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ON A CHARACTERISTIC PROPERTY OF PERIODIC ENTIRE FUNCTIONS

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Introduction. We shall pursue an investigation on a certain functional equation treated in [8] with some overmuch restrictions. The functional equation is related to the following problem: If two entire functions in a certain class have the same zero-sets (including multiplicities), then what can be said about these functions?

Denoting by $G(b)$ the class of all entire functions each of which is periodic with period b $(\neq 0)$ mod a non-constant entire function of order less than one (cf. Def. in § 1), in this paper, we shall prove that if two entire functions belonging to $G(b_j)$ $(j=1, 2)$ have the same zero-sets (essentially), then they must coincide up to a non-zero multiplicative constant (Theorem 1). In the proof, we use the Borel-Nevanlinna type unicity theorem.

Note that, together with $G(b)$, the class $J(b)$, consisting of the entire functions each of which is periodic mod a non-constant polynomial of degree one, is significant in factorization theory (under composition) of transcendental entire functions (cf. for example, [1] or [7]).

Now recall some of the results of Gross ([2]). Among others, he proved that any non-constant, periodic, entire function $H(z)$ has an infinite number of fixed points, that is, the zeros of $H(z)-z$. Further the fixed points play an important role especially in cases concerning periodic entire functions (in factorization theory, etc.). So one might expect that periodic entire functions would be uniquely determined by the sets of their fixed points. In this paper, we'll show that this is the case (Theorem 2).

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1. Statement of results.

At first, we'll give the definition of the class $G(b)$ explicitly.

DEFINITION. For a non-zero constant b, we denote by $G(b)$ the class of entire functions of the form;

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$$f(z)=h(z)+H(z),$$

where $H(z)$ and $h(z)$ are non-constant entire functions such that $H(z)$ is periodic with period b, $H(z+b)\equiv H(z)$, and that $h(z)$ is of order less than one.

After we've written a short summary [9], we get a generalization. Our generalized results are stated as follows.

THEOREM 1. *Let $f(z)\in \boldsymbol{G}(b_1)$ and $g(z)\in \boldsymbol{G}(b_2)$ for some non-zero constants b_j $(j=1, 2)$. Assume that the sets of the zeros of $f(z)$ and $g(z)$ are identical (including multiplicities) except at most a (sequence) set whose exponent of convergence is less than one. Then we must have*

$$f(z)\equiv c\cdot g(z) \tag{1}$$

for some non-zero constant c and b_1/b_2 is a rational number.

Remark. Let $f(z)\in \boldsymbol{G}(b_1)$ and $g(z)\in \boldsymbol{G}(b_2)$ be represented as

$$f(z)=h(z)+H(z), \quad g(z)=k(z)+K(z), \tag{2}$$

where $H(z+b_1)\equiv H(z)$, $K(z+b_2)\equiv K(z)$, $h(z)$ and $k(z)$ are non-constant entire functions such that h and k are both of order less than one. Then the condition of Theorem 1 concerning the zero-sets of f and g means that the identical relation

$$h(z)+H(z)=(k(z)+K(z))R(z)e^{p(z)} \tag{3}$$

is valid for some meromorphic function $R(z)$ $(\not\equiv 0)$ of order less than one and for some entire function $p(z)$.

The conclusion (1) is equivalent to show that $R(z)$ and $p(z)$ are both constant.

Also note that any meromorphic function $R(z)$ $(\not\equiv 0)$ of order less than one can be represented as

$$R(z)=v(z)/u(z) \tag{4}$$

for some entire functions $u(z)$ and $v(z)$ ($\not\equiv 0$ and without common zeros), both of which are of order less than one.

Periodic entire functions have the following characteristic property, as is mentioned in Introduction.

THEOREM 2. *Let $H_j(z)$ $(j=1, 2)$ be non-constant periodic entire functions with period b_j (resp.). Assume that the sets of the fixed points of $H_j(z)$ are identical (including multiplicities) except at most a (sequence) set whose exponent of convergence is less than one. Then we have necessarily that*

$$H_1(z)\equiv H_2(z) \tag{5}$$

and b_1/b_2 is a rational number.

2. Lemmas.

For the proof of Theorem 1, we shall need the following unicity theorem of Borel-Nevanlinna type due to Niino-Ozawa [5].

LEMMA A. *Let $G_j(z)$ be a transcendental entire function and c_j be a non-zero constant, and let $g(z)$ be an entire function, $\not\equiv 0$ such that $T(r, g)=o(T(r, G_j))$ as r tends to infinity for any j with $1 \leqq j \leqq n$. Assume that there exists an identical relation such as*

$$\sum_{j=1}^{n} c_j G_j(z)=g(z),$$

then we have necessarily

$$\sum_{j=1}^{n} \delta(0, G_j) \leqq n-1.$$

Here $T(r, *)$ and $\delta(a, *)$ denote the Nevanlinna characteristic function and deficiency, respectively.

Remark. Compared with the original Niino-Ozawa's lemma, the above Lemma A will seem slightly general. But the proof is essentially same.

Also we use the following simple fact.

LEMMA B. *Let $w(z)$ be a meromorphic function of order less than one and assume that*

$$w(z+b) \equiv e^c \cdot w(z) \tag{6}$$

for some constants b $(\neq 0)$ and c. Then $w(z)$ must be constant.

For completeness, we prove this. Indeed, if we consider the function

$$F(z)=w(z) \cdot \exp\left[-\frac{c}{b} z\right], \tag{7}$$

then the assumption (6) implies

$$F(z+b) \equiv F(z). \tag{8}$$

If $w(z)$ is non-constant, then $w(z)$ has at least one zero or one pole, since $w(z)$ is of order less than one by assumption. Then from (7) and (8), the exponent of convergence of the zeros or poles of $w(z)$ cannot be less than one, so that the order of $w(z)$ cannot be so. This is a contradiction. Hence $w(z)$ must be constant.

3. To prove Theorem 1, at the first step, we note here the following facts.

PROPOSITION 1. *Let the identity* (3) *be valid. And assume that $p(z)$ is con-*

stant. Then $R(z)$ is constant ($\neq 0$) and b_1/b_2 is a rational number.

PROPOSITION 2. *Let the identity* (3) *be valid. And assume that b_1/b_2 is a rational number. Then $R(z)$ and $p(z)$ are constants.*

Proof of Prop. 1. In this case, the identity (3) may be written as

$$h(z)+H(z)=(k(z)+K(z))R(z)\,. \tag{9}$$

We shall introduce here the notations;

$$h_j(z)=h(z+jb_1)\,,\quad k_j(z)=k(z+jb_1)\,,\quad R_j(z)=R(z+jb_1)\,, \tag{10}$$

for any natural number j.

Since $H(z+b_1)=H(z)$ by assumption, noting (10), from (9) we have

$$h_1-h=(k_1+K(z+b_1))R_1-(k+K(z))R\,,$$

so that

$$K(z+b_1)R_1-K(z)R=(h_1-h)-(k_1R_1-kR)\,. \tag{11}$$

Setting

$$S(z)=R(z+b_2)\,,\quad r(z)=h(z+b_2)\,,\quad s(z)=k(z+b_2)\,, \tag{12}$$

and noting $K(z+b_2)=K(z)$, from (11) we obtain

$$K(z+b_1)S_1-K(z)S=(r_1-r)-(s_1S_1-sS)\,. \tag{13}$$

Here we put

$$r_j(z)=r(z+jb_1)\,,\quad s_j(z)=s(z+jb_1)\,,\quad S_j(z)=S(z+jb_1) \tag{14}$$

for any natural number j as in (10). Note that

$$R_1(z+b_2)=R(z+b_1+b_2)=S(z+b_1)=S_1(z)$$

etc., by (10), (12) and (14).

Cancelling $K(z+b_1)$ from the identities (11) and (13), we get

$$(RS_1-R_1S)K(z)=R_1[(r_1-r)-(s_1S_1-sS)] -S_1[(h_1-h)-(k_1R_1-kR)]\,. \tag{15}$$

Now the right hand side of this identity is of order less than one, while if (RS_1-R_1S) is not identically zero, the left hand side is of order not less than one, since $K(z)$ is a non-constant periodic entire function (by assumption). Therefore we conclude

$$RS_1-R_1S\equiv 0\,,\quad \text{or}\quad S/R-S_1/R_1\equiv 0\,.$$

Noting (10) and (14), this means that S/R is periodic with period b_1. Since S/R is a meromorphic function of order less than one, an application of Lemma B gives

$$S/R=\text{const.}=c\,, \quad \text{or} \quad cR-S\equiv 0\,. \tag{16}$$

Setting $R(z)=v(z)/u(z)$ as in (4), then (16) can be written as

$$cu(z+b_2)v(z)-u(z)v(z+b_2)\equiv 0\,. \tag{17}$$

Assume that $R(z)$ is non-constant. Then we may assume without loss of generality that $u(z)$ is non-constant. In this case, since $u(z)$ is of order less than one, $u(z)$ has zeros. Assume $u(z_0)=0$. From (17), dividing the relation by $u(z)$, we have the identity

$$cu(z+b_2)v(z)/u(z)=v(z+b_2)\,. \tag{18}$$

Here the right hand side of (18) is entire, and since $u(z)$ and $v(z)$ have no common zeros, $u(z_0)=0$ implies $u(z_0+b_2)=0$, and so again by (18), changing the variable if necessary, $u(z_0+2b_2)=0$. Repeating this argument, we obtain that

$$u(z_0+mb_2)=0\,, \quad \text{for any natural number } m. \tag{19}$$

Then from (19) we conclude that the exponent of convergence of the zeros of $u(z)$ is not less than one, and hence the order of $u(z)$ is so. This is a contradiction. Thus we have proved that $R(z)$ is constant.

Putting $R(z)=\text{const.}=c$ $(\neq 0)$, from (11) we have (since $R_1(z)=\text{const.}=c$ also)

$$c(K(z+b_1)-K(z))=(h_1-h)-c(k_1-k)\,. \tag{20}$$

Here the left hand side of (20) is periodic with period b_2, while the right hand side is of order less than one, so that we conclude again that

$$K(z+b_1)-K(z)=\text{const.} \tag{21}$$

Then $K(z+b_2)=K(z)$ and (21) imply that $K'(z)$ (the first derived function of $K(z)$, non-constant since $K(z)$ is non-constant and periodic) is periodic with periods b_1 and b_2. Since non-constant entire function cannot be doubly periodic, b_1/b_2 must be a rational number, which is to be proved.

Proof of Prop. 2. By the assumption, we may put

$$b=mb_1=nb_2$$

for some non-zero integers m and n. In this case,

$$H(z+jb)=H(z)\,, \quad K(z+jb)=K(z) \tag{22}$$

for any natural number j.

Putting

$$h_j(z)=h(z+jb)\,, \quad k_j(z)=k(z+jb)\,, \tag{23}$$

$$R_j(z)=R(z+jb)\,, \quad p_j(z)=p(z+jb)\,,$$

from (3) we have

$$h_j-h=(k_j+K(z))R_je^{p_j}-(k+K(z))Re^p\,,$$

where j is a natural number. Hence

$$(R_j e^{p_j} - Re^p)K(z) = (h_j - h) - (k_j R_j e^{p_j} - kRe^p). \tag{24}$$

By using (24) for $j=1, 2$, and cancelling $K(z)$, we obtain the following identity

$$R_2(h_1-h)e^{p_2} + R_1R_2(k_2-k_1)e^{p_1+p_2} - RR_2(k_2-k)e^{p+p_2}$$
$$+RR_1(k_1-k)e^{p+p_1} - R_1(h_2-h)e^{p_1} + R(h_2-h_1)e^p = 0.$$

Dividing this relation by $\exp(p_2)$, we deduce

$$R_1R_2(k_2-k_1)e^{p_1} - RR_2(k_2-k)e^p + RR_1(k_1-k)e^{p+p_1-p_2}$$
$$-R_1(h_2-h)e^{p_1-p_2} + R(h_2-h_1)e^{p-p_2} = -R_2(h_1-h) \ (\not\equiv 0). \tag{25}$$

Assume

$$p(z) \neq \text{constant}, \tag{26}$$

in which case, if we can prove

$$p-p_1, \quad p-p_2, \quad p+p_1-p_2 \neq \text{const.}, \tag{27}$$

then, applying Lemma A to the identity (25)*), clearly we have a contradiction.

