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ON A CHARACTERISATION OF MATRIX FUNCTIONS WHICH ARE DIFFERENCES OF TWO MONOTONE MATRIX FUNCTIONS[1]

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ABSTRACT. The class of matrix functions of 'bounded variation' was introduced by O. Dobsch in a paper published in 1937 [2]. The consideration of this class of functions immediately gives rise to the consideration of those matrix functions of order n on an interval $[a, b]$ that are representable as the difference of two monotone matrix functions on that interval. Such a difference will have high regularity properties when n is large and is therefore much more than simply a function of bounded variation. The characterization of this class was sought in the paper of Dobsch [2]. The purpose of this paper is to give a complete description of a related class: the functions defined on $(-1, 1)$ which have restrictions to any closed subinterval which are such differences.

1. **Introduction.** This paper concerns the study of the class of matrix functions of 'bounded variation' corresponding to the monotone functions introduced by Charles Loewner in a paper published in 1934 [3]. O. Dobsch tried to find a characterisation of functions given on a closed interval $[a, b]$ which were the differences of monotone matrix functions on that interval. He did not succeed in finding this characterisation, and neither have we. However, we give a complete description of the functions on an open interval which appear on any closed subinterval as the differences of two monotone matrix functions on that subinterval.

2. If A is an $n \times n$ real symmetric or Hermitian matrix, $f(A)$ is the matrix resulting from A by leaving the eigenvectors fixed while the corresponding eigenvalues λ are replaced by $f(\lambda)$. Thus, if $A = T'DT$ where T is a unitary matrix, T' its conjugate transpose, and D a diagonal matrix, then $F(A) = T'f(D)T$. The function $f(A)$ on all $n \times n$ Hermitian matrices with eigenvalues in the domain of $f(x)$ is called a matrix function of order n generated by f.

An operator function f associated with I is monotone provided

$$H \geqq 0 \Rightarrow f(A + H) \geqq f(A).$$

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When considering $n\times n$ Hermitian matrices, a monotone operator function is called a monotone matrix function of order n. A convenient summary of the results about monotone matrix functions may be found in [1]. Here we shall state a known result due to Dobsch [2].

THEOREM 1. *Let $f(x)$ be a real-valued function defined on an open interval $(-1, 1)$.*[2] *$f(x)$ is a monotone matrix function of order n in $(-1, 1)$ iff f is of class C^{2n-3}, its $(2n-3)$rd derivative is convex and the matrix*

$$M'_n(x;f) = \begin{bmatrix} f'(x) & \dfrac{f''(x)}{2!} & \cdots & \dfrac{f^{(n)}(x)}{n!} \\ \dfrac{f''(x)}{2!} & \dfrac{f'''(x)}{3!} & & \dfrac{f^{(n+1)}(x)}{(n+1)!} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \dfrac{f^{(n)}(x)}{n!} & \dfrac{f^{(n+1)}(x)}{(n+1)!} & & \dfrac{f^{(2n-1)}(x)}{(2n-1)!} \end{bmatrix}$$

(*which makes sense almost everywhere*) *is nonnegative definite.*

3. DEFINITION. Let $f(x)$ be a real-valued function defined on a closed interval $[a, b]$; $f(x)$ is called a matrix function of bounded variation if there exists a constant K such that

$$\sum_{v=1}^{p} \|f(A_v) - f(A_{v-1})\| \leqq K$$

for all partitions $aI \leqq A_0 \leqq A_1 \leqq \cdots \leqq A_p \leqq bI$.

Since the norms on the finite-dimensional space of matrices of order n are equivalent, it is not necessary to specify exactly what norm occurs in the definition. For the special case $n=1$, it is clear that the matrix functions of bounded variation coincide with the usual functions of bounded variation over $[a, b]$. An important theorem due to Dobsch [2] asserts that this is also the case for larger values of n.

Below we give a complete description of a class: the functions defined on $(-1, 1)$ whose restrictions to any closed subinterval generate matrix functions of order n which are differences of two monotone matrix functions.

[2] Throughout this paper the restriction to the interval $(-1, 1)$ is unessential. All results can easily be transformed to the case of an arbitrary open interval.

LEMMA 2. *Let C be a compact subset of the* $n\times n$ *Hermitian matrices such that every element of C has its spectrum in* $(-1, 1)$. *Let* $h(x)=\sqrt{(x+2)}$; *then there exists a* $K>0$ *so that the matrix*

$$A + KM_n'(x; h)$$

is positive definite for every A in C and every x in $(-1, 1)$.

