $a \in \pi$ and $c \in \pi_n(Y, y_0)$.*

During the course of the proof of this proposition we will make use of the following well known

LEMMA 1.4. *Let* $\varphi\colon A \to B$ *be a continuous map of topological spaces. Let* σ *be any path in* A *with* $\sigma(0) = a_0$, $\sigma(1) = a_1$. *Let* $b_i = \varphi(a_i)$ $(i = 0, 1)$. *Then the following diagram is commutative*

$$\begin{array}{ccc} \pi_n(A, a_1) & \xrightarrow{\sigma_\#} & \pi_n(A, a_0) \\ \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \varphi_*} \\ \pi_n(B, b_1) & \xrightarrow{\varphi\circ\sigma_\#} & \pi_n(B, b_0) \end{array}$$

DIAGRAM 1

Proof of Proposition 1.3. Let $f\colon (S^n, *) \to (Y, y_0)$ represent $c \in \pi_n(Y, y_0)$. Let $a \in \pi$ choose any path $\tilde{\theta}\colon I \to \tilde{X}$ joining $\tilde{x}_0$ to $a\tilde{x}_0$ in $\tilde{X}$. Then $\tilde{\alpha}_a\colon I \to \tilde{X} \times Y$ defined by $\tilde{\alpha}_a(t) = (\tilde{\theta}(t), y_0)$ is a path in $\tilde{X} \times Y$ joining $\tilde{\alpha}_a(0) = (\tilde{x}_0, y_0)$ to $\tilde{\alpha}_a(1) = (a\tilde{x}_0, y_0) = (a\tilde{x}_0, ay_0) = a(\tilde{x}_0, y_0)$. By the definition of λ_Q we have $\lambda_Q(a) = [\alpha_a]$ where $\alpha_a = q \circ \tilde{\alpha}_a$. By Lemma 1.4 we see that

$$\begin{array}{ccc} \pi_n(\tilde{X} \times Y, (a\tilde{x}_0, y_0)) & \xrightarrow[\cong]{\tilde{\alpha}_{a\#}} & \pi_n(\tilde{X} \times Y, (\tilde{x}_0, y_0)) \\ \downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_*} \\ \pi_n(Q, u_0) & \xrightarrow[\cong]{\alpha_{a\#}} & \pi_n(Q, u_0) \end{array}$$

DIAGRAM 2

is a commutative diagram. Since $\lambda_Q(a) = [\alpha_a]$, by definition of the action $\pi_1(Q, u_0)$ on $\pi_n(Q, u_0)$ we have

$$\lambda_Q(a) \cdot q'_*(c) = \alpha_{a\#}(q'_*(c)) \,. \tag{3}$$

The element $q'_*(c) \in \pi_n(Q, u_0)$ is represented by the map $q' \circ f\colon (S^n, *) \to (Q, u_0)$. Denote by d the element in $\pi_n(\tilde{X} \times Y, \tilde{x}_0 \times y_0)$ represented by the map $h\colon (S^n, *) \to (\tilde{X} \times Y, \tilde{x}_0 \times y_0)$ given by $h(z) = (a\tilde{x}_0, af(z))$ for all $z \in S^n$. Then

$$q \circ h(z) = q(a\tilde{x}_0, af(z)) = q(\tilde{x}_0, f(z)) = q \circ j(f(z)) = q' \circ f(z) \quad \text{for all } z \in S^n \,.$$

$q \circ h$ represents $q_*(d)$ in $\pi_n(Q, u_0)$ and $q' \circ f$ represents $q'_*(c)$ in $\pi_n(Q, u_0)$. Since $q \circ h = q' \circ f$ we have

$$q'_*(c) = q_*(d) \,. \tag{4}$$

Consider the map $H\colon S^n \times I \to \tilde{X} \times Y$ given by $H(z, t) = (\tilde{\theta}(t), af(z))$. We have

$$H(*, t) = (\tilde{\theta}(t), ay_0) = (\tilde{\theta}(t), y_0) = \tilde{\alpha}_a(t)$$

and

$$H(z, 1) = (\tilde{\theta}(1), af(z)) = (a\tilde{x}_0, af(z)) = h(z) .$$

Hence $\tilde{\alpha}_{a\sharp}(d) \in \pi_n(X \times Y, (\tilde{x}_0, y_0))$ is represented by $g: (S^n, *) \to (X \times Y, (\tilde{x}_0, y_0))$ where $g(z) = H(z, 0)$. Now $H(z, 0) = (\tilde{\theta}(0), af(z)) = \tilde{x}_0, af(z))$. Hence