We wish to prove that

$$\text{when (26) holds, then } p - p_j \neq \text{const.}, \tag{28}$$

for any natural number j. Assume

$$p_j - p = \text{const.} = c, \quad \text{or} \quad e^{p_j} = e^c e^p, \tag{29}$$

then (24) can be rewritten as

$$K(z) = \frac{h_j - h}{e^c R_j - R} \cdot e^{-p} - \frac{e^c k_j R_j - kR}{e^c R_j - R}, \tag{30}$$

if

$$e^c R_j - R \not\equiv 0. \tag{31}$$

Then from (22), (29) and (30), using the notations (23), we have

$$\frac{h_j - h}{e^c R_j - R} = e^{-c} \cdot \frac{h_{2j} - h_j}{e^c R_{2j} - R_j},$$
$$\frac{e^c k_j R_j - kR}{e^c R_j - R} = \frac{e^c k_{2j} R_{2j} - k_j R_j}{e^c R_{2j} - R_j}.$$

Hence by Lemma B, we have

$$\frac{h_j - h}{e^c R_j - R} = \text{const.} = c' \ (\neq 0),$$
$$\frac{e^c k_j R_j - kR}{e^c R_j - R} = \text{const.} = c'' \ (\neq 0),$$

*) we'd multiply a common (entire) denominator, if necessary.

or

(32)
$$\begin{cases} h_j-h=c'(e^cR_j-R), \quad \text{and} \\ e^ck_jR_j-kR=c''(e^cR_j-R). \end{cases}$$

Putting $R=v/u$ as in (4), and $R_j=v_j/u_j$ ($u_j(z)=u(z+jb)$, etc.), the upper relation of (32) reduces to

$$c'(e^cuv_j-u_jv)=uu_j(h_j-h),$$

which is rewritten as

$$-c'\cdot\frac{u_jv}{u}=u_j(h_j-h)-c'e^cv_j.$$

By using the quite similar argument around (18), we can conclude that $u(z)$ is constant. Hence we may assume $R(z)=v(z)$ without loss of generality, since $R=v/u$. Then the lower relation of (32) becomes

$$e^ck_jv_j-kv=c''(e^cv_j-v),$$

or

$$(c''-k)v=e^c(c''-k_j)v_j.$$

Again by Lemma B, we conclude

$$(c''-k)v=\text{const.},$$

which is impossible. Thus

(33) $$e^cR_j-R=0 \quad \text{identically}$$

is the only possibility remained. In this case, by Lemma B, $R(z)$ is constant. Under (33), the identity (30) must reduce to

$$(h_j-h)e^{-p}-(e^ck_jR_j-kR)\equiv 0.$$

But this will be clearly impossible, since $p(z)$ is non-constant and others (h, k and R, $\cdots$) are of order less than one (h_j-h cannot be identically zero). Thus we have proved the assertion (28).

Next assume

(34) $$p+p_1-p_2=\text{const.}=c.$$

Then

$$p_1-p_2=-p+c, \quad p-p_2=-p_1+c,$$

hence (25) reduces to the following identity

$$R_1R_2(k_2-k_1)e^{p_1}-RR_2(k_2-k)e^{p}-e^cR_1(h_2-h)e^{-p}$$
$$+e^cR(h_2-h_1)e^{-p_1}+[e^cRR_1(k_1-k)+R_2(h_1-h)]=0.$$

Deviding this by $\exp(p_1)$, we obtain

(35) $$-RR_2(k_2-k)e^{p-p_1}+[e^cRR_1(k_1-k)+R_2(h_1-h)]e^{-p_1}$$
$$-e^cR_1(h_2-h)e^{-p-p_1}+e^cR(h_2-h_1)e^{-2p_1}=-R_1R_2(k_2-k_1).$$

Here, by (26) and (28)

$$p-p_1, \quad -p_1, \quad -2p_1 \neq \text{const.},$$

and

$$-R_1R_2(k_2-k_1) \not\equiv 0,$$

since $k(z)$ is a non-constant entire function of order less than one. Also

$$-p-p_1=-(p+p_1) \neq \text{const.}$$

Indeed if

(36) $$p+p_1=\text{const.}=c,$$

then

(37) $$p_1+p_2=\text{const.}=c.$$

Hence, subtracting (37) from (36), we get

$$p-p_2=\text{const.}=0,$$

which contradicts the fact noted in (28).

Therefore we can apply Lemma A to the identity (35), and derive a contradiction. Hence (34) is impossible.

Thus we have checked (27). Hence (26) is impossible,*) which is to be proved.

4. In order to complete the proof of Theorem 1, by noting Propositions 1 and 2, it is enough to show that the identity (3) is impossible to hold under the additional assumptions

(38) $p(z)$ is non-constant, and
b_1/b_2 is not a rational number
(b_1/b_2 is a non-real complex number or real and irrational number).

Here we'll note the following fact, which is needed later.

PROPOSITION 3. *Let the identity* (3) *be valid, and assume the condition* (38). *Then, for any natural number j, the functions*

$$p(z+jb_1)-p(z) \quad \text{and} \quad p(z+jb_2)-p(z)$$

are both non-constant.

Proof. Without loss of generality, it is sufficient to show

(39) $$p(z+jb_1)-p(z) \neq \text{const.}$$

Indeed, for the proof of the fact

*) In this case, the fact that $R(z)$ is constant follows easily from (24) (without using Proposition 1).

$$p(z+jb_2)-p(z)\neq\text{const.},$$

we need only to start the subsequent argument from the identity

$$k(z)+K(z)=(h(z)+H(z))\cdot 1/R(z)\cdot e^{-p(z)}.$$

We use the notations (10), (12), (14) and further

$$q(z)=p(z+b_2), \quad q_j(z)=q(z+jb_1). \tag{40}$$

Assume (39) is not true, that is, for some natural number j,

$$p_j(z)-p(z)=\text{const.}=c, \tag{41}$$

then also from (40) and (41)

$$q_j(z)-q(z)=\text{const.}=c.$$

Hence, noting $H(z+jb_1)=H(z)$, from (3) we have

$$h_j-h=[e^c(k_j+K(z+jb_1))R_j-(k+K(z))R]e^p,$$

and noting $K(z+b_2)=K(z)$,

$$r_j-r=[e^c(s_j+K(z+jb_1))S_j-(s+K(z))S]e^q,$$

so that

$$\begin{cases} Re^pK(z)-e^cR_je^pK(z+jb_1)=(e^ck_jR_j-kR)e^p-(h_j-h), \\ Se^qK(z)-e^cS_je^qK(z+jb_1)=(e^cs_jS_j-sS)e^q-(r_j-r). \end{cases}$$

From the above system of equations, we get

$$(*) \quad \begin{cases} e^{p+q}(RS_j-R_jS)K(z)=S_je^q[(e^ck_jR_j-kR)e^p-(h_j-h)] \\ \qquad\qquad -R_je^p[(e^cs_jS_j-sS)e^q-(r_j-r)], \\ e^{p+q}e^c(RS_j-R_jS)K(z+jb_1)=Se^q[(e^ck_jR_j-kR)e^p-(h_j-h)] \\ \qquad\qquad -Re^p[(e^cs_jS_j-sS)e^q-(r_j-r)]. \end{cases}$$

Hence

$$\begin{aligned} &e^{p+q}e^c(RS_j-R_jS)\{S_{2j}e^{q_j}[(e^ck_{2j}R_{2j}-k_jR_j)e^{p_j}-(h_{2j}-h_j)] \\ &\qquad -R_{2j}e^{p_j}[(e^cs_{2j}S_{2j}-s_jS_j)e^{q_j}-(r_{2j}-r_j)]\} \\ &=e^{p_j+q_j}(R_jS_{2j}-R_{2j}S_j)\{Se^q[(e^ck_jR_j-kR)e^p-(h_j-h)] \\ &\qquad -Re^p[(e^cs_jS_j-sS)e^q-(r_j-r)]\}. \end{aligned}$$

Using

$$\begin{aligned} &p+q+p_j+q_j=2p+2q+2c, \\ &p+q+q_j=p+2q+c, \\ &p+q+p_j=2p+q+c, \\ &p_j+q_j+q=p+2q+2c, \\ &p+p_j+q_j=2p+q+2c, \end{aligned}$$

we have the following identical relation

$$e^{2c}[(RS_j-R_jS)R_{2j}(r_{2j}-r_j)-(R_jS_{2j}-R_{2j}S_j)R(r_j-r)]e^{2p+q}$$
$$-e^{2c}[(RS_j-R_jS)S_{2j}(h_{2j}-h_j)-(R_jS_{2j}-R_{2j}S_j)S(h_j-h)]e^{p+2q}$$
$$+e^{2c}[e^c(RS_j-R_jS)S_{2j}(e^ck_{2j}R_{2j}-k_jR_j)-e^c(RS_j-R_jS)R_{2j}(e^cs_{2j}S_{2j}$$
$$-s_jS_j)-(R_jS_{2j}-R_{2j}S_j)S(e^ck_jR_j-kR)$$
$$+(R_jS_{2j}-R_{2j}S_j)R(e^cs_jS_j-sS)]e^{2p+2q}=0$$

Dividing the above identity by e^{2p+q} or e^{p+2q}, and applying Lemma A, we conclude that three [*****] are all identically zero, since now because $p_j=p+c$ and b_1/b_2 is not a rational number, $p-q$ is non-constant.*)₁

From the fact that the first [***]=0 and the second [***]=0 we obtain

$$\frac{RS_j-R_jS}{R(r_j-r)}=\frac{R_jS_{2j}-R_{2j}S_j}{R_{2j}(r_{2j}-r_j)} \tag{42}$$

and

$$\frac{RS_j-R_jS}{S(h_j-h)}=\frac{R_jS_{2j}-R_{2j}S_j}{S_{2j}(h_{2j}-h_j)}\,. \tag{43}$$

Multiplying $1/R_j$ to the both side of (42), by Lemma B, we have

$$\frac{RS_j-R_jS}{RR_j(r_j-r)}=\text{const.}=c\,. \tag{44}$$

Rewriting (44) as $S_j/R_j-S/R=c(r_j-r)$, we obtain

$$S_j/R_j-cr_j=S/R-cr\,,$$

whence, using Lemma B, we conclude

$$S/R-cr=\text{const.}=c', \quad \text{or} \quad S/R=cr+c'. \tag{45}$$

We note here that the constant c in (44) is zero if and only if $R(z)$ is constant. Indeed if $c=0$, then from (44) we have $S_j/R_j=S/R$. Lemma B leads us S/R is constant, and so again we know that $R(z)$ is constant (cf. the notation (12)).

From (43), similarly we have

$$\frac{RS_j-R_jS}{SS_j(h_j-h)}=\text{const.}=c'', \tag{46}$$

and hence $R/S-R_j/S_j=c''(h_j-h)$, so that as before we conclude

$$R/S+c''h=\text{const.}=c''', \quad \text{or} \quad R/S=-c''h+c'''. \tag{47}$$

Here also $c''=0$ if and only if $R(z)$ is constant.