PROOF. Let $\|A\|$ be a norm on the space of matrices which is at least as large as the usual operator norm so that the spectrum of A is contained in the interval $(-\|A\|, \|A\|)$. This norm is clearly a continuous function on the compact C and bounded there by some positive μ. It follows that $-\mu I\leqq A\leqq \mu I$ for every A in C.

Consider next the family of matrices $M_n'(x; h)$ as x varies over the closed interval $[-1, 1]$. This is evidently a compact set of matrices depending continuously on x. Every matrix in the set is positive and so there exists a positive α so that $M_n'(x; h)\geqq \alpha I$ for every x in $(-1, 1)$.

Note that if $\phi(x)=Kh(x)$ then $A+M_n'(x; \phi)$ is positive definite for all A in C and all x in $(-1, 1)$.

THEOREM 3. *A real-valued function* $f(x)$ *defined on* $(-1, 1)$ *is the difference of two monotone matrix functions of order* $n>1$ *on every closed subinterval* $[a, b]$ *if and only if* f *is* C^{2n-3}, *the derivative* $f^{(2n-3)}(x)$ *is absolutely continuous on* $[a, b]$ *and its derivative,* $f^{(2n-2)}(x)$ *is of bounded variation there.*

PROOF. Let $[a, b]$ be an arbitrary but fixed closed subinterval of $(-1, 1)$. Then there exists a closed subinterval $[a', b']$ of $(-1, 1)$ such that $[a, b]$ is contained in the interior of $[a', b']$. If $f(x)$ is the difference of two monotone matrix functions of order $n>1$ on $[a', b']$, then in view of Theorem 1, it follows that f is C^{2n-3}, the derivative $f^{(2n-3)}(x)$ is absolutely continuous and its derivative, $f^{(2n-2)}(x)$ is of bounded variation on at least $[a, b]$.

Suppose $f(x)$ is C^{2n-3}, the derivative $f^{(2n-3)}$ is absolutely continuous on $[a, b]$ and $f^{(2n-2)}(x)$ is of bounded variation there. We shall show that f can be expressed as the difference of two monotone functions of order n. Since $f^{(2n-2)}$ is of bounded variation, it follows that $f=G_1-G_2$ where $G_i^{(2n-2)}$ $(i=1, 2)$ are monotone increasing. The polynomial $p(x)$ which appears in the process of obtaining f from $f^{(2n-2)}$ may be absorbed into G_1. Hence it is enough to prove the theorem when $f^{(2n-2)}$ is monotone increasing. To show that f is the difference of two monotone matrix functions, we must find a convenient $h\in P(-1, 1)$ [3] so that the matrix

[3] $P(-1, 1)$ denotes the functions in the Pick class which are real and regular on the interval $(-1, 1)$ and which therefore admit analytic continuation into the lower half-plane which is given by reflection.

$M_n'(x;f+h)$ is a positive matrix for almost all $x\in[a,b]$. Setting $f(x)+h(x)=g(x)$, we obtain $f(x)=g(x)-h(x)$ where $g(x)$ and $h(x)$ are such that $M_n'(x;g)$ and $M_n'(x;h)$ are positive matrices for almost all $x\in[a,b]$. By a theorem of Dobsch [**1**, Theorem 2.4] and remarks preceding Lemma 2, it follows that $g(x)$ and $h(x)$ are monotone matrix functions of order n on $[a,b]$. We write $M_n'(x;f)=A(x)+B(x)$ where

$$B(x)=\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & \dfrac{f^{(2n-1)}(x)}{(2n-1)!} \end{bmatrix}$$

and $A(x)$ is the remainder of $M_n'(x;f)$.

Since $A(x)$ does not contain $f^{(2n-1)}(x)$, it is a uniformly bounded matrix function of x in $[a,b]$. It follows from Lemma 2 that

$$A(x)+\lambda M_n'(x;h)>0, \qquad x\in[a,b],$$

if λ is sufficiently large. On the other hand $B(x)\geqq 0$ almost everywhere, because $f^{(2n-1)}(x)>0$ almost everywhere. It follows that

$$M_n'(x;f)+\lambda M_n'(x;h)>0$$

almost everywhere, as required.

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