ON A CHARACTERIZATION OF THE TRIANGULAR ASSOCIATION SCHEME[1]

By S. S. Shrikhande

University of North Carolina

1. Introduction. A partially balanced incomplete block design with two associate classes [1] is said to be triangular [2] if the number of treatments is $v = n(n - 1)/2$ and the association scheme is an array of n rows and n columns with the following properties:

(a) The positions in the principal diagonal are blank.

(b) The $n(n - 1)/2$ positions above the principal diagonal are filled by the numbers $1, 2, \cdots, n(n - 1)/2$ corresponding to the treatments.

(c) The array is symmetric about the principal diagonal.

(d) For any treatment x the first associates are exactly those treatments which lie in the same row and same column as x.

It is then obvious that in the notation of [1]

(1) the number of first associates of any treatment is $n_1 = 2n - 4$;

(2) with respect to any two treatments x_1 and x_2 which are first associates (denoted by $(x_1, x_2) = 1$), the number of treatments which are first associates of both x_1 and x_2 is

$$p_{11}^1(x_1, x_2) = n - 2;$$

(3) with respect to any two treatments x_3 and x_4 which are second associates (denoted by $(x_3, x_4) = 2$) the number of treatments which are first associates of both x_3, x_4 is

$$p_{11}^2(x_3, x_4) = 4.$$

In an interesting paper Connor [3] has shown that if $n \geqq 9$, (1), (2) and (3) above imply (a), (b), (c) and (d), i.e., the association scheme is triangular. In this paper we derive a theorem and utilize it to prove that Connor's result is true for the cases $n = 5, 6$.

2. A characterization of the triangular association scheme. We prove a theorem which is equivalent to the theorem proved by Connor [3].

Theorem: *A necessary and sufficient condition that a partially balanced incomplete block design with two associate classes for $n(n - 1)/2$ treatments with $p_{11}^1(x_1, x_2) = n - 2$, where $(x_1, x_2) = 1$, has a triangular association scheme, is that all the first associates of any treatment x whatsoever can be divided into two*

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sets $(y_1, y_2, \cdots, y_{n-2})$, *and* $(z_1, z_2, \cdots, z_{n-2})$ *such that* $(y_i, y_j) = (z_i, z_j) = 1$ for $i \neq j = 1, 2, \cdots, n-2$.

PROOF. Necessity is obvious. We now prove the sufficiency.

Since y_i has $(n-3)$ first associates y_j and $p^1_{11}(x, y_i) = n - 2$, y_i has just one treatment from the other set, say z_i such that $(y_i, z_i) = 1$ and $(y_i, z_j) = 2$ for $j \neq i$. Now suppose that $(y_{i_1}, z_i) = 1$ for $i_1 \neq i$. Then z_i has y_i, y_{i_1} and z_j, $j \neq i$ for its first associates giving the value $p^1_{11}(x, z_i) = n - 1$ which is a contradiction. Hence we can pair off the treatments of the two sets such that

$$(y_i, z_i) = 1, \qquad (y_i, z_j) = 2, \qquad i \neq j = 1, 2, \cdots, n-2.$$

We will use this fact repeatedly. Further it is obvious that if the first associates of any treatment can be divided into sets as above, this division into two sets can be done in a unique manner.

For simplicity let us assume that the first associates of 1 are given by the two sets,

$$\begin{array}{llll} (2, & 3, & \cdots, & (n-1)) \quad \text{and} \\ (n, & n+1, & \cdots, & (2n-3)), \end{array}$$

where any two treatments in the same column are first associates and two treatments from different columns are second associates. We will adopt this method of writing to indicate the relationship of the two treatments from different sets.

We now write the rows,

$$\begin{array}{cccccc} * & 1 & 2 & 3 & \cdots & (n-1) \\ 1 & * & n & n+1 & \cdots & (2n-3), \end{array}$$

where $*$ denotes that the corresponding position is blank.