Now from (45) and (47), we know

$$(cr+c')(-c''h+c''')\equiv 1\,,$$

which is possible only when $c=0$ and $c''=0$. Hence $R(z)$ must be constant.*)₁

Thus we may assume

*)₁ See the final remark (added in the proof).

$$R(z)\equiv 1 \quad \text{and hence} \quad S(z)=R(z+b_2)\equiv 1.$$

In this case, the system of equations becomes

$$\begin{cases} K(z)-e^cK(z+jb_1)=(e^ck_j-k)-(h_j-h)e^{-p}, \\ K(z)-e^cK(z+jb_1)=(e^cs_j-s)-(r_j-r)e^{-q}. \end{cases}$$

Hence

$$(h_j-h)e^{-p}-(r_j-r)e^{-q}=(e^ck_j-k)-(e^cs_j-s),$$

so that we have the identity

$$(48) \qquad (r_j-r)e^{p-q}+[(e^ck_j-k)-(e^cs_j-s)]e^p=h_j-h.$$

Here $h_j-h\not\equiv 0$ and $p\not\equiv$const. And also

$$p-q\not\equiv \text{const.},{*})_2$$

since now $p_j-p=$const., and b_1/b_2 is not a rational number (non-constant function $p(z)$ cannot doubly periodic). So by Lemma A the identity (48) is impossible to hold, a contradiction. Hence the assertion (39) follows, which completes the proof of Prop. 3.

5. In this section, we wish to deduce the new identical relation starting the identity (3). We assume the condition (38) from now on.

We use the former notations (10), (12), (14) and (40). For example,

$$h_j(z)=h(z+jb_1), \quad k_j(z)=k(z+jb_1),$$
$$R_j(z)=R(z+jb_1), \quad p_j(z)=p(z+jb_1).$$

As $H(z+b_1)=H(z)$, from (3);

$$h(z)+H(z)=(k(z)+K(z))R(z)e^{p(z)},$$

we have

$$h_1-h=(k_1+K(z+b_1))R_1e^{p_1}-(k+K(z))Re^p.$$

Hence

$$(49) \qquad Re^pK(z)-R_1e^{p_1}K(z+b_1)=(k_1R_1e^{p_1}-kRe^p)-(h_1-h).$$

And, noting $K(z+b_2)=K(z)$ and $r(z)=h(z+b_2)$ etc., from (49) we have

$$(50) \qquad Se^qK(z)-S_1e^{q_1}K(z+b_1)=(s_1S_1e^{q_1}-sSe^q)-(r_1-r).$$

Setting

$$R=v/u, \quad \text{and} \quad S=y/x$$

($x(z)=u(z+b_2)$, $y(z)=v(z+b_2)$) as in (4), the identities (49) and (50) can be rewritten as

$$(51) \qquad \begin{cases} u_1ve^pK(z)-uv_1e^{p_1}K(z+b_1)=k_1uv_1e^{p_1}-ku_1ve^p-(h_1-h)uu_1, \\ x_1ye^qK(z)-xy_1e^{q_1}K(z+b_1)=s_1xy_1e^{q_1}-sx_1ye^q-(r_1-r)xx_1. \end{cases}$$

${*})_2$ See the final remark (added in the proof).

We solve this system of equations as follows:

(52) $$K(z)=D_1(z)/D(z)\,,\quad K(z+b_1)=D_2(z)/D(z)\,,$$

where

(53) $$D(z)=\det\begin{pmatrix} u_1ve^{p}, & -uv_1e^{p_1} \\ x_1ye^{q}, & -xy_1e^{q_1}\end{pmatrix},$$

$$D_1(z)=\det\begin{pmatrix} k_1uv_1e^{p_1}-ku_1ve^{p}-(h_1-h)uu_1, & -uv_1e^{p_1} \\ s_1xy_1e^{q_1}-sx_1ye^{q}-(r_1-r)xx_1, & -xy_1e^{q_1}\end{pmatrix},$$

$$D_2(z)=\det\begin{pmatrix} u_1ve^{p}, & k_1uv_1e^{p_1}-ku_1ve^{p}-(h_1-h)uu_1 \\ x_1ye^{q}, & s_1xy_1e^{q_1}-sx_1ye^{q}-(r_1-r)xx_1\end{pmatrix}.$$

We note here that $D(z)\not\equiv 0$. Otherwise

(54) $$D(z)=x_1yuv_1e^{p_1+q}-xy_1u_1ve^{p+q_1}\equiv 0\,.$$

Then $(p+q_1)-(p_1+q)=\text{const.}=c$ (say), or

(55) $$q_1=-p+p_1+q+c\,.$$

If $D(z)\equiv 0$, then from (52) $D_1(z)\equiv 0$. This means

(56) $$kxy_1u_1ve^{p+q_1}+(s_1-k_1)xy_1uv_1e^{p_1+q_1}$$
$$-sx_1yuv_1e^{p_1+q}+(h_1-h)xy_1uu_1e^{q_1}$$
$$-(r_1-r)xx_1uv_1e^{p_1}=0.$$

From (55),

$$p+q_1=p_1+q+c\,,\quad p_1+q_1=-p+2p_1+q+c\,,$$

so that (56) becomes

$$(e^{c}kxy_1u_1v-sx_1yuv_1)e^{p_1+q}+e^{c}(s_1-k_1)xy_1uv_1e^{-p+2p_1+q}$$
$$+e^{c}(h_1-h)xy_1uu_1e^{-p+p_1+q}-(r_1-r)xx_1uv_1e^{p_1}=0\,.$$

Dividing this by $\exp(-p+p_1+q)$, we obtain

(57) $$(e^{c}kxy_1u_1v-sx_1yuv_1)e^{p}+e^{c}(s_1-k_1)xy_1uv_1e^{p_1}$$
$$-(r_1-r)xx_1uv_1e^{p-q}=-e^{c}(h_1-h)xy_1uu_1\ (\not\equiv 0)\,.$$

Here p, $p_1\neq$const., and also $p-q\neq$const. by Prop. 3 (cf. (38)). Then applying Lemma A to (57), we have a contradiction. Thus

(58) $$D(z)\not\equiv 0\,.$$

From (52), we know

(59) $$D(z)\cdot D_1(z+b_1)\equiv D(z+b_1)\cdot D_2(z)\,.$$

Now

$$D(z)=x_1yuv_1e^{p_1+q}-xy_1u_1ve^{p+q_1}\,,$$

$$D(z+b_1)=x_2y_1u_1v_2e^{p_2+q_1}-x_1y_2u_2v_1e^{p_1+q_2},$$

$$D_1(z+b_1)=k_1x_1y_2u_2v_1e^{p_1+q_2}+(s_2-k_2)x_1y_2u_1v_2e^{p_2+q_2}$$
$$-s_1x_2y_1u_1v_2e^{p_2+q_1}+(h_2-h_1)x_1y_2u_1u_2e^{q_2}$$
$$-(r_2-r_1)x_1x_2u_1v_2e^{p_2},$$

$$D_2(z)=-(s-k)x_1yu_1ve^{p+q}+s_1xy_1u_1ve^{p+q_1}$$
$$-k_1x_1yuv_1e^{p_1+q}-(r_1-r)xx_1u_1ve^{p}$$
$$+(h_1-h)x_1yuu_1e^{q}.$$

Hence from (59), we have the following identity after arrangement

$$(r_2-r_1)X_1e^{p_1+p_2+q}+(r_1-r)X_2e^{p+p_1+q_2}+(h_2-h_1)X_3e^{p+q_1+q_2}$$
$$-(r_2-r)X_4e^{p+p_2+q_1}-(h_2-h)X_5e^{p_1+q+q_2}+(h_1-h)X_6e^{p_2+q+q_1}$$
$$+(s-k)X_7e^{p+p_1+q+q_2}-(s-k)X_8e^{p+p_2+q+q_1}+(s_2-k_2)X_9e^{p+p_2+q_1+q_2}$$
$$-(s_1-k_1)X_{10}e^{p+p_1+q_1+q_2}-(s_2-k_2)X_{11}e^{p_1+p_2+q+q_2}$$
$$+(s_1-k_1)X_{12}e^{p_1+p_2+q+q_1}=0. \tag{60}$$

Here we put

$$X_1=x_1^2x_2yuu_1v_1v_2,\quad X_2=xx_1^2y_2u_1u_2vv_1,$$
$$X_3=xx_1y_1y_2u_1^2u_2v,\quad X_4=xx_1x_2y_1u_1^2vv_2,$$
$$X_5=x_1^2yy_2uu_1u_2v_1,\quad X_6=x_1x_2yy_1uu_1^2v_2,$$
$$X_7=x_1^2yy_2u_1u_2vv_1,\quad X_8=x_1x_2yy_1u_1^2vv_2,$$
$$X_9=xx_1y_1y_2u_1^2vv_2,\quad X_{10}=xx_1y_1y_2u_1u_2vv_1,$$
$$X_{11}=x_1^2yy_2uu_1v_1v_2,\quad X_{12}=x_1x_2yy_1uu_1v_1v_2, \tag{61}$$

each of which is an entire function and not identically zero such that the order is less than one.

Dividing the relation (60) by $\exp(p_1+p_2+q)$, we obtain

$$(r_2-r_1)X_1+(r_1-r)X_2e^{p-p_2-q+q_2}+(h_2-h_1)X_3e^{p-p_1-p_2-q+q_1+q_2}$$
$$-(r_2-r)X_4e^{p-p_1-q+q_1}-(h_2-h)X_5e^{-p_2+q_2}+(h_1-h)X_6e^{-p_1+q_1}$$
$$+(s-k)X_7e^{p-p_2+q_2}-(s-k)X_8e^{p-p_1+q_1}+(s_2-k_2)X_9e^{p-p_1-q+q_1+q_2}$$
$$-(s_1-k_1)X_{10}e^{p-p_2-q+q_1+q_2}-(s_2-k_2)X_{11}e^{q_2}+(s_1-k_1)X_{12}e^{q_1}=0. \tag{62}$$

This is the new identity which we mention at the beginning of this section. In (62), we may transfer the term $(r_2-r_1)X_1$ to the right hand side if necessary.

6. In (62), as $r(z)=k(z+b_2)$ is a non-constant entire function of order less than one and X_1 is not identically zero,

(63) $$(r_2-r_1)X_1\not\equiv 0\,,$$

and as $p(z)$ is non-constant by assumption,

(64) $$q_j(z)=p(z+jb_1+b_2)\not\equiv \text{const.}$$

Also by Proposition 3, noting (40) etc., we know

(65) $$-p_2+q_2\,,\quad -p_1+q_1\not\equiv \text{const.},$$

and further, for any natural numbers m and n $(m\neq n)$,

(66) $$\begin{aligned}&p-p_m\,,\quad p+p_m\,,\quad p_m-p_n\,,\quad p_m+p_n\,,\\ &q-q_m\,,\quad q+q_m\,,\quad q_m-q_n\,,\quad q_m+q_n\,,\\ &p-q\,,\quad p+q\,,\quad p_m-q_m\,,\quad p_m+q_m\end{aligned}$$

are all non-constant by Proposition 3.

Indeed, for example if

(67) $$p+p_m=\text{const.}=c \text{ (say)},$$

then also

(68) $$p_m+p_{2m}=\text{const.}=c\,,$$

since $p_j(z)=p(z+jb_1)$. From (67) and (68), subtracting,

$$p_{2m}-p=\text{const.}=0\,,$$

or

$$p(z+2mb_1)-p(z)=\text{const.}=0\,,$$

which contradicts the fact proved in Proposition 3.

Also note that

(69) $$p-p_1-q+q_1\neq \text{const., if } p-p_2-q+q_2\neq \text{const.}$$

Because, if

(70) $$p-p_1-q+q_1=\text{const.}=c \text{ (say)},$$

also, noting the notations,

(71) $$p_1-p_2-q_1+q_2=\text{const.}=c\,,$$

so that, by adding (70) and (71), we have

$$p-p_2-q+q_2=\text{const.}=2c\,.$$

Hence if we can prove the following six functions

(72) $$\begin{aligned}&p-p_2-q+q_2\,,\quad p-p_1-p_2-q+q_1+q_2\,,\\ &p-p_2+q_2\,,\quad p-p_1+q_1\,,\\ &p-p_1-q+q_1+q_2\,,\quad p-p_2-q+q_1+q_2\end{aligned}$$

are non-constant, because of the facts (63), (64), (65) and (69), by applying Lemma

A to the identity (62), we get a contradiction. Therefore the proof of Theorem 1 has become done.