$$(5) \qquad g(z) = (\tilde{x}_0, af(z)) .$$

The element $a * c$ of $\pi_n(Y, y_0)$ is represented by $l: (S^n, *) \to (Y, y_0)$ where $l(z) = af(z)$. Hence $q'_*(a * c)$ is represented by $q' \circ l$. But

$$q' \circ l(z) = q(\tilde{x}_0, af(z)) = q \circ g(z) \qquad \text{by } (5) .$$

Since g represents $\tilde{\alpha}_{a\sharp}(d)$ it follows that $q \circ g$ represents $q_*(\tilde{\alpha}_{a\sharp}(d))$ in $\pi_n(Q, u_0)$. From $q' \circ l = q \circ g$ we immediately get $q'_*(a * c) = q_*(\tilde{\alpha}_{a\sharp}(d))$. But $q_*(\tilde{\alpha}_{a\sharp}(d)) = \alpha_{a\sharp}(q_*(d))$ by commutativity of Diagram 2. Hence

$$\begin{aligned} q'_*(a * c) &= \alpha_{a\sharp}(q_*(d)) \\ &= \alpha_{a\sharp}(q'_*(c)) && \text{by } (4) \\ &= \lambda_Q(a) \cdot q'_*(c) && \text{by } (3) . \end{aligned}$$

This completes the proof of Proposition 1.3

2. Study of $U(\pi) \times |K(M, n)|$ for a π-module M. Let π be a a given group and K a $K(\pi, 1)$ CW-complex. The universal covering $\tilde{K}$ of K is a contractible CW-complex and the Deck transformation group of the covering $\tilde{K} \xrightarrow{p} K$ is π. We will denote the contractible complex $\tilde{K}$ by $U(\pi)$.

Let M be any left π-module and n any integer $\geqq 2$. Consider the Eilenberg-Maclane semi-simplicial complex $K(M, n)$. The action of π on M gives rise to an action of π on the semi-simplicial complex $K(M, n)$. This in turn gives rise to an action of π on the geometric realization $|K(M, n)|$ (Milnor's geometric realization [5]). We would like to apply the results of § 1 to the case when $X = K$ and $Y = |K(M, n)|$. For that purpose we should make sure that there exists a base point $y_0 \in Y$ such that $a \cdot y_0 = y_0$ for all $a \in \pi$. For this purpose we briefly recall the definition of the semi-simplicial complex $K(M, n)$.

For any integer $k \geqq 0$ let Δ_k be the set consisting of the integers j such that $0 \leqq j \leqq k$. Let $C^n(\Delta_k, M)$ denote the set of "n-cochains" of Δ_k in M, namely functions $\varphi: \Delta_k \to M$. For any integer $k \geqq 1$ let $\varepsilon_i: \Delta_{k-1} \to \Delta_k$ for $0 \leqq i \leqq k$ and $s_i: \Delta_k \to \Delta_{k-1}$ for $0 \leqq i \leqq k-1$ be given by

$$\varepsilon_i(\mu) = \begin{cases} \mu & \text{for } 0 \leqq \mu \leqq i-1 \\ \mu+1 & \text{for } i \leqq \mu \leqq k-1 \end{cases}$$

$$s_i(\mu) = \begin{cases} \mu & \text{for } 0 \leqq \mu \leqq i \\ \mu-1 & \text{for } i+1 \leqq \mu \leqq k\,. \end{cases}$$

Let $\varepsilon_i^*: C^n(\Delta_k, M) \to C^n(\Delta_{k-1}, M)$ and $s_i^*: C^n(\Delta_{k-1}, M) \to C^n(\Delta_k, M)$ be induced by ε_i and s_i. More specifically

for any $v_0, v_1, \cdots, v_n$ in Δ_{k-1} and $\varphi \in C^n(\Delta_k, M)$

$$\varepsilon_i^*\varphi(v_0, \cdots, v_n) = \varphi(\varepsilon_i(v_0), \varepsilon_i(v_0), \varepsilon_i(v_1), \cdots, \varepsilon_i(v_n))$$

and

for any $x_0, x_1, \cdots, x_n$ in Δ_k and $\theta \in C^n(\Delta_{k-1}, M)$

$$s_i^*\theta(x_0, \cdots, x_n) = \theta(s_i(x_0), s_i(x_1), \cdots, s_i(x_n))\,.$$