Now amongst the treatments occurring so far the first associates of 2 are $1, 3, 4, \cdots, n-1$ and n. Let the remaining first associates be $(2n-2), (2n-1), \cdots, (3n-6)$. Assume without loss of generality that

$$(3, 2n-2) = (4, 2n-1) = \cdots = (n-1, 3n-6) = 1;$$

then we form the third row by putting 2, n and $*$ in the first three positions respectively and placing $(2n-2), (2n-1), \cdots, (3n-6)$ below $3, 4, \cdots, (n-1)$ respectively. Thus, we have the three following rows:

$$\begin{array}{ccccccc} * & 1 & 2 & 3 & 4 & \cdots & (n-1) \\ 1 & * & n & n+1 & n+2 & \cdots & (2n-3) \\ 2 & n & * & 2n-2 & 2n-1 & \cdots & (3n-6). \end{array}$$

We note that $*$ occurs in the principal diagonal positions and the array written so far is symmetric and that the new first associates of 2 are written after the position of the $*$ in the third row.

Now consider the first associates of treatment 3. The only treatments till now which are first associates of 3 are $1, 2, 4, \cdots, n-1, n+1, 2n-2$. Let the remaining $(n-4)$ first associates be $(3n-5), (3n-4), \cdots, (4n-10)$. The two sets of first associates of 3 are

$$(1, \quad 2, \quad 4, \quad 5, \quad \cdots \quad (n-1))$$

and

$$(n+1, \quad 2n-2, \quad 3n-5, \quad 3n-4, \quad \cdots \quad (4n-10)),$$

where we can assume without loss of generality that the treatments in the same column are first associates. We now write down the fourth row to give

$$\begin{array}{llllll} * & 1 & 2 & 3 & 4 & \cdots & (n-1) \\ 1 & * & n & n+1 & n+2 & \cdots & (2n-3) \\ 2 & n & * & 2n-2 & 2n-1 & \cdots & (3n-6) \\ 3 & n+1 & 2n-2 & * & 3n-5 & \cdots & (4n-10). \end{array}$$

The same method can be used to write down the other arrays corresponding to 4, 5, $\cdots$, $(n-1)$ respectively. It is easy to see that all the positions above the principal diagonal are filled in with the numbers 1, 2, $\cdots$, $n(n-1)/2$ occurring just once. Thus, conditions (a), (b) and (c) are satisfied. Further, any treatment x occurs just in one position above the principal diagonal, say in row i and column $j(\neq i)$. Then, it also occurs in row j and column i. Hence, the first associates of x are all the treatments of row i and all the treatments of row j. By symmetry the treatments of column j are exactly those occurring in row j. Hence, the first associates of x are exactly those treatments which occur in the same row and same column as x. Thus (d) is also satisfied. Hence the association scheme is triangular. This completes the proof.

As stated previously, this theorem is equivalent to one given by Connor [3]. In the present form, however, it is more directly useful.

3. Uniqueness of the triangular scheme for $n = 5$. LEMMA 1. *The first associates of any treatment whatsoever for the design with parameters,*

$$v = 10, \quad n_1 = 6, \quad n_2 = 3, \tag{3.1}$$

$$p_{ij}^1 = \begin{pmatrix} 3 & 2 \\ & 1 \end{pmatrix}, \quad p_{ij}^2 = \begin{pmatrix} 4 & 2 \\ & 0 \end{pmatrix},$$

can be divided into two sets of three each such that any two treatments of the same set are first associates.

PROOF: Assume that first associates of 1 are 2, 3, 4, 5, 6, and 7 of which 3, 4 and 5 are first associates of 2, and 6 and 7 are second associates of 2. We then have

$$\begin{aligned} (1, 2) = (1, 3) = (1, 4) = (1, 5) = (1, 6) = (1, 7) &= 1 \\ (2, 3) = (2, 4) = (2, 5) &= 1 \\ (2, 6) = (2, 7) &= 2. \end{aligned}$$

We show that $(6, 7) = 1$. Suppose not, then $(6, 7) = 2$ and 2 is second associate of both 6 and 7 contradicting $p_{22}^2(6, 7) = 0$. Thus, we must have $(6, 7) = 1$.