In fact, in the following section, we'll prove that the six functions in (72) are all non-constant, one by one, applying Lemma A repeatedly.

7. The non-constancy of the six functions in (72).

[I] Non-constancy of $p-p_2-q+q_2$.

Assume

(73) $$p-p_2-q+q_2=\text{const.}=c\,.$$

Then cancelling q_2, we have

$$p-p_1-p_2-q+q_1+q_2=-p_1+q_1+c\,,$$

$$-p_2+q_2=-p+q+c\,,\quad p-p_2+q_2=q+c\,,$$

$$p-p_1-q+q_1+q_2=-p_1+p_2+q_1+c\,,$$

$$p-p_2-q+q_1+q_2=q_1+c\,,\quad q_2=-p+p_2+q+c\,.$$

Hence the identity (62) can be written as

$$\begin{aligned}&[(r_2-r_1)X_1+(r_1-r)X_2e^c]+[(h_2-h_1)X_3e^c+(h_1-h)X_6]e^{-p_1+q_1}\\&-(r_2-r)X_4e^{p-p_1-q+q_1}-(h_2-h)X_5e^ce^{-p+q}+(s-k)X_7e^ce^q\\&-(s-k)X_8e^{p-p_1+q_1}+(s_2-k_2)X_9e^ce^{-p_1+p_2+q_1}\\&-(s_1-k_1)(X_{10}e^c-X_{12})e^{q_1}-(s_2-k_2)X_{11}e^ce^{-p+p_2+q}=0\,.\end{aligned}$$

Dividing this relation by $\exp(-p_1+p_2+q_1)$, we obtain the following identity.

(74) $$\begin{aligned}&(s_2-k_2)X_9e^c+[(r_2-r_1)X_1+(r_1-r)X_2e^c]e^{p_1-p_2-q_1}\\&+[(h_2-h_1)X_3e^c+(h_1-h)X_6]e^{-p_2}-(r_2-r)X_4e^{p-p_2-q}\\&-(h_2-h)X_5e^ce^{-p+p_1-p_2+q-q_1}+(s-k)X_7e^ce^{p_1-p_2+q-q_1}\\&-(s-k)X_8e^{p-p_2}-(s_1-k_1)(X_{10}e^c-X_{12})e^{p_1-p_2}\\&-(s_2-k_2)X_{11}e^ce^{-p+p_1+q-q_1}=0\,.\end{aligned}$$

Here, noting the notations (12) etc.,

$$(s_2-k_2)X_9e^c\not\equiv 0\,,\quad\text{and}$$

$$-p_2,\quad p-p_2,\quad p_1-p_2\neq\text{const.}$$

Further we shall prove that

$$p_1-p_2-q_1\,,\quad p-p_2-q\,,\quad -p+p_1-p_2+q-q_1\,,$$

$$p_1-p_2+q-q_1\,,\quad -p+p_1+q-q_1$$

are non-constant, under (73). At first, we note that the former three are non-constant, and next we prove that the lower two are also non-constant.

Indeed if $p_1-p_2-q_1$ is constant, then by the notations given before, $p-p_1-q=p_1-p_2-q_1=\text{const.}=c'$ (say). Adding these, $p-p_2-q-q_1=2c'$, which together with (73) show us that $q_1+q_2=\text{const.}=c-2c'$. But this is contrary to the fact noted in (66).

If $p-p_2-q$ would be constant, then from (73) q_2 and hence $p(z)$ must be constant ($q_2(z)\equiv p(z+2b_1+b_2)$), a contradiction.

If $-p+p_1-p_2+q-q_1=\text{const.}=c'$, then $-p_1+p_2-p_3+q_1-q_2=c'$ and so by adding we get $-p-p_3+q-q_2=\text{const.}=2c'$ as above. Hence, using (73), we have $p_2+p_3=\text{const.}=-(c+2c')$. By (66) this is not valid, a contradiction.

[I. 1] Assume

(75) $$p_1-p_2+q-q_1=\text{const.}=c'.$$

Then, cancelling q_1,

$$p_1-p_2-q_1=-q+c', \quad -p+p_1-p_2+q-q_1=-p+c',$$
$$-p+p_1+q-q_1=-p+p_2+c'.$$

Hence (74) can be written as

$$\begin{aligned}&[(s_2-k_2)X_9+(s-k)X_7e^{c'}]e^c+[(r_2-r_1)X_1+(r_1-r)X_2e^c]e^{c'}e^{-q}\\&+[(h_2-h_1)X_3e^c+(h_1-h)X_6]e^{-p_2}-(r_2-r)X_4e^{p-p_2-q}\\&-(h_2-h)X_5e^{c+c'}e^{-p}-(s-k)X_8e^{p-p_2}-(s_1-k_1)(X_{10}e^c-X_{12})e^{p_1-p_2}\\&-(s_2-k_2)X_{11}e^{c+c'}e^{-p+p_2}=0.\end{aligned}$$

Dividing this relation by $\exp(p-p_2)$, we obtain

(76) $$\begin{aligned}&-(s-k)X_8+[(s_2-k_2)X_9+(s-k)X_7e^{c'}]e^ce^{-p+p_2}\\&+[(r_2-r_1)X_1+(r_1-r)X_2e^c]e^{c'}e^{-p+p_2-q}\\&+[(h_2-h_1)X_3e^c+(h_1-h)X_6]e^{-p}-(r_2-r)X_4e^{-q}\\&-(h_2-h)X_5e^{c+c'}e^{-2p+p_2}-(s_1-k_1)(X_{10}e^c-X_{12})e^{-p+p_1}\\&-(s_2-k_2)X_{11}e^{c+c'}e^{-2(p-p_2)}=0.\end{aligned}$$

Here

(77) $$-(s-k)X_8\not\equiv 0, \quad \text{and}$$

(78) $$-p+p_2, \quad -p, \quad -q, \quad -p+p_1, \ -2(p-p_2)\neq\text{const.}$$

Also we note that

(79) $$-p+p_2-q \quad \text{and} \quad -2p+p_2 \quad \text{are non-constant.}$$

Indeed if

(80) $$-p+p_2-q=\text{const.}=c'' \text{ (say)},$$

then from (75) and (80), by adding, we have

$$-p+p_1-q_1=\text{const.}=c'+c'', \quad \text{and so}$$

$$-p_1+p_2-q_2=c'+c''.$$

Again by adding

$$-p+p_2-q_1-q_2=\text{const.}=2(c'+c''),$$

which together with (73) imply $q+q_1=\text{const.}$, contrary to (66).

If

(81) $$-2p+p_2=\text{const.}=c'' \text{ (say)},$$

then in this case from (75) and (81)

$$-2p+p_1+q-q_1=-2p_1+p_2+q_1-q_2=c'+c'',$$

and hence, by adding,

$$-2p-p_1+p_2+q-q_2=\text{const.}=2(c'+c'').$$

Using (73), from this we deduce $p+p_1=\text{const.}$, a contradiction.

As we have checked the facts (77), (78) and (79), applying Lemma A to the identity (76), a contradiction follows. Thus (75) is impossible.

[I. 2] Assume now

(82) $$-p+p_1+q-q_1=\text{const.}=c' \text{ (say)}.$$

Then cancelling q_1,

$$p_1-p_2-q_1=p-p_2-q+c', \quad -p+p_1-p_2+q-q_1=-p_2+c',$$

$$p_1-p_2+q-q_1=p-p_2+c'.$$

Hence (74) can be written as

(83) $$\begin{aligned}&(s_2-k_2)e^c(X_9-e^{c'}X_{11})\\&+[\{(r_2-r_1)X_1+(r_1-r)X_2e^c\}e^{c'}-(r_2-r)X_4]e^{p-p_2-q}\\&+[(h_2-h_1)X_3e^c+(h_1-h)X_6-(h_2-h)X_5e^{c+c'}]e^{-p_2}\\&+(s-k)(X_7e^{c+c'}-X_8)e^{p-p_2}-(s_1-k_1)(X_{10}e^c-X_{12})e^{p_1-p_2}=0.\end{aligned}$$

Here

(84) $$(s_2-k_2)e^c(X_9-e^{c'}X_{11})\not\equiv 0.$$

If otherwise

(85) $$X_9-e^{c'}X_{11}\equiv 0,$$

since $s_2-k_2\not\equiv 0$. (Note that $s_2(z)-k_2(z)=k(z+2b_1+b_2)-k(z+2b_1)$, by definition, and that $k(z)$ is a non-constant entire function of order less than one.)

Noting (82), (85) implies that

(86) $$X_9e^{p+q_1}-X_{11}e^{p_1+q}\equiv 0.$$

Recall

$$X_9 = x x_1 y_1 y_2 u_1^2 v v_2 , \quad X_{11} = x_1^2 y y_2 u u_1 v_1 v_2 ,$$

cf. (61). Then (86) reduces to

$$x_1 y_2 u_1 v_2 (x y_1 u_1 v e^{p+q_1} - x_1 y u v_1 e^{p_1+q}) \equiv 0 . \tag{87}$$

Since $x_1 y_2 u_1 v_2 \not\equiv 0$, (87) shows

$$D(z) \equiv 0 \tag{88}$$

(cf. (53)). However, in §5, we have ruled out the possibility (88). Thus (84) must be valid.

Now, $-p_2$, $p-p_2$, $p_1-p_2 \neq$ const., and noting (73)

$$p - p_2 - q = c - q_2$$

is also non-constant, since q_2 is so.

Then, under (84), by applying Lemma A to (83), we obtain a contradiction. Hence (82) is not valid.

Therefore, again, by applying Lemma A to the identity (74), we know that it cannot hold. This contradiction shows us that (73) is impossible. Thus the proof of the case [I] is complete.

[II]. Non-constancy of $p-p_1-p_2-q+q_1+q_2$.

Assume

$$p - p_1 - p_2 - q + q_1 + q_2 = \text{const.} = c . \tag{89}$$

Then cancelling q_2,

$$p - p_2 - q + q_2 = p_1 - q_1 + c , \quad -p_2 + q_2 = -p + p_1 + q - q_1 + c ,$$
$$p - p_2 + q_2 = p_1 + q - q_1 + c , \quad p - p_1 - q + q_1 + q_2 = p_2 + c ,$$
$$p - p_2 - q + q_1 + q_2 = p_1 + c , \quad q_2 = -p + p_1 + p_2 + q - q_1 + c .$$

Hence (62) can be written as

$$\begin{aligned}
&[(r_2-r_1)X_1 + (h_2-h_1)X_3 e^c] + (r_1-r)X_2 e^c e^{p_1-q_1} \\
&-(r_2-r)X_4 e^{p-p_1-q+q_1} - (h_2-h)X_5 e^c e^{-p+p_1+q-q_1} + (h_1-h)X_6 e^{-p_1+q_1} \\
&+(s-k)X_7 e^c e^{p_1+q-q_1} - (s-k)X_8 e^{p-p_1+q_1} + (s_2-k_2)X_9 e^c e^{p_2} \\
&-(s_1-k_1)X_{10} e^c e^{p_1} - (s_2-k_2)X_{11} e^c e^{-p+p_1+p_2+q-q_1} + (s_1-k_1)X_{12} e^{q_1} = 0.
\end{aligned}$$

Dividing this relation by $\exp(p_1-q_1)$, we have

$$\begin{aligned}
&(r_1-r)X_2 e^c + [(r_2-r_1)X_1 + (h_2-h_1)X_3 e^c] e^{-p_1+q_1} \\
&-(r_2-r)X_4 e^{p-2p_1-q+2q_1} - (h_2-h)X_5 e^c e^{-p+q} + (h_1-h)X_6 e^{-2(p_1-q_1)} \\
&+(s-k)X_7 e^c e^q - (s-k)X_8 e^{p-2p_1+2q_1} + (s_2-k_2)X_9 e^c e^{-p_1+p_2+q_1} \\
&-(s_1-k_1)X_{10} e^c e^{q_1} - (s_2-k_2)X_{11} e^c e^{-p+p_2+q} + (s_1-k_1)X_{12} e^{-p_1+2q_1} = 0 .
\end{aligned} \tag{90}$$

Here,

$$(r_1-r)X_2e^c\not\equiv 0\,,$$

$$-p_1+q_1\,,\quad -p+q\,,\quad -2(p_1-q_1)\,,\quad q\,,\quad q_1\neq\text{const.}$$

Next, under (89), we shall prove the non-constancy of the following five functions;

$$p-2p_1-q+2q_1\,,\quad p-2p_1+2q_1\,,\quad -p_1+p_2+q_1\,,$$
$$-p+p_2+q\,,\quad -p_1+2q_1\,.$$

Then by Lemma A we have a contradiction from (90).