Let $Z^n(\Delta_k, M)$ be the subset of $C^n(\Delta_k, M)$ consisting of element φ satisfying the following two conditions:

(i) $\varphi(x_0, \cdots, x_n) = 0$ if $x_0, \cdots, x_n$ are elements of Δ_k not all distinct

(ii) $\sum_{i=0}^{n+1} (-1)^i \varphi(x_0, \cdots, \hat{x}_i, \cdots, x_{n+1}) = 0$ for any $x_0, x_1, \cdots x_{n+1}$ in Δ_k. It is known that

$$\left.\begin{array}{l} \varepsilon_i^*(Z^n(\Delta_k, M)) \subset Z^n(\Delta_{k-1}, M) \\ s_i^*(Z^n(\Delta_{k-1}, M) \subset Z^n(\Delta_k, M) \end{array}\right\} \text{ for any } k \geqq 1\,.$$

For $k \geqq 0$ the set of k-simplices $K_k(M, n)$ of the semi-simplicial complex $K(M, n)$ is the same as $Z^n(\Delta_k, M)$. For any $k \geqq 1$ the face maps $K_k(M, n) \to K_{k-1}(M, n)$ and the degeneracy maps $K_{k-1}(M, n) \to K_k(M, n)$ are given by ε_i^* for $0 \leqq i \leqq k$ and s_i^* for $0 \leqq i \leqq k-1$ respectively. It is clear that the set $K_k(M, n)$ for any k in $0 \leqq k \leqq n-1$ has only one element, namely the zero element of $C^n(\Delta_k, M)$.

The action of π on M gives rise to an action of π on $K(M, n)$ as follows. For any $\varphi \in K_k(M, n)$ let $a\varphi$ be the function $\Delta_k^{n+1} \to M$ defined by

$$(a\varphi)(x_0, \cdots, x_n) = a\{\varphi(x_0, \cdots, x_n)\}\,.$$

Then it is easy to see that $a\varphi \in K_k(M, n)$ and that the map

$$\beta_a: K(M, n) \longrightarrow K(M, n)$$

given by $\beta_a(\varphi) = a\varphi$ for any $\varphi \in K_k(M, n)$ is a semi-simplicial isomorphism. Moreover for any a, a' in π it is clear that $\beta_{aa'} = \beta_a \circ \beta_{a'}$. If we denote the only 0-simplex of $K(M, n)$ by e_0 clearly $\beta_a(e_0) = e_0$ for all $a \in \pi$. If we take $Y = |K(M, n)|$, $y_0 = |e_0|$ and define $ay = |\beta_a|(y)$ for any $y \in Y$ it is clear that we get an action of π on Y

satisfying $a \cdot y_0 = y_0$ for all $a \in \pi$. The resulting action of π on $\pi_n(|K(M, n)|, y_0) = M$ agrees with the original π-module structure on M that we started with.

Let $\tilde{x}_0 \in U(\pi)$, $Q := U(\pi) \times_\pi Y$, $q: U(\pi) \times Y \to Q$ the canonical quotient map and $u_0 = q(\tilde{x}_0, y_0)$. Let $\lambda_Q: \pi \to \pi_1(Q, u_0)$ be the isomorphism given by (2). Let $H = \{a \in \pi \mid am = m$ for all $m \in M\}$

$$H_1 = \{v \in \pi_1(Q, u_0) \mid v \cdot w = w \text{ for all } w \in \pi_n(Q, u_0)\} .$$

As an immediate consequence of Proposition 1.3 we get the following

PROPOSITION 2.1. *The isomorphism $\lambda_Q: \pi \to \pi_1(Q, u_0)$ carries the subgroup H of π onto the subgroup H_1 of $\pi_1(Q, u_0)$.*

3. A subgroup of the Deck transformation group. Let X be 0-connected topological space and $\tilde{X} \xrightarrow{p} X$ a covering space of X ($\tilde{X}$ not necessarily simply connected). Let us denote the Deck transformation group of the covering $\tilde{X} \xrightarrow{p} X$ by $\mathscr{D}$. Recall the usual

DEFINITION 3.1. A homotopy $F: \tilde{X} \times I \to \tilde{X}$ is said to be fibre preserving if there exists a homotopy $H: X \times I \to X$ such that

$$\begin{array}{ccc} \tilde{X} \times I & \xrightarrow{F} & \tilde{X} \\ \downarrow{\scriptstyle p \times Id} & & \downarrow{\scriptstyle p} \\ X \times I & \xrightarrow{H} & X \end{array}$$

DIAGRAM 3

is commutative.