Consider the pair $(1, 6) = 1$. 7 is first associate and 2 is second associate of 6. Hence 6 has two first associates and one second associate from the set (3, 4, 5). Assume that $(6, 4) = 2$ and hence, $(6, 3) = (6, 5) = 1$. Similarly, 7 has two first associates and one second associate from the set (3, 4, 5). We show that 4 cannot be second associate of 7. For if $(7, 4) = 2$, then $(6, 7) = 1$ and 2 and 4 are common second associates of both 6 and 7, contradicting $p_{22}^1(6, 7) = 1$. Hence, we must have $(7, 4) = 1$, and hence 7 has one first associate and one second associate from the set (3, 5). We can assume without loss of generality that $(7, 3) = 2$, $(7, 5) = 1$. Then we have the set (5, 6, 7) such that any two treatments of the set are first associates. Now we consider the set (2, 3, 4). We already know that $(2, 3) = (2, 4) = 1$. We now show that $(3, 4) = 1$. Now the common first associates of 1 and 5 are 2, 6, 7. Hence, 3 and 4 are second associates of 5. Hence, we must have $(3, 4) = 1$ in the same way as we obtained $(6, 7) = 1$ above. Thus, (2, 3, 4) is another set of first associates of 1 such that any two members of the set are first associates. A similar result is true for any other treatment. This proves the Lemma.

An appeal to the theorem now gives the corollary:

COROLLARY. *A partially balanced design with parameters* (3.1) *has triangular association scheme.*

4. Uniqueness of the triangular association scheme for $n = 6$. LEMMA 2. *The first associates of any treatment whatsoever for the design with parameters,*

$$v = 15, \quad n_1 = 8, \quad n_2 = 6 \tag{4.1}$$

$$p_{ij}^1 = \begin{pmatrix} 4 & 3 \\ & 3 \end{pmatrix}, \quad p_{ij}^2 = \begin{pmatrix} 4 & 4 \\ & 1 \end{pmatrix},$$

can be broken up into two sets of four each such that any two treatments of the same set are first associates.

PROOF: Assume without loss of generality that the second associates of treatment 1 are the treatments 10, 11, 12, 13, 14, 15, and those of 10 are 1, 4, 5, 8, 9, 15, so that 15 is the only common second associate (since $p_{22}^2 = 1$) of both 1 and 10. Hence, any two treatments of the set (1, 10, 15) are second associates. Then, considering the pairs (1, 10) and (1, 15), it is easy to see that 11, 12, 13, 14 are first associates of both 10 and 15. Now, $(1, 11) = 2$ and 10 and 15 are first associates of 11. Hence, from the value $p_{11}^2\ (1, 11) = 4$, we see that 11 has two first associates from the set (12, 13, 14). Let these be 12 and 13 so that

$$(11, 12) = (11, 13) = 1, \qquad (11, 14) = 2.$$

Now, $(1, 14) = 2$ and, as before, 14 has two first associates from the set (11, 12, 13). These are obviously 12 and 13, since $(14, 11) = 2$. Hence, we have

$$(12, 14) = (13, 14) = 1.$$

Similarly considering the pair (1, 12) and noting that $(12, 11) = (12, 14) = 1$, we get $(12, 13) = 2$

All the above information can be easily read by writing the second associates of 1 in the following scheme:

$$S_1: \quad \begin{array}{lll} 1 & 10 & 15 \\ \left|\begin{array}{l} 11 \\ 14 \end{array}\right. & & \\ \left|\begin{array}{l} 12 \\ 13 \end{array}\right. & & \end{array}$$