[II. 1]. If

(91) $$p-2p_1-q+2q_1=\text{const.}=c'.$$

Then $p_1-2p_2-q_1+2q_2=c'$ also, and hence by adding, $p-p_1-2p_2-q+q_1+2q_2=2c'$. This together with (89) imply that $p_2-q_2=\text{const.}$, which is a contradiction (cf. (66)).

[II. 2]. Assume

(92) $$p-2p_1+2q_1=\text{const.}=c'.$$

Then, changing the variable and adding, we have as before $p-p_1-2p_2+2q_1+2q_2=2c'$, whence by (89),

(93) $$p-p_1-2q=\text{const.}=2c-2c'.$$

Now (92) and (93) give us that

(94) $$p_1-2q-2q_1=p_2-2q_1-2q_2=2c-3c'.$$

Also (93) and (94) give

(95) $$p-4q-2q_1=4c-5c'.$$

Using (94) and (95), by cancelling p, p_1, and p_2,

$$-p_1+q_1=-2q-q_1-(2c-3c')\,,\quad p-2p_1-q+2q_1=-q+c',$$
$$-p+q=-3q-2q_1-(4c-5c')\,,\quad -2p_1+2q_1=-4q-2q_1-2(2c-3c')\,,$$
$$-p_1+p_2+q_1=-2q+q_1+2q_2\,,\quad -p+p_2+q=-3q+2q_2-(2c-2c')\,,$$
$$-p_1+2q_1=-2q-(2c-3c')\,.$$

Hence (90) reduces to

$$[(r_1-r)X_2e^c-(s-k)X_8e^{c'}]+[(r_2-r_1)X_1+(h_2-h_1)X_3e^c]e^{-(2c-3c')}e^{-2q-q_1}$$
$$-(r_2-r)X_4e^{c'}e^{-q}-(h_2-h)X_5e^ce^{-(4c-5c')}e^{-3q-2q_1}$$
$$+(h_1-h)X_6e^{-2(2c-3c')}e^{-4q-2q_1}+(s-k)X_7e^ce^q$$
$$+(s_2-k_2)X_9e^ce^{-2q+q_1+2q_2}-(s_1-k_1)X_{10}e^ce^{q_1}$$
$$-(s_2-k_2)X_{11}e^ce^{-(2c-2c')}e^{-3q+2q_2}+(s_1-k_1)X_{12}e^{-(2c-3c')}e^{-2q}=0\,.$$

Dividing the above relation by $\exp(-q)$, we have

$$
\begin{aligned}
(96)\qquad & -(r_2-r)X_4e^{c'}+[(r_1-r)X_2e^c-(s-k)X_8e^{c'}]e^q \\
& +[(r_2-r_1)X_1+(h_2-h_1)X_3e^c]e^{-(2c-3c')}e^{-q-q_1} \\
& -(h_2-h)X_5e^{-3c+5c'}e^{-2q-2q_1}+(h_1-h)X_6e^{-(2c-3c')}e^{-3q-2q_1} \\
& +(s-k)X_7e^ce^{2q}+(s_2-k_2)X_9e^ce^{-q+q_1+2q_2}-(s_1-k_1)X_{10}e^ce^{q+q_1} \\
& -(s_2-k_2)X_{11}e^{-c+2c'}e^{-2q+2q_2}+(s_1-k_1)X_{12}e^{-(2c-3c')}e^{-q}=0\,.
\end{aligned}
$$

Now, from (92) and (93) (changing the variable), we have

$$
\begin{aligned}
(97)\qquad p-p_1-p_2 &=(p-2p_1+2q_1)+(p_1-p_2-2q_1) \\
&=\text{const.}=2c-c'.
\end{aligned}
$$

This together with (89) show

$$(98)\qquad q-q_1-q_2=\text{const.}=c-c'.$$

Using this fact, we can easily show that

$$(99)\qquad -3q-2q_1\,,\quad -q+q_1+2q_2\neq\text{const.}$$

Indeed if $-(3q+2q_1)=\text{const.}=c''$ (say), then $-(3q_1+2q_2)=c''$ so that by (98) $2q+q_1=2(q-q_1-q_2)+(3q_1+2q_2)$ becomes constant, and hence $q=2(2q+q_1)-(3q+2q_1)$ must be so, a contradiction.

Also, $-q+q_1+2q_2=-(q-q_1-q_2)+q_2$ cannot be constant by (98).

Further,

$$-(r_2-r)X_4e^{c'}\not\equiv 0\,,$$

$$q\,,\quad -q-q_1\,,\quad -2q-2q_1\,,\quad 2q\,,\quad q+q_1\,,\quad -2q+2q_2\,,\quad -q\neq\text{const.}$$

and, noting (99), by Lemma A, the identity (96) leads us to a contradiction. Thus (92) is impossible.

[II. 3]. Assume

$$(100)\qquad -p_1+p_2+q_1=\text{const.}=c',$$

and so,

$$(101)\qquad -p+p_1+q=c'.$$

Then by subtracting (101) from (100), we have $p-2p_1+p_2-q+q_1=\text{const.}=0$. This and (89) give

$$-p_1+2p_2-q_2=-c \quad\text{and hence}\quad -p+2p_1-q_1=-c\,.$$

Then this and (100) give, by adding, $-p+p_1+p_2=-c+c'$, so that by (89) we have

$$(102)\qquad q-q_1-q_2=\text{const.}=-c'.$$

While by (89) and (101),

(103) $$p_2-q_1-q_2=p_1-q-q_1=\text{const.}=-c-c'.$$

Also by (101) and (103) we have

(104) $$p-2q-q_1=\text{const.}=-c-2c'.$$

By using (103) and (104), and cancelling p, p_1 and p_2, from (90) we deduce

$$\begin{aligned}&[(r_1-r)X_2+(s_2-k_2)X_9e^{c'}]e^c+[(r_2-r_1)X_1+(h_2-h_1)X_3e^c]e^{c+c'}e^{-q}\\&-[(r_2-r)X_4-(s_1-k_1)X_{12}e^{c'}]e^ce^{-q+q_1}\\&-(h_2-h)X_5e^{2c+2c'}e^{-q-q_1}+(h_1-h)X_6e^{2c+2c'}e^{-2q}\\&+(s-k)X_7e^ce^q-[(s-k)X_8+(s_1-k_1)X_{10}]e^ce^{q_1}\\&-(s_2-k_2)X_{11}e^{c+c'}e^{-q+q_2}=0.\end{aligned}$$

Dividing this by $\exp(q)$, we have

(105) $$\begin{aligned}&(s-k)X_7e^c+[(r_1-r)X_2+(s_2-k_2)X_9e^{c'}]e^ce^{-q}\\&+[(r_2-r_1)X_1+(h_2-h_1)X_3e^c]e^{c+c'}e^{-2q}\\&-[(r_2-r)X_4-(s_1-k_1)X_{12}e^{c'}]e^ce^{-2q+q_1}\\&-(h_2-h)X_5e^{2c+2c'}e^{-2q-q_1}+(h_1-h)X_6e^{2c+2c'}e^{-3q}\\&-[(s-k)X_8+(s_1-k_1)X_{10}]e^ce^{-q+q_1}-(s_2-k_2)X_{11}e^{c+c'}e^{-2q+q_2}=0.\end{aligned}$$

Here

$$(s-k)X_7e^c\neq 0,$$

$$-q,\quad -2q,\quad -3q,\quad -q+q_1\neq\text{const.}$$

Also, by using (102), we can easily show as before that

$$-2q+q_1,\quad -2q-q_1,\quad -2q+q_2\neq\text{const.}$$

Hence, applying Lemma A to (105), we have that (100) is not valid.

[II. 4]. Assume

(106) $$-p+p_2+q=\text{const.}=c'.$$

Then, using (89), from (106) we have

$$p_1-q_1-q_2=p-q-q_1=-c-c',$$

and hence by (106),

(107) $$p_2-q_1=p_1-q=\text{const.}=-c.$$

Again by (106) and (107)

(108) $$p-p_1-p_2=\text{const.}=c-c'.$$

Using (107), by cancelling q and q_1, from (90) we deduce

$$\begin{aligned}&[(r_1-r)X_2-(s_2-k_2)X_{11}e^{c'}]e^c+[(r_2-r_1)X_1+(h_2-h_1)X_3e^c]e^ce^{-p_1+p_2}\\&-(r_2-r)X_4e^ce^{p-3p_1+2p_2}-(h_2-h)X_5e^{2c}e^{-p+p_1}+(h_1-h)X_6e^{2c}e^{-2p_1+2p_2}\end{aligned}$$

$$+(s-k)X_7e^{2c}e^{p_1}-(s-k)X_8e^{2c}e^{p-2p_1+2p_2}$$
$$+[(s_2-k_2)X_9+(s_1-k_1)X_{12}]e^{2c}e^{-p_1+2p_2}$$
$$-(s_1-k_1)X_{10}e^{2c}e^{p_2}=0\,.$$

Dividing this by $\exp(-p+p_1)$, we obtain

$$\begin{aligned}(109)\qquad &-(h_2-h)X_5e^{2c}+[(r_1-r)X_2-(s_2-k_2)X_{11}e^{c'}]e^ce^{p-p_1}\\ &+[(r_2-r_1)X_1+(h_2-h_1)X_3e^c]e^ce^{p-2p_1+p_2}\\ &-(r_2-r)X_4e^ce^{2(p-2p_1+p_2)}+(h_1-h)X_6e^{2c}e^{p-3p_1+2p_2}\\ &+(s-k)X_7e^{2c}e^p-(s-k)X_8e^{2c}e^{2p-3p_1+2p_2}\\ &+[(s_2-k_2)X_9+(s_1-k_1)X_{12}]e^{2c}e^{p-2p_1+2p_2}\\ &-(s_1-k_1)X_{10}e^{2c}e^{p-p_1+p_2}=0\,.\end{aligned}$$

Here

$$-(h_2-h)X_5e^{2c}\not\equiv 0\,,$$
$$p-p_1\,,\quad p\neq\text{const.}$$

Further, using (108), we can show arithmetically that

$$p-2p_1+p_2\,,\quad 2(p-2p_1+p_2)\,,\quad p-3p_1+2p_2\,,$$
$$2p-3p_1+2p_2\,,\quad p-2p_1+2p_2\,,\quad p-p_1+p_2\neq\text{const.}$$

Indeed, for example, if

$$(110)\qquad p-2p_1+p_2=\text{const.}=c''\ \text{(say)},$$

then, using (108), $2p-3p_1=\text{const.}=c-c'+c''$. Hence from (110) $2(p-2p_1+p_2)-(2p-3p_1)=-p_1+2p_2=\text{const.}$, so that $-p+2p_1=\text{const.}$ Then $p-p_1=(2p-3p_1)+(-p+2p_1)$ must become constant, which is impossible by (66). Thus $p-2p_1+p_2\neq\text{const.}$ Other cases are treated quite similarly.