Let $\mathscr{G}$ denote the set of Deck transformations of $\tilde{X}$ which are homotopic to $Id_{\tilde{X}}$ through fibre preserving homotopies.

LEMMA 3.2. *$\mathscr{G}$ is a subgroup of $\mathscr{D}$.*

Proof. Let a, b be arbitrary elements of $\mathscr{G}$. Let

$$\begin{array}{ccc} \tilde{X} \times I & \xrightarrow{F} & \tilde{X} \\ \downarrow{\scriptstyle p \times Id} & & \downarrow{\scriptstyle p} \\ X \times I & \xrightarrow{H} & X \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \tilde{X} \times I & \xrightarrow{F'} & \tilde{X} \\ \downarrow{\scriptstyle p \times Id} & & \downarrow{\scriptstyle p} \\ X \times I & \xrightarrow{H'} & X \end{array}$$

DIAGRAM 4

be fibre preserving homotopies satisfying

$$\left.\begin{array}{l} F(\tilde{x}, 0) = \tilde{x},\ F(\tilde{x}, 1) = a\tilde{x} \\ F'(\tilde{x}, 0) = \tilde{x},\ F'(\tilde{x}, 1) = b\tilde{x} \end{array}\right\} \text{ for all } \tilde{x} \in \tilde{X} .$$

It follows that

$$\left.\begin{array}{l} H(x, 0) = x = H(x, 1) \\ H'(x, 0) = x = H'(x, 1) \end{array}\right\} \text{ for all } x \in X .$$

Let $F''\colon \tilde{X} \times I \to \tilde{X}$ and $H''\colon X \times I \to X$ be given by

$$F''(\tilde{x}, t) = \begin{cases} F'(\tilde{x}, 2t) & \text{for } 0 \leqq t \leqq 1/2 \\ F(b\tilde{x}, 2t - 1) & \text{for } 1/2 \leqq t \leqq 1 \end{cases}$$

$$H''(x, t) = \begin{cases} H'(x, 2t) & \text{for } 0 \leqq t \leqq 1/2 \\ H(x, 2t - 1) & \text{for } 1/2 \leqq t \leqq 1 . \end{cases}$$

Then $F''(\tilde{x}, 0) = \tilde{x}$, $F''(\tilde{x}, 1) = ab\tilde{x}$ and

$$\begin{CD} \tilde{X} \times I @>{F''}>> \tilde{X} \\ @V{p \times Id}VV @VV{p}V \\ X \times I @>{H''}>> X \end{CD}$$

DIAGRAM 5

is commutative. Hence $ab \in \mathscr{G}$.

Let $A\colon \tilde{X} \times I \to \tilde{X}$ and $B\colon X \times I \to X$ be defined by

$$A(\tilde{x}, t) = F(a^{-1}\tilde{x}, 1 - t);\ B(x, t) = H(x, 1 - t)$$

Then $A(\tilde{x}, 0) = \tilde{x}$, $A(\tilde{x}, 1) = a^{-1}\tilde{x}$ and

$$\begin{CD} \tilde{X} \times I @>{A}>> \tilde{X} \\ @V{p \times Id}VV @VV{p}V \\ X \times I @>{B}>> X \end{CD}$$

DIAGRAM 6

is commutative. Hence $a^{-1} \in \mathscr{G}$.

Thus for any a, b in $\mathscr{G}$ the elements ab and a^{-1} are in $\mathscr{G}$. This proves that $\mathscr{G}$ is a subgroup of $\mathscr{D}$.

Let $S(X)$ denote the singular $C.S.S.$ complex of X and $|S(X)|$ the geometric realization of $S(X)$. Let $j_X\colon |S(X)| \to X$ be the canonical map [5]. Let

$$\begin{CD} E @>{r}>> \tilde{X} \\ @V{p'}VV @VV{p}V \\ |S(X)| @>>{j_X}> X \end{CD}$$

DIAGRAM 7

denote the pull-back of the covering space $\tilde{X} \xrightarrow{p} X$. Then $E \xrightarrow{p'} |S(X)|$ is a covering space with the same Deck transformation group $\mathscr{D}$ as $\tilde{X} \xrightarrow{p} X$. Let $\mathscr{G}'$ be the subgroup of those Deck transformations of E which are homotopic to Id_E through fibre preserving homotopies.