The explanation of the scheme is as follows: Treatments 10, 11, $\cdots$, 15 are second associates of 1, where 11, 12, 13, 14 are first associates and 15 is a second associate of 10. We write 10 and 15 in the row in the second and third positions respectively. Treatments 11, 12, 13, 14 are also first associates of 15. Further any two treatments of the set (1, 10, 15) are second associates. Of the six pairs from the set (11, 12, 13, 14), only those marked by straight lines on the left are second associates, while the remaining four pairs are first associates. The relations implied by S_1 are written completely as follows:

$$\begin{array}{ll} (4.2) & (1, 10) = (1, 11) = (1, 12) = (1, 13) = (1, 14) = (1, 15) = 2 \\ & (10, 15) = 2, \quad (10, 11) = (10, 12) = (10, 13) = (10, 14) = 1 \\ & (15, 11) = (15, 12) = (15, 13) = (15, 14) = 1 \\ & (11, 14) = 2, \quad (11, 12) = (11, 13) = 1 \\ & (14, 12) = (14, 13) = 1, \quad (12, 13) = 2. \end{array}$$

Now, among the seven treatments above the only second associates of 15 are 1 and 10. Let the remaining four second associates of 15 be 2, 3, 6, 7. Then, as before 2, 3, 6, 7 are first associates of both 1 and 10. Without loss of generality assume that $(2, 7) = (3, 6) = 2$ and hence, $(2, 3) = (2, 6) = (7, 3) = (7, 6) = 1$. Hence, we can represent the second associates of 15 in the following scheme:

$$S_2: \quad \begin{array}{lll} 15 & 1 & 10 \\ \left|\begin{array}{l} 2 \\ 7 \end{array}\right. & & \\ \left|\begin{array}{l} 3 \\ 6 \end{array}\right. & & \end{array}$$

The new relations implied by S_2 are

$$\begin{array}{ll} (4.3) & (15, 2) = (15, 3) = (15, 6) = (15, 7) = 2 \\ & (1, 2) = (1, 3) = (1, 6) = (1, 7) = 1 \\ & (10, 2) = (10, 3) = (10, 6) = (10, 7) = 1 \\ & (2, 7) = 2, \quad (2, 3) = (2, 6) = 1 \\ & (7, 3) = (7, 6) = 1, \quad (3, 6) = 2. \end{array}$$

We now consider the relation of any treatment from the set (2, 3, 6, 7) with any treatment of the set (11, 12, 13, 14).

Now, $(1, 2) = 1$ and 10, 15 are, respectively, first and second associates of 2.

Hence, from the value $p_{12}^{1}(2, 1) = 3$ and $p_{22}^{1}(2, 1) = 3$, we see that 2 has exactly two first associates and exactly two second associates from the set (11, 12, 13, 14). Suppose we have $(2, 11) = (2, 14) = 1$ and hence, $(2, 12) = (2, 13) = 2$. Then, since $(12, 13) = 2$ and the common second associates of both 12 and 13 are 1 and 2, we get $p_{22}^{2} = 2$. Hence, a contradiction. We get a similar contradiction, if we assume that 11, 14 are second associates of 2. Hence, the only possible case is that 2 has just one first associate and just one second associate from each set (11, 14) and (12, 13). We can assume without loss of generality that

$$(2, 11) = (2, 12) = 1, \qquad (2, 14) = (2, 13) = 2. \tag{4.4}$$

Now, consider the pair $(15, 11) = 1$. Here 1 and 10 are respectively second and first associates of 11. Hence, as before, of the remaining four second associates of 15, i.e., 2, 3, 6, 7, exactly two are first associates and exactly two are second associates of 11. A similar argument shows that 11 has exactly one first associate and exactly one second associate from the sets (2, 7) and (3, 6). But, we already have $(11, 2) = 1$, and hence we must have