Hence an application of Lemma A to (109) leads us to a contradiction Therefore (106) is impossible.

[II. 5]. Assume that

$$(111)\qquad -p_1+2q_1=\text{const.}=c',$$

which implies that

$$(112)\qquad -p+2q=-p_2+2q_2=\text{const.}=c'.$$

Then by these, we have $-p+p_1+p_2+2q-2q_1-2q_2=\text{const.}=-c'$, from which, together with (89), we have

$$(113)\qquad q-q_1-q_2=\text{const.}=c-c'.$$

Using (111) and (112), by cancelling p, p_1 and p_2, the identity (90) becomes

$$[(r_1-r)X_2e^c+(s_1-k_1)X_{12}e^{c'}]+[(r_2-r_1)X_1+(h_2-h_1)X_3e^c]e^{c'}e^{-q_1}$$

$$-(r_2-r)X_4e^{c'}e^{q-2q_1}-(h_2-h)X_5e^ce^{c'}e^{-q}+(h_1-h)X_6e^{2c'}e^{-2q_1}$$
$$+(s-k)X_7e^ce^q-(s-k)X_8e^{c'}e^{2q-2q_1}+(s_2-k_2)X_9e^ce^{-q_1+2q_2}$$
$$-(s_1-k_1)X_{10}e^ce^{q_1}-(s_2-k_2)X_{11}e^ce^{-q+2q_2}=0\,.$$

Dividing this identity by $\exp(-q)$, we deduce

$$\begin{aligned}(114)\qquad &-(h_2-h)X_5e^{c+c'}+[(r_1-r)X_2e^c+(s_1-k_1)X_{12}e^{c'}]e^q\\ &+[(r_2-r_1)X_1+(h_2-h_1)X_3e^c]e^{c'}e^{q-q_1}-(r_2-r)X_4e^{c'}e^{2(q-q_1)}\\ &+(h_1-h)X_6e^{2c'}e^{q-2q_1}+(s-k)X_7e^ce^{2q}-(s-k)X_8e^{c'}e^{3q-2q_1}\\ &+(s_2-k_2)X_9e^ce^{q-q_1+2q_2}-(s_1-k_1)X_{10}e^ce^{q+q_1}\\ &-(s_2-k_2)X_{11}e^ce^{2q_2}=0\,.\end{aligned}$$

Here

$$-(h_2-h_1)X_5e^{c+c'}\not\equiv 0\,,$$
$$q\,,\quad q-q_1\,,\quad 2(q-q_1)\,,\quad 2q\,,\quad q+q_1\,,\quad 2q_2\neq\text{const.}$$

Further, noting (113), we can easily show that

$$q-2q_1\,,\quad 3q-2q_1\,,\quad q-q_1+2q_2\neq\text{const.},$$

arithmetically, as before.

Hence, applying Lemma A to (114), we obtain that (111) is impossible.

Thus we've checked the assertion [II].

[III] Non-constancy of $p-p_2+q_2$.

Assume

$$(115)\qquad p-p_2+q_2=\text{const.}=c\,.$$

Then cancelling q_2,

$$p-p_2-q+q_2=-q+c\,,\quad p-p_1-p_2-q+q_1+q_2=-p_1-q+q_1+c\,,$$
$$-p_2+q_2=-p+c\,,\quad p-p_1-q+q_1+q_2=-p_1+p_2-q+q_1+c\,,$$
$$p-p_2-q+q_1+q_2=-q+q_1+c\,,\quad q_2=-p+p_2+c\,,$$

so that (62) can be rewritten as, after dividing the reduced relation by $\exp(-q)$,

$$\begin{aligned}(116)\qquad &(r_1-r)X_2e^c+[(r_2-r_1)X_1+(s-k)X_7e^c]e^q+(h_2-h_1)X_3e^ce^{-p_1+q_1}\\ &-(r_2-r)X_4e^{p-p_1+q_1}-(h_2-h)X_5e^ce^{-p+q}+(h_1-h)X_6e^{-p_1+q+q_1}\\ &-(s-k)X_8e^{p-p_1+q+q_1}+(s_2-k_2)X_9e^ce^{-p_1+p_2+q_1}\\ &-(s_1-k_1)X_{10}e^ce^{q_1}-(s_2-k_2)X_{11}e^ce^{-p+p_2+q}+(s_1-k_1)X_{12}e^{q+q_1}=0\,.\end{aligned}$$

Here, $(r_1-r)X_2e^c\not\equiv 0$, and that q, $-p_1+q_1$, $-p+q$, q_1 and $q+q_1$ are non-constant. Also

$$p-p_1+q_1\,,\quad -p_1+p_2+q_1\,,\quad -p+p_2+q\neq\text{const.}$$

Indeed if $p-p_1+q_1$ is constant, then $p_1-p_2+q_2$ is so, which together with (115) give $p-p_1=\text{const.}$, a contradiction. If $-p_1+p_2+q_1$ is constant, then $-p_2+p_3+q_2$ is so. Then (115) shows $p-p_3=\text{const.}$, a contradiction. If $-p+p_2+q$ is constant, then (115) means $q+q_2=\text{const.}$, also a contradiction (cf. (66)).

Further we wish to prove that the following two functions

$$-p_1+q+q_1, \quad p-p_1+q+q_1$$

are non-constant under (115).

[III. 1]. Assume

$$-p_1+q+q_1=\text{const.}=c'. \tag{117}$$

Then $-p_2+q_1+q_2=c'$ also and hence by (115) we have

$$p-q_1=\text{const.}=c-c', \tag{118}$$

which together with (117) give

$$p-p_1+q=\text{const.}=c. \tag{119}$$

Using (118) and (119), by cancelling q and q_1 from (116), and then dividing the new identity by $\exp(p-p_1)$, we deduce

$$\begin{aligned}(h_2-h_1)X_3e^{c'}&+[(r_1-r)X_2e^c+(h_1-h)X_6e^{c'}]e^{-p+p_1}\\ &+[(r_2-r_1)X_1+(s-k)X_7e^c]e^ce^{-2(p-p_1)}-(r_2-r)X_4e^{-(c-c')}e^p\\ &-(h_2-h)X_5e^{2c}e^{-3p+2p_1}-[(s-k)X_8+(s_1-k_1)X_{10}]e^{c'}e^{p_1}\\ &-(s_2-k_2)X_9e^{c'}e^{p_2}-(s_2-k_2)X_{11}e^{2c}e^{-3p+2p_1+p_2}+(s_1-k_1)X_{12}e^{c'}e^{-p+2p_1}=0.\end{aligned} \tag{120}$$

Here $(h_2-h_1)X_3e^{c'}\not\equiv 0$, and that $-p+p_1$, $-2(p-p_1)$, p, p_1 and p_2 are non-constant. Now, from (119), $p_1-p_2+q_1=c$, which together with (118) give

$$p+p_1-p_2=\text{const.}=2c-c'. \tag{121}$$

Noting this fact (121), we can easily show that

$$-3p+2p_1, \quad -3p+2p_1+p_2, \quad -p+2p_1\neq\text{const.}$$

Then, applying Lemma A, the identity (120) leads us to a contradiction. Hence (117) is impossible.

[III. 2]. Assume

$$p-p_1+q+q_1=\text{const.}=c'. \tag{122}$$

Then $p_1-p_2+q_1+q_2=c'$ also, with which and (115) we have

$$p-p_1-q_1=c-c'. \tag{123}$$

Hence by (122)

$$2p-2p_1+q=c. \tag{124}$$

Using (123) and (124), by cancelling q and q_1 from (116), and then dividing the rewritten identity by $\exp(-p+p_1)$, we obtain

$$(125)\qquad (s_1-k_1)X_{12}e^{c'}+[(r_1-r)X_2e^c-(s-k)X_8e^{c'}]e^{p-p_1}$$
$$+[(r_2-r_1)X_1+(s-k)X_7e^c]e^ce^{-p+p_1}+(h_2-h_1)X_3e^{c'}e^{2p-3p_1}$$
$$-(r_2-r)X_4e^{-c+c'}e^{3(p-p_1)}-(h_2-h)X_5e^{2c}e^{-2p+p_1}$$
$$+(h_1-h)X_6e^{c'}e^{-p_1}+(s_2-k_2)X_9e^{c'}e^{2p-3p_1+p_2}$$
$$-(s_1-k_1)X_{10}e^{c'}e^{2(p-p_1)}-(s_2-k_2)X_{11}e^{2c}e^{-2p+p_1+p_2}=0\,.$$

Now from (123) and (124), we have

$$(126)\qquad p+p_1-2p_2=\text{const.}=2c-c',$$

since $2p_1-2p_2+q_1=c$ also by (124). Using (126), we can easily show that

$$2p-3p_1\,,\quad -2p+p_1\,,\quad 2p-3p_1+p_2\quad\text{and}\quad -2p+p_1+p_2$$

are non-constant, as before. Further $(s_1-k_1)X_{12}e^{c'}\not\equiv 0$, and that $p-p_1$ and $-p_1$ are non-constant. Hence applying Lemma A to (125), we deduce a contradiction. Thus (122) is not valid.

Therefore again applying Lemma A to the identity (116), we have a contradiction. Hence (115) is impossible, which is to be proved.

[IV]. Non-constancy of $p-p_1+q_1$.

Assume

$$(127)\qquad p-p_1+q_1=\text{const.}=c\,.$$

Then,

$$p_1-p_2+q_2=c\quad\text{and}\quad p-p_2+q_1+q_2=2c\,.$$

Hence cancelling q_1 and q_2, we have

$$p-p_2-q+q_2=p-p_1-q+c\,,\quad p-p_1-p_2-q+q_1+q_2=-p_1-q+2c\,,$$
$$p-p_1-q+q_1=-q+c\,,\quad -p_2+q_2=-p_1+c\,,\quad -p_1+q_1=-p+c\,,$$
$$p-p_2+q_2=p-p_1+c\,,\quad p-p_1-q+q_1+q_2=-p_1+p_2-q+2c\,,$$
$$p-p_2-q+q_1+q_2=-q+2c\,,\quad q_2=-p_1+p_2+c\,,\quad q_1=-p+p_1+c\,.$$

Hence the identity (62) reduces to the following one, after dividing the rewritten relation by $\exp(-p_1-q)$,

$$(128)\qquad (h_2-h_1)X_3e^{2c}+[(r_2-r_1)X_1-(s-k)X_8e^c]e^{p_1+q}$$
$$+(r_1-r)X_2e^ce^p-[(r_2-r)X_4+(s_1-k_1)X_{10}e^c]e^ce^{p_1}$$
$$-(h_2-h)X_5e^ce^q+(h_1-h)X_6e^ce^{-p+p_1+q}+(s-k)X_7e^ce^{p+q}$$
$$+(s_2-k_2)X_9e^{2c}e^{p_2}-(s_2-k_2)X_{11}e^ce^{p_2+q}+(s_1-k_1)X_{12}e^ce^{-p+2p_1+q}=0\,.$$

Here, $(h_2-h_1)X_3e^{2c}\not\equiv 0$, and that

$$p, \quad p_1, \quad q, \quad -p+p_1+q=-c+q+q_1, \quad p+q, \quad p_2 \neq \text{const.}$$

We must check the non-constancy of the following three functions

$$p_1+q, \quad p_2+q, \quad -p+2p_1+q.$$

[IV. 1]. Assume

$$p_1+q=\text{const.}=c', \tag{129}$$

and hence $p_2+q_1=\text{const.}=c'$ also. Cancelling q from (128) and then dividing the new identity by $\exp(p)$, we obtain

$$\begin{aligned}(r_1-r)X_2e^c+[(h_2-h_1)X_3e^{2c}+(r_2-r_1)X_1e^{c'}-(s-k)X_8e^{c+c'}]e^{-p}\\ -[(r_2-r)X_4+(s_1-k_1)X_{10}e^c]e^ce^{-p+p_1}-(h_2-h)X_5e^{c+c'}e^{-p-p_1}\\ +(h_1-h)X_6e^{c+c'}e^{-2p}+(s-k)X_7e^{c+c'}e^{-p_1}+(s_2-k_2)X_9e^{2c}e^{-p+p_2}\\ -(s_2-k_2)X_{11}e^{c+c'}e^{-p-p_1+p_2}+(s_1-k_1)X_{12}e^{c+c'}e^{-2p+p_1}=0.\end{aligned} \tag{130}$$

Here, $-(r_1-r)X_2e^c\not\equiv 0$, and that $-p$, $-p+p_1$, $-p-p_1$, $-2p$, $-p_1$ and $-p+p_2$ are non-constant.