THEOREM 3.1. $\mathscr{G} \subset \mathscr{G}'$.

For the proof of this theorem we need the following

LEMMA 3.3. *Let A, B be topological spaces and h_0, h_1 homotopic maps of A into B under a homotopy $H: A \times I \to B$. Then there exists a homotopy $P: |S(A)| \times I \to |S(B)|$ between $|S(h_0)|$ and $|S(h_1)|$ such that*

$$\begin{array}{ccc} |S(A)| \times I & \xrightarrow{P} & |S(B)| \\ \downarrow{\scriptstyle j_A \times Id} & & \downarrow{\scriptstyle j_B} \\ A \times I & \xrightarrow{H} & B \end{array}$$

DIAGRAM 8

is comutative.

Proof. Let Δ_1 be the $C.S.S.$ complex whose p-simplices are $(p+1)$ tuples $(a_0, \cdots, a_p)$ with $a_i = 0$ or 1, the face and degeneracy operations being the usual ones.

$$\varepsilon_i(a_0, \cdots, a_p) = (a_0, \cdots, \hat{a}_i, \cdots, a_p)$$
$$s_i(a_0, \cdots, a_{p-1}) = (a_0, \cdots, a_{i-1}, a_i, a_{i+1}, \cdots, a_{p-1}) .$$

Let $|\Delta_p|$ denote the standard Euclidean p-simplex in $\boldsymbol{R}^{p+1}$ with the usual unit vectors $e_0, \cdots, e_p$ as its vertices. For any p-simplex $\sigma = (a_0, \cdots, a_p)$ of $\underset{\sim}{\Delta}_1$ let $\theta_\sigma: |\Delta_p| \to I$ be the simplicial map determined by $\theta_\sigma(e_i) = a_i$. Let $\iota: S(A) \times \underset{\sim}{\Delta}_1 \to S(A \times I)$ be defined as follows. For any $\varphi \in S_p(A)$ and any p-simplex σ of $\underset{\sim}{\Delta}_1$ let $\iota(\varphi \times \sigma)$ be that singular p-simplex in $A \times I$ which satisfies $\iota(\varphi \times \sigma)(x) = (\varphi(x), \theta_0(x))$ for all $x \in |\Delta_p|$. Then $\iota: S(A) \times \underset{\sim}{\Delta}_1 \to S(A \times I)$ is a $C.S.S.$ inclusion. It is easily checked that $P = |S(H)| \circ |\iota|$ satisfies the requirements of the Lemma.

Proof of Theorem 3.1. Observe that

$$E = \{(\tilde{x}, w) \in \tilde{X}\,|S(X)|\,|\,p(\tilde{x}) = j_X(w)\}$$

and $p'(\tilde{x}, w) = w$. For any $a \in \mathscr{D}$ the action of a on E is given by $a(\tilde{x}, w) = (a\tilde{x}, w)$.

Let now $b \in \mathscr{G}$. Then there exists a commutative diagram

$$\begin{array}{ccc} \tilde{X} \times I & \xrightarrow{F} & \tilde{X} \\ \Big\downarrow{\scriptstyle p \times Id} & & \Big\downarrow{\scriptstyle p} \\ X \times I & \xrightarrow{H} & X \end{array}$$

DIAGRAM 9

with $F(\tilde{x}, 0) = \tilde{x}$ and $F(\tilde{x}, 1) = b\tilde{x}$. Then $H(x, 0) = x = H(x, 1)$ for all $x \in X$. By Lemma 3.3 there exists a homotopy $P: |S(X)| \times I \to |S(X)|$ between $|S(Id_X)|$ and $|S(Id_X)|$ such that

$$\begin{array}{ccc} |S(X)| \times I & \xrightarrow{P} & |S(X)| \\ \Big\downarrow{\scriptstyle j_X \times Id} & & \Big\downarrow{\scriptstyle j_X} \\ X \times I & \xrightarrow{H} & X \end{array}$$

DIAGRAM 10

is commutative. But $|S(Id_X)| = Id_{|S(X)|}$. Hence $P(w, 0) = w = P(w, 1)$ for all $w \in |S(X)|$.