$$(11, 7) = 2. \tag{4.5}$$

A similar argument considering the pair $(1, 7) = 1$ shows that

$$(7, 14) = 1. \tag{4.6}$$

In the same manner we also get

$$(7, 13) = 1, \qquad (7, 12) = 2. \tag{4.7}$$

We, thus, get the relationship of any treatment from the set (2, 7) with any treatment of the set (11, 12, 13, 14). A similar argument shows that 11 has just one first associate and just one second associate from the set (3, 6). Without loss of generality we can assume that

$$\begin{aligned} (11, 6) &= 1, \qquad (11, 3) = 2, \\ (14, 3) &= 1, \qquad (14, 6) = 2. \end{aligned} \tag{4.8}$$

Now, the relationship of 3, 6 with 12, 13 remains to be determined. Obviously, we have the two following possibilities: Either

$$\text{(A)}: (6, 12) = (3, 13) = 1, \qquad (6, 13) = (3, 12) = 2, \tag{4.9}$$

or

$$\text{(B)}: (6, 12) = (3, 13) = 2, \qquad (6, 13) = (3, 12) = 1. \tag{4.10}$$

We now proceed to show that case (B) is impossible.

Amongst the eleven treatments occurring so far, the only second associates of 10 are 1 and 15. Hence, the remaining second associates of 10 are 4, 5, 8, 9. Of the six possible pairs just two of them are second associates. Assume without loss of generality that

$$\begin{aligned} &(4, 9) = (5, 8) = 2 \\ &(4, 5) = (4, 8) = (9, 5) = (9, 8) = 1. \end{aligned} \tag{4.11}$$

TABLE 1

Treatment	First Associates			Second Associates		
Column 1	Column 2	Column 3	Column 4	Column 5	Column 6	Column 7
1	2 3 4 5 6 7 8 9			10 11 12 13 14 15		
2	1 3 6 10 11 12			7 13 14 15		
3	1 2 7 10 12 14			6 11 13 15		
4	1 5 8 11 15	13	12	9 10 14	12	13
5	1 4 9 14 15	13	12	8 10 11	12	13
6	1 2 7 10 11 13			3 12 14 15		
7	1 3 6 10 13 14			2 11 12 15		
8	1 4 9 11 15	12	13	5 10 14	13	12
9	1 5 8 14 15	12	13	4 10 11	13	12
10	2 3 6 7 11 12 13 14			1 4 5 8 9 15		
11	2 4 6 8 10 12 13 15			1 3 5 7 9 14		
12	2 3 10 11 14 15	8 9	4 5	1 6 7 13	4 5	8 9
13	6 7 10 11 14 15	4 5	8 9	1 2 3 12	8 9	4 5
14	3 5 7 9 10 12 13 15			1 2 4 6 8 11		
15	4 5 8 9 11 12 13 14			1 2 3 6 7 10		

We can represent this by

$$S_3: \quad \begin{array}{lll} 10 & 15 & 1 \\ \vert\ 4 & & \\ \vert\ 9 & & \\ \vert\ 8 & & \\ \vert\ 5 & & \end{array}$$

Also, we can assume with loss of generality by considering the pair (10, 11) = 1, that

$$\begin{aligned} (11, 4) = (11, 8) = (14, 9) = (14, 5) = 1 \\ (11, 9) = (11, 5) = (14, 4) = (14, 8) = 2. \end{aligned} \tag{4.12}$$

We note that the relations (4.11) and (4.12) do not depend in any manner on the relation (B). We summarize the information given by (4.2), $\cdots$, (4.8), (4.10), (4.11) and (4.12) in Table 1 in columns 1, 2 and 5.