Now from (127) and (129) (since $p_2+q_1=c'$), we have

$$p-p_1-p_2=\text{const.}=c-c'. \tag{131}$$

Using (131), we can easily show that $-p-p_1+p_2$ and $-2p+p_1$ are non-constant. Hence, applying Lemma A to (130), we get a contradiction. Thus (129) is impossible.

[IV. 2]. Assume

$$p_2+q=\text{const.}=c'. \tag{132}$$

In this case, cancelling q from (128) and then dividing by $\exp(p)$, we deduce the following identity.

$$\begin{aligned}(r_1-r)X_2e^c+[(h_2-h_1)X_3e^c-(s_2-k_2)X_{11}e^{c'}]e^ce^{-p}\\ +[(r_2-r_1)X_1-(s-k)X_8e^c]e^{c'}e^{-p+p_1-p_2}\\ -[(r_2-r)X_4+(s_1-k_1)X_{10}e^c]e^ce^{-p+p_1}\\ -(h_2-h)X_5e^{c+c'}e^{-p-p_2}+(h_1-h)X_6e^{c+c'}e^{-2p+p_1-p_2}\\ +(s-k)X_7e^{c+c'}e^{-p_2}+(s_2-k_2)X_9e^{2c}e^{-p+p_2}\\ +(s_1-k_1)X_{12}e^{c+c'}e^{-2p+2p_1-p_2}=0.\end{aligned} \tag{133}$$

Now from (132), $p_3+q_1=c'$, from which and (127), we have

$$p-p_1-p_3=\text{const.}=c-c'. \tag{134}$$

Using (134), we can easily show that

$$-p+p_1-p_2, \quad -2p+p_1-p_2, \quad -2p+2p_1-p_2 \neq \text{const.}$$

Indeed, for example, if

(135) $$-2p+2p_1-p_2=\text{const.},$$

then, using (134), by cancelling $p-p_1$, we get $p_2+2p_3=\text{const.}$ Hence $p+2p_1$ and p_1+2p_2 are constant, so that, by subtracting these, we have

$$p+p_1-2p_2=\text{const.}$$

Then from (135), cancelling p_2, we have $5p-3p_1=\text{const.}$ From this fact and $p+2p_1=\text{const.}$, we conclude that $p=\text{const.}$, which is a contradiction. Thus (135) is impossible.

Further, clearly, $(r_1-r)X_2e^c\not\equiv 0$, and that $-p$, $-p+p_1$, $-p-p_2$, $-p_2$ and $-p+p_2$ are non-constant. Hence applying Lemma A to (133), we conclude as before that (132) is not valid.

[IV. 3]. Assume

(136) $$-p+2p_1+q=\text{const.}=c'.$$

Then, cancelling q from (128) and then dividing by $\exp(p)$, we obtain

(137) $$\begin{aligned}&(r_1-r)X_2e^c+[(h_2-h_1)X_3e^c+(s_1-k_1)X_{12}e^{c'}]e^ce^{-p}\\&+[(r_2-r_1)X_1-(s-k)X_8e^c]e^{c'}e^{-p_1}\\&-[(r_2-r)X_4+(s_1-k_1)X_{10}e^c]e^ce^{-p+p_1}-(h_2-h)X_5e^{c+c'}e^{-2p_1}\\&+(h_1-h)X_6e^{c+c'}e^{-p-p_1}+(s-k)X_7e^{c+c'}e^{p-2p_1}\\&+(s_2-k_2)X_9e^{2c}e^{-p+p_2}-(s_2-k_2)X_{11}e^{c+c'}e^{-2p_1+p_2}=0\,.\end{aligned}$$

Now from (136), $-p_1+2p_2+q_1=c'$ and hence by (127)

$$p-2p_2=\text{const.}=c-c'.$$

Using this, we can easily show that $p-2p_1$ and $-2p_1+p_2$ are non-constant. Other conditions are clearly satisfied. Hence, applying Lemma A to (137), we have that (136) is impossible.

Again applying Lemma A to (128), we obtain a contradiction. Hence (127) is not valid, which is to be proved.

[V]. Non-constancy of $p-p_1-q+q_1+q_2$.

Assume

(138) $$p-p_1-q+q_1+q_2=\text{const.}=c\,.$$

Then, cancelling q_2,

$$\begin{aligned}&p-p_2-q+q_2=p_1-p_2-q_1+c\,,\quad p-p_1-p_2-q+q_1+q_2=-p_2+c\,,\\&-p_2+q_2=-p+p_1-p_2+q-q_1+c\,,\quad p-p_2+q_2=p_1-p_2+q-q_1+c\,,\\&p-p_2-q+q_1+q_2=p_1-p_2+c\,,\quad q_2=-p+p_1+q-q_1+c\,.\end{aligned}$$

Hence the identity (62) becomes, after dividing by $\exp(-p_2)$

(139) $$(h_2-h_1)X_3e^c+[(r_2-r_1)X_1+(s_2-k_2)X_9e^c]e^{p_2}$$

$$+(r_1-r)X_2e^c e^{p_1-q_1}-(r_2-r)X_4e^{p-p_1+p_2-q+q_1}$$
$$-(h_2-h)X_5e^c e^{-p+p_1+q-q_1}+(h_1-h)X_6e^{-p_1+p_2+q_1}$$
$$+(s-k)X_7e^c e^{p_1+q-q_1}-(s-k)X_8e^{p-p_1+p_2+q_1}$$
$$-(s_1-k_1)X_{10}e^c e^{p_1}-(s_2-k_2)X_{11}e^c e^{-p+p_1+p_2+q-q_1}$$
$$+(s_1-k_1)X_{12}e^{p_2+q_1}=0\,.$$

Here,

$$(h_2-h_1)X_3e^c\neq 0\,,$$
$$p_2\,,\quad p_1-q_1\,,\quad p-p_1+p_2-q+q_1=c+p_2-q_2\,,\quad p_1\,,$$
$$-p+p_1+q-q_1=-c+q_2\,,\quad -p+p_1+p_2+q-q_1=-c+p_2+q_2\neq \text{const.}$$

Also

$$-p_1+p_2+q_1\neq \text{const.},$$

since otherwise $-p+p_1+q$ is constant, which together with (138) give $q_1+q_2=$ const., a contradiction.

Further we must show that

$$p_1+q-q_1\,,\quad p-p_1+p_2+q_1\,,\quad p_2+q_1\neq \text{const.}$$

[V. 1]. Assume

(140) $$p_1+q-q_1=\text{const.}=c'.$$

Then $p_2+q_1-q_2=c'$ also, and by adding these, $p_1+p_2+q-q_2=2c'$, which together with (138) give

(141) $$p+p_2+q_1=c+2c'.$$

Hence from (140) and (141), we have

(142) $$p+p_1+p_2+q=\text{const.}=c+3c'.$$

Using (141) and (142), by cancelling q and q_1 from (139) and then dividing the rewritten relation by $\exp(-p_1)$, we obtain the following identity

(143) $$-(s-k)X_8e^{c+2c'}+[(h_2-h_1)X_3+(s-k)X_7e^{c'}]e^c e^{p_1}$$
$$+[(r_2-r_1)X_1+(s_2-k_2)X_9e^c]e^{p_1+p_2}+(r_1-r)X_2e^{-2c'}e^{p+2p_1+p_2}$$
$$-(r_2-r)X_4e^{-c'}e^{p+p_1+p_2}-[(h_2-h)X_5-(s_1-k_1)X_{12}e^{c'}]e^{c+c'}e^{-p+p_1}$$
$$+(h_1-h)X_6e^{c+2c'}e^{-p}-(s_1-k_1)X_{10}e^c e^{2p_1}$$
$$-(s_2-k_2)X_{11}e^{c+c'}e^{-p+p_1+p_2}=0\,.$$

Here

$$-(s-k)X_8e^{c+2c'}\neq 0\,,$$
$$p_1\,,\quad p_1+p_2\,,\quad -p+p_1\,,\quad -p\,,\quad 2p_1\neq \text{const.}$$

Now note that from (141) and (142) we get

(144) $$p-p_1-p_3=\text{const.}=-c'.$$

Using this, we can show that

$$p+2p_1+p_2,\quad p+p_1+p_2,\quad -p+p_1+p_2\neq\text{const.}$$

We check here the first case. Other two cases are quite easy. Assume

(145) $$p+2p_1+p_2=\text{const.}=c''\ \text{(say)}.$$

Then also $p_1+2p_2+p_3=c''$, which together with (144) give

$$p+2p_2=\text{const.}=-c'+c''.$$

From this and (145), by cancelling p, we have

$$2p_1-p_2=2p-p_1=\text{const.}=c'.$$

Hence from the above two facts, we have that $p+4p_1=(p+2p_2)+2(2p_1-p_2)$ is constant, so that $9p=4(2p-p_1)+(p+4p_1)=\text{const.}$, a contradiction.

Hence applying Lemma A to (143), we conclude that (140) is impossible.

[V. 2]. Assume

(146) $$p-p_1+p_2+q_1=\text{const.}=c'.$$

Then $p_1-p_2+p_3+q_2=c'$ also and hence by adding these

$$p+p_3+q_1+q_2=\text{const.}=2c'.$$

From this and (138) we have

(147) $$p_1+p_3+q=\text{const.}=-c+2c'.$$

Note that from (146) and (147) we get

(148) $$p-p_1-p_4=\text{const.}=c-c',$$

since $p_2+p_4+q_1=-c+2c'$ by (147).

Using (146) and (147), by cancelling q and q_1 from (139) and then dividing by $\exp(p_1)$, we obtain

(149) $$\begin{aligned}&-(s_1-k_1)X_{10}e^{c}+[(h_2-h_1)X_3e^{c}-(s-k)X_8e^{c'}]e^{-p_1}\\&+[(r_2-r_1)X_1+(s_2-k_2)X_9e^{c}]e^{-p_1+p_2}+(r_1-r)X_2e^{c-c'}e^{p-p_1+p_2}\\&-(r_2-r)X_4e^{c-c'}e^{p_3}-(h_2-h)X_5e^{c'}\cdot e^{-2p_1+p_2-p_3}\\&+(h_1-h)X_6e^{c'}e^{-p-p_1}+(s-k)X_7e^{c'}\cdot e^{p-2p_1+p_2-p_3}\\&-(s_2-k_2)X_{11}e^{c'}\cdot e^{-2p_1+2p_2-p_3}+(s_1-k_1)X_{12}e^{c'}e^{-p}=0\,.\end{aligned}$$

Here $-(s_1-k_1)X_{10}e^{c}\not\equiv0$ and that $-p_1$, $-p_1+p_2$, p_3, $-p-p_1$ and $-p$ are non-constant. Further, using (148), we can easily prove that

$$p-p_1+p_2,\quad -2p_1+p_2-p_3,\quad p-2p_1+p_2-p_3\ \text{and}\ -2p_1+2p_2-p_3$$

are non-constant. Hence applying Lemma A to (149), we can conclude that (146) is impossible.