Let $L: E \times I \to E$ be defined by $L((\tilde{x}, w), t) = (F(\tilde{x}, t), P(w, t))$. To make sure that $L((\tilde{x}, w), t) \in E$ we have to check that $j_X(P(w, t)) = p(F(\tilde{x}, t))$. But commutativity of diagram 10 yields $j_X(P(w, t)) = H(j_X(w), t)$. Since $(\tilde{x}, w) \in E$ we get $j_X(w) = p(\tilde{x})$. Hence

$$\begin{aligned} H(j_X(w), t) &= H(p(\tilde{x}), t) \\ &= p(F(\tilde{x}, t)) \quad \text{by commutativity of Diagram 9}. \end{aligned}$$

Hence

$$j_X(P(w, t)) = H(j_X(w), t) = p(F(\tilde{x}, t)).$$

It is easy to see that

$$\begin{array}{ccc} E \times I & \xrightarrow{L} & E \\ \Big\downarrow{\scriptstyle p' \times Id} & & \Big\downarrow{\scriptstyle p'} \\ |S(X)| \times I & \xrightarrow[P]{} & |S(X)| \end{array}$$

DIAGRAM 11

is commutative. In fact

$$P(p'(\tilde{x}, w), t) = P(w, t)$$

and

$$p' \circ L((\tilde{x}, w), t) = p'(F(\tilde{x}, t), P(w, t)) = P(w, t)$$

Moreover

$$L((\tilde{x}, w), 0) = (F(\tilde{x}, 0), P(w, 0)) = (\tilde{x}, w)$$
$$L((\tilde{x}, w), 1) = (F(\tilde{x}, 1), P(w, 1)) = (b\tilde{x}, w) = b(\tilde{x}, w)$$

This proves that $b \in \mathscr{G}'$. Since b is arbitrary in $\mathscr{G}$ we get $\mathscr{G} \subset \mathscr{G}'$.

4. The main result. As stated in the introduction the main result proved in this paper is:

THEOREM 4.1. *Let π be any group and G any subgroup of the centre π. Then there exists a 0-connected CW-Complex X such that $\pi_1(X) \cong \pi$ under an isomorphism carrying the Gottlieb subgroup $G_1(X)$ of $\pi_1(X)$ onto G.*

Let M be any nonzero free left π/G-module, for instance $M = Z(\pi/G)$ the integral group ring of π/G. Using the canonical quotient map $\eta: \pi \to \pi/G$ we consider M as a left π-module. More specifically for any $a \in \pi$ and any $m \in M$ we set $am = \eta(a)m$. Then it is clear that $G = \{a \in \pi \mid am = m \text{ for all } m \in M\}$. Consider the covering space $U(\pi) \times Y \xrightarrow{q} Q$ where $Y = |K(M, n)|$ (with n an integer $\geqq 2$) constructed in § 2. Since $a \cdot m = m$ for $a \in G$ and all $m \in M$ the isomorphism $\beta_a: K(M, n) \to K(M, n)$ reduces to the Identity of $K(M, n)$ whenever $a \in G$. Hence $|\beta_a| = Id_Y$. It follows that $ay = y$ for all $y \in Y$ whenever $a \in G$. For the covering space $U(\pi) \times Y \xrightarrow{q} Q$ let Γ be the subgroup of π consisting of those Deck transformations which are homotopic to $Id_{U(\pi)\times Y}$ through fibre preserving homotopies. As a first step towards proving Theorem 4.1 we show

PROPOSITION 4.2. $G \subset \Gamma$.

For the proof of this proposition we need the following well known results. Let A be a 0-connected space and $\tilde{A} \xrightarrow{p} A$ a covering space of A. Let $\mathscr{D}$ be the Deck transformation group of this covering.

PROPOSITION 4.3. *A homotopy $F: \tilde{A} \times I \to \tilde{A}$ is fibre preserving if and only if $aF(\tilde{x}, t) = F(a\tilde{x}, t)$ for all $(\tilde{x}, t) \in \tilde{A} \times I$ and all $a \in \mathscr{D}$.*

This is Theorem II.2 of [2]. The proof given there is valid for a covering space of any 0-connected space A. (A need not be a CW-complex).

PROPOSITION 4.4. *If A is a CW-complex and $\tilde{A} \xrightarrow{p} A$ the universal covering of A, the subgroup of $\mathscr{D}(A)$ which corresponds to the*

Gottlieb group $G_1(A, a0)$ *under the isomorphism* $\mathscr{D}(A) \to \pi_1(A, a0)$ *consists precisely of all those Deck transformations which are homotopic to* $Id_{\tilde{A}}$ *under fibre preserving homotopics.*

This is Theorem II.1 of [2].