We now consider the possible relationship of the treatments 12, 13 with the treatments 4, 5, 8, 9. We have the four possible cases,

(i)
$$\begin{aligned} (12, 4) = (12, 8) = (13, 9) = (13, 5) = 1 \\ (12, 9) = (12, 5) = (13, 4) = (13, 8) = 2, \end{aligned}$$

(ii)
$$\begin{aligned} (12, 9) = (12, 5) = (13, 4) = (13, 8) = 1 \\ (12, 4) = (12, 8) = (13, 9) = (13, 5) = 2, \end{aligned}$$

(iii)
$$\begin{aligned} (12, 8) = (12, 9) = (13, 4) = (13, 5) = 1 \\ (12, 4) = (12, 5) = (13, 9) = (13, 8) = 2, \end{aligned}$$

(iv)
$$\begin{aligned} (12, 8) = (12, 9) = (13, 4) = (13, 5) = 2 \\ (12, 4) = (12, 5) = (13, 9) = (13, 8) = 1. \end{aligned}$$

Of these case (i) is impossible, since otherwise from Table 1 and columns 1 and 2 we see that (11, 12) = 1 and 11 and 12 would have five common first associates contradicting $p_{11}^1 = 4$. Similarly, case (ii) gives (11, 12) = 1 and $p_{11}^1(11, 12) = 3$. We are thus left with only case (iii) and case (iv).

We now consider case (iii). The information given by this is entered in columns 3 and 6. We now consider the possible relationships of 2 and 7 with 4, 5, 8, 9. We have the following cases to be considered:

(α) $\quad (2, 9) = (2, 5) = (7, 4) = (7, 8) = 1$
$\quad (2, 4) = (2, 8) = (7, 5) = (7, 9) = 2,$

(β) $\quad (2, 9) = (2, 5) = (7, 4) = (7, 8) = 2$
$\quad (2, 4) = (2, 8) = (7, 5) = (7, 9) = 1,$

(γ) $\quad (2, 9) = (2, 8) = (7, 4) = (7, 5) = 1$
$\quad (2, 4) = (2, 5) = (7, 9) = (7, 8) = 2,$

(δ) $\quad (2, 9) = (2, 8) = (7, 4) = (7, 5) = 2$
$\quad (2, 4) = (2, 5) = (7, 9) = (7, 8) = 1.$

Referring to Table 1 and columns 1, 2, 3, 5 and 6 we see that (14, 2) = 2, and cases (α) and (β) give p_{22}^2 (14, 2) = 2 and 0 respectively, giving a contradiction since $p_{22}^2 = 1$. Similarly, (13, 2) = 2 and (γ) and (δ) give p_{22}^2 (13, 2) = 0 and 2 respectively, again a contradiction. Hence, we see that case (iii) is impossible.

We now suppress the information in columns 3 and 6 and put down the information given by case (iv) in columns 4 and 7. With case (iv) we again consider the cases (α), (β), (γ) and (δ). We now look up columns 1, 2, 4, 5, 7 of Table 1. Again, (14, 2) = 2 and (α) and (β) give p_{22}^2 (14, 2) = 2 and 0 respectively. Similarly, (13, 2) = 2 and (γ) and (δ) give p_{22}^2 (13, 2) = 2 and 0 respectively. Hence, a contradiction again. Thus, case (iv) is also impossible. It is now clear that case (B) is impossible, and we are left with case (A) alone. The relations (4.2), . . . , (4.9) now give the following two sets of first associates of treatment 10:

$$(11, 12, 2, 6)$$

and

$$(14, 13, 7, 3),$$

where any two treatments from each of the two sets are first associates.

A similar result can be proved for any treatment x by considering its two second associates y and z where $(y, z) = 2$ and taking the four remaining second associates of y and z which will be the eight first associates of x. This completes the proof of Lemma 2.

The application of the theorem now gives the corollary.

COROLLARY. *A design with parameters* (4.1) *has triangular association scheme.*

5. Uniqueness of Triangular Association Scheme for $n \geqq 9$. A lemma similar to Lemmas 1 and 2 can be proved for this case which implies that the association scheme is triangular if $n \geqq 9$. The proof is omitted, as another proof has already been given by Connor [3].

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