[V. 3]. Assume

(150) $$p_2+q_1=\text{const.}=c'.$$

Then also

(151) $$p_1+q=p_3+q_2=\text{const.}=c'.$$

From (150) and (151),

$$p_1-p_2-p_3+q-q_1-q_2=\text{const.}=-c',$$

and hence by (138) we have

(152) $$p-p_2-p_3=\text{const.}=c-c'.$$

Using (150) and (151), by cancelling q and q_1 from (139) and then dividing by $\exp(p)$, we deduce

(153)
$$\begin{aligned}&-(r_2-r)X_4+[(h_2-h_1)X_3e^c+(s_1-k_1)X_{12}e^{c'}]e^{-p}\\&+[(r_2-r_1)X_1+(s_2-k_2)X_9e^c+(s-k)X_7e^c]e^{-p+p_2}\\&+(r_1-r)X_2e^{c-c'}e^{-p+p_1+p_2}-(h_2-h)X_5e^ce^{-2p+p_2}\\&+(h_1-h)X_6e^{c'}e^{-p-p_1}-(s-k)X_8e^{c'}e^{-p_1}\\&-(s_1-k_1)X_{10}e^ce^{-p+p_1}-(s_2-k_2)X_{11}e^ce^{-2p+2p_2}=0.\end{aligned}$$

Here $-(r_2-r)X_4\not\equiv 0$ and that $-p$, $-p+p_2$, $-p-p_1$, $-p_1$, $-p+p_1$ and $-2(p-p_2)$ are non-constant. Further, using (152), we can easily show that

$$-p+p_1+p_2 \quad\text{and}\quad -2p+p_2$$

are non-constant. Hence applying Lemma A to (153) we get a contradiction, which shows that (150) is impossible.

Therefore again applying Lemma A to the identity (139), we obtain a contradiction. Thus (138) is not valid.

[VI]. Non-constancy of $p-p_2-q+q_1+q_2$.

Assume

(154) $$p-p_2-q+q_1+q_2=\text{const.}=c.$$

Then cancelling q_2,

$$\begin{aligned}&p-p_2-q+q_2=-q_1+c, \quad p-p_1-p_2-q+q_1+q_2=-p_1+c,\\&-p_2+q_2=-p+q-q_1+c, \quad p-p_2+q_2=q-q_1+c,\\&p-p_1-q+q_1+q_2=-p_1+p_2+c, \quad q_2=-p+p_2+q-q_1+c.\end{aligned}$$

In this case, using the above facts and then dividing by $\exp(-q_1)$, from (62) we obtain the following identity

(155) $$(r_1-r)X_2e^c+[(r_2-r_1)X_1-(s_1-k_1)X_{10}e^c]e^{q_1}$$

$$+(h_2-h_1)X_3e^ce^{-p_1+q_1}-(r_2-r)X_4e^{p-p_1-q+2q_1}$$
$$-(h_2-h)X_5e^ce^{-p+q}+(h_1-h)X_6e^{-p_1+2q_1}+(s-k)X_7e^ce^q$$
$$-(s-k)X_8e^{p-p_1+2q_1}+(s_2-k_2)X_9e^ce^{-p_1+p_2+q_1}$$
$$-(s_2-k_2)X_{11}e^ce^{-p+p_2+q}+(s_1-k_1)X_{12}e^{2q_1}=0\,.$$

Here $(r_1-r)X_2e^c\not\equiv0$, and that q_1, $-p_1+q_1$, $-p+q$, q and $-p+p_2+q=q_1+q_2-c$ are non-constant. Also

$$p-p_1-q+2q_1\neq\text{const.},$$

since otherwise

$$p-p_2-q+q_1+2q_2=(p-p_1-q+2q_1)+(p_1-p_2-q_1+2q_2)$$

is constant, and hence from (154) q_2 becomes so, which is a contradiction.

Subsequently, we'll prove that the following three functions

$$-p_1+2q_1\,,\quad p-p_1+2q_1\,,\quad -p_1+p_2+q_1$$

are non-constant.

[VI. 1]. Assume

$$-p_1+2q_1=\text{const.}=c'. \tag{156}$$

Then also

$$-p+2q=-p_2+2q_2=\text{const.}=c'. \tag{157}$$

From (157), $p-p_2-2q+2q_2=\text{const.}=0$, and hence by (154),

$$q+q_1-q_2=\text{const.}=c\,. \tag{158}$$

Using (156) and (157), by cancelling p, p_1 and p_2 from (155), and then dividing by $\exp(-q_1)$, we obtain

$$\begin{aligned}(159)\qquad &(h_2-h_1)X_3e^{c+c'}+[(r_1-r)X_2e^c+(h_1-h)X_6e^{c'}]e^{q_1}\\ &+[(r_2-r_1)X_1-(s_1-k_1)X_{10}e^c]e^{2q_1}\\ &-[(r_2-r)X_4-(s-k)X_7e^c]e^{q+q_1}-(h_2-h)X_5e^{c+c'}e^{-q+q_1}\\ &-(s-k)X_8e^{2q+q_1}+(s_2-k_2)X_9e^ce^{2q_2}-(s_2-k_2)X_{11}e^ce^{-q+q_1+2q_2}\\ &+(s_1-k_1)X_{12}e^{3q_1}=0\,.\end{aligned}$$

Since, using (158), we can easily show that

$$2q+q_1 \quad\text{and}\quad -q+q_1+2q_2$$

are non-constant, by applying Lemma A to (159) we have a contradiction. Hence (156) is impossible.

[VI. 2]. Assume

$$p-p_1+2q_1=\text{const.}=c'. \tag{160}$$

Then, $p_1-p_2+2q_2=c'$ also and hence

(161) $$p-p_2+2q_1+2q_2=\text{const.}=2c'.$$

Now from (154) and (161), we obtain

(162) $$q+q_1+q_2=\text{const.}=-c+2c'.$$

Then, from (161) and (162), cancelling q_1+q_2, we have

$$p-p_2-2q=\text{const.}=2c-2c',$$

and so

(163) $$p_1-p_3-2q_1=\text{const.}=2c-2c'.$$

From (160) and (163), we get

$$p-p_3=\text{const.}=2c-c',$$

which is contrary to the fact noted in (66). Thus (160) is not valid.

[VI. 3]. Assume

(164) $$-p_1+p_2+q_1=\text{const.}=c'.$$

Then also

(165) $$-p+p_1+q=\text{const.}=c'$$

and so by adding (164) and (165)

(166) $$-p+p_2+q+q_1=\text{const.}=2c'.$$

From (154) and (166), we conclude

(167) $$2q_1+q_2=2q+q_1=\text{const.}=c+2c'.$$

Also from (164) and (165) we have

(168) $$-2p+p_1+p_2+2q+q_1=2(-p+p_1+q)+(-p_1+p_2+q_1)$$
$$=\text{const.}=3c'.$$

Then by (167) and (168), we obtain

(169) $$-2p+p_1+p_2=\text{const.}=-c+c'.$$

Using (164) and (165), by cancelling q and q_1 from (155) and then dividing by $\exp(-p_1+p_2)$, we have

(170) $$-(s_2-k_2)X_{11}e^{c+c'}+[(r_1-r)X_2+(s_2-k_2)X_9e^{c'}]e^c e^{p_1-p_2}$$
$$+[(r_2-r_1)X_1-(s_1-k_1)X_{10}e^c]e^{c'}e^{2(p_1-p_2)}$$
$$+(h_2-h_1)X_3e^{c+c'}e^{p_1-2p_2}-[(r_2-r)X_4-(s_1-k_1)X_{12}e^{c'}]e^{c'}e^{3(p_1-p_2)}$$
$$-(h_2-h)X_5e^{c+c'}e^{-p_2}+(h_1-h)X_6e^{2c'}e^{2p_1-3p_2}$$
$$+(s-k)X_7e^{c+c'}e^{p-p_2}-(s-k)X_8e^{2c'}e^{p+2p_1-3p_2}=0\,.$$

Here,

$$-(s_2-k_2)X_{11}e^{c+c'}\not\equiv 0\,,$$

$$p_1-p_2,\quad 2(p_1-p_2),\quad 3(p_1-p_2),\quad -p_2,\quad p-p_2\neq\text{const}.$$

Further, using (169), we can easily show that

$$p_1-2p_2,\quad 2p_1-3p_2\quad\text{and}\quad p+2p_1-3p_2$$

are non-constant. Hence, by applying Lemma A to (170), we have a contradiction. This implies that (164) is impossible.

Again, applying Lemma A to (155), we have a contradiction, so that (154) is not valid.

We have checked the facts [I], [II], [III], [IV], [V] and [VI] as above. Therefore the proof of Theorem 1 is now complete.

8. Proof of Theorem 2.

By assumption, we have the following identical relation

$$z-H_1(z)=(z-H_2(z))R(z)e^{p(z)}, \tag{171}$$

where $R(z)$ is a meromorphic function ($\not\equiv 0$) of order less than one and $p(z)$ is an entire function as in (3).

By Theorem 1, $R(z)$ and $p(z)$ are constant and that b_1/b_2 is a rational number. Hence (171) reduces to

$$z-H_1(z)=c\cdot(z-H_2(z)), \tag{172}$$

where c is a non-zero constant. We rewrite (172) as

$$(1-c)z=H_1(z)-c\cdot H_2(z). \tag{173}$$

Since b_1/b_2 is a rational number, $mb_1=nb_2$ ($=b$, say) for some non-zero integers m and n, so that the right hand side of (173) is periodic with period b. Hence we conclude that $c=1$. Then the identity (173) implies that

$$H_1(z)\equiv H_2(z).$$

Thus the assertion of Theorem 2 follows.

9. Remark.

Our results can be generalized to the case of entire functions in several complex variables. For example, for the non-zero constants b and b', we consider the following class

$$G(b, b')\equiv\{F(z, w)=f(z)+g(w)\,;\ f(z)\in G(b)\quad\text{and}\quad g(z)\in G(b')\}.$$

Then we obtain from Theorem 1 that, when $F(z, w)\in G(b_1, b_1')$ and $E(z, w)\in G(b_2, b_2')$ for some non-zero constants b_j, b_j' ($j=1, 2$), if further the following identical relation

$$F(z, w)=E(z, w)e^{p(z, w)}$$

holds for some entire function $p(z, w)$ in two complex variables z and w, then $p(z, w)$ must be constant and that b_1/b_2 and b_1'/b_2' are both rational numbers.

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Remark (added in the proof). About $^{*)}{}_1$: Under $p_j=p+c$, if in addition $p-q=\text{const.}=c'$, then using two relations (※) in p. 261, we represent $K(z)$ and $K(z+jb_1)$ as

$$K(z)=L_1+L_2\cdot e^{-p}\,,\qquad K(z+jb_1)=L_3+L_4\cdot e^{-p}\,,$$

where meromorphic functions L_i $(1\leqq i\leqq 4)$ are all of order less than one. Since $K(z)$ is periodic with period b_2, by Lemma B, we know that L_i must be constant such that $L_1=L_3$ and $L_2=L_4$ (cf. around (30)). In view of these facts, by applying Lemma B, we can conclude that ($R(z)$ is constant or) $e^c=1$ and $e^{c'}=1$. For instance, from $L_1=L_3$, if $RS_j-R_jS\not\equiv 0$ (otherwise R is constant), we deduce $(e^cS_j-S)(e^ck_jR_j-kR)=(e^cR_j-R)(e^cs_jS_j-sS)$ so that $(e^ck_jR_j-kR)/(e^cR_j-R)$ $=\text{const.}=c''$ (say), which is rewritten as $e^c(k_j-c'')R_j=(k-c'')R$ (cf. the notations (10), (12) and (14)). Hence $(k-c'')R$ is constant $(\neq 0)$ so that we have $e^c=1$. Similarly, from $L_2=L_4$ we can conclude $e^{c'}=1$. Thus $e^{p(z)}$ becomes doubly periodic, a contradiction.

About $^{*)}{}_2$: More simply as above, we can show that $p-q\not\equiv\text{const.}$ In fact, if otherwise, using (48), we get also $e^c=e^{c'}=1$ by Lemma B.