PROPOSITION 4.5. *If* A *is a* $K(\pi, 1)$ *CW-complex then* $G_1(A)$ *is the centre of* π.

This is Corollary I.13 of [2].

Proof of Proposition 4.2. Since G is in the centre of π and by Proposition 4.5 $G_1(K(\pi, 1)) =$ the centre of π we see that $G \subset G_1(K(\pi, 1))$. By Proposition 4.4, for any $b \in G$ the Deck transformation $\tilde{x} \to b\tilde{x}$ of $U(\pi) \xrightarrow{p} K(\pi, 1)$ is homotopic to $Id_{U(\pi)}$ by fibre preserving homotopy $F: U(\pi) \times I \to U(\pi)$. By Proposition 4.3 we have $aF(\tilde{x}, t) = F(a\tilde{x}, t)$ for every $a \in \pi$ and $(\tilde{x}, t) \in U(\pi) \times I$.

Consider the homotopy $L: U(\pi) \times Y \times I \to U(\pi) \times Y$ given by $L((\tilde{x}, y), t) = (F(\tilde{x}, t), y)$. We have $L(\tilde{x}, y), 0) = (F(\tilde{x}, 0), y) = (\tilde{x}, y)$ and $(L(\tilde{x}, y), 1) = (F(\tilde{x}, 1), y) = (b\tilde{x}, y) = (b\tilde{x}, by)$ since $by = y$ for any $y \in Y$ whenever $b \in G$. Hence $L((\tilde{x}, y), 1) = (b\tilde{x}, by) = b(\tilde{x}, y)$.
Also if $a \in \pi$ we have

$$\begin{aligned} aL((\tilde{x}, y), t) &= a(F(\tilde{x}, t), y) = (aF(\tilde{x}, t), ay) = (F(a\tilde{x}, t), ay) \\ &= L((a\tilde{x}, ay), t) = L(a(\tilde{x}, y), t) . \end{aligned}$$

It now follows from Proposition 4.3 applied to the covering $U(\pi) \times Y \xrightarrow{q} Q$ that $L: U(\pi) \times Y \times I \to U(\pi) \times Y$ is a fibre preserving homotopy. Since $L((\tilde{x}, y), 0) = ((\tilde{x}, y), 0)$ and $L((\tilde{x}, y), 1) = b(\tilde{x}, y)$ we see that $b \in \Gamma$ and b is arbitrary in G. Hence $G \subset \Gamma$.

Proof of Theorem 4.1. Consider the 0-connected space $Q = U(\pi) \times_\pi |K(M, n)|$ where $n \geqq 2$ and M a nonzero free π/G-module converted into a π-module using the natural homomorphism $\eta: \pi \to \pi/G$. Since $G = \{a \in \pi \mid am = m \text{ for all } m \in M\}$ by Proposition 2.1 we get

$$\lambda_Q(G) = H_1 = \{v \in \pi_1(Q, u_0) \mid v \cdot w = \text{ for all } w \in \pi_n(Q, u_0)\} \tag{6}$$

If Q happens to be CW-complex Propositions 4.2 and 4.4 immediately yield

$$\lambda_Q(G) \subset G_1(Q, u_0) \tag{7}$$

We now recall the following

PROPOSITION 4.6. *Let X be a CW-complex and $x_0 \in X$.*

Let $P(X, x_0) = \{v \in \pi_1(X, x_0) \mid v \cdot w = w$ for all $w \in \pi_n(X, x_0)$ and all $n \geqq 1\}$. Then $G_1(X, x_0) \subset P(X, x_0)$.

This is Proposition I.4 of [2]

It follows from (6) and Proposition 4.6 that if Q happens to be a CW-complex

$$G_1(Q, u_0) \subset \lambda_Q(G) \tag{8}$$

(7) and (8) yield $\lambda_Q(G) = G_1(Q, u_0)$. Thus if Q happens to be a CW-complex $X = Q$ will satisfy the requirements of Theorem 4.1. Even though $U(\pi)$ and $|K(M, n)|$ are CW-complexes the product $U(\pi) \times |K(M, n)|$ with the product topology need not be. Even if it is (or even if we alter the topology to the weak topology and get a CW-structure on the set $U(\pi) \times |K(M, n)|$) there is no guarantee that Q will be CW. However it is possible to rectify the situation. It turns out that $X = |S(Q)|$ satisfies the requirements of Theorem 4.1.

First of all observe that there exists a point $c_0 \in X = |S(Q)|$ such that $j_Q(c_0) = u_0$. In fact if we take the 0-singular simplex α_{u_0} corresponding to the point u_0 then $c_0 = |\alpha_{u_0}| \in |S(Q)|$ satisfies $j_Q(c_0) = u_0$. It is known that $j_{Q*}: \pi_i(|S(Q)|, c_0) \to \pi_i(Q, u_0)$ is an isomorphism for all $i \geqq 1$. [5].

Let $H_1(X, c_0) = \{v \in \pi_1(X, c_0) \mid v \cdot w = w$ for all $w \in \pi_n(X, c_0)\}$. Using Lemma 1.4 and the fact that $j_{Q*}: \pi_i(X, c_0) \to \pi_i(Q, u_0)$ is an isomorphism for all i we see that

$$j_{Q*}(H_1(X, c_0)) = H_1 \ . \tag{9}$$

Let

$$\begin{array}{ccc} E & \xrightarrow{r} & U(\pi) \times Y \\ \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q} \\ |S(Q)| & \xrightarrow{j_Q} & Q \end{array}$$

DIAGRAM 12

denote the pull back of the covering space $U(\pi) \times Y \xrightarrow{q} Q$. Let Γ' be the subgroup of π consisting of those Deck transformations of the covering $E \xrightarrow{q'} |S(Q)|$ which are homotopic to I_{d_E} through fibre preserving homotopies. By Theorem 3.1 we get $\Gamma \subset \Gamma'$ We take $e_0 = ((\tilde{x}_0, y_0), c_0)$ as the base point in E and use it to get the isomorphism

$$\lambda_X: \pi \longrightarrow \pi_1(X, c_0) \ .$$

It is easy to see that

$$\begin{array}{ccc} & \pi & \\ {\scriptstyle \lambda_X} \swarrow \cong & & \cong \searrow {\scriptstyle \lambda_Q} \\ \pi_1(X, c_0) & \xrightarrow[j_{Q*}]{\cong} & \pi_1(Q, u_0) \end{array}$$

DIAGRAM 13

is a commutative diagram. In fact if $\tilde{\sigma}$ is a path in E joining $e_0 = ((\tilde{x}_0, y_0), c_0)$ to $a \cdot e_0 = (a(\tilde{x}_0, y_0), c_0)$ it is clear that $r\tilde{\sigma}$ is a path in $U(\pi) \times Y$ joining $(\tilde{x}_0, y_0)$ to $a(\tilde{x}_0, y_0)$. This fact yields commutativity of Diagram 13.

From Proposition 4.4 we have $G_1(X, c_0) = \lambda_X(\Gamma')$. Commutativity of Diagram 13, the fact that j_{Q_*} is an isomorphism together with (6) and (9) yield

$$\lambda_X(G) = H_1(X, c_0) . \tag{10}$$

From Proposition 4.6 we get $G_1(X, c_0) \subset H_1(X, c_0)$. Hence

$$G_1(X, c_0) \subset \lambda_X(G) . \tag{11}$$

Also from $G \subset \Gamma \subset \Gamma'$ we get $\lambda_X(G) \subset \lambda_X(\Gamma) \subset \lambda_X(\Gamma')$ and $\lambda_X(\Gamma') = G_1(X, c_0)$. Hence

$$\lambda_X(G) \subset G_1(X, c_0) \tag{12}$$

(11) and (12) together yield $\lambda_X(G) = G_1(X, c_0)$.

Thus $\lambda_X^{-1}: \pi_1(X, c_0) \to \pi$ is an isomorphism carrying $G_1(X, c_0)$ onto the subgroup G of π.

This completes the proof of Theorem 4.1.

Finally we end the paper by raising a question. Let

$$\begin{array}{ccc} E & \xrightarrow{r} & \tilde{A} \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ |S(A)| & \xrightarrow[j_A]{} & A \end{array}$$

DIAGRAM 14

be the pullback of a covering $\tilde{A} \xrightarrow{p} A$ of a 0-connected space A. Let $\mathscr{G}$ and $\mathscr{G}'$ be the subgroups associated to the covering $\tilde{A} \xrightarrow{p} A$ and $E \xrightarrow{p'} |S(A)|$ respectively of the Deck transformation group $\mathscr{D}$. We have proved in § 3 that $\mathscr{G} \subset \mathscr{G}'$ (Theorem 3.1).

PROBLEM. Is $\mathscr{G} = \mathscr{G}'$?

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