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ON A CLASS OF ALIGNED RANK ORDER TESTS IN TWO-WAY LAYOUTS[1]

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1. Summary and introduction. The present investigation is concerned with the formulation of a multivariate approach for the construction of a class of aligned rank order tests for the analysis of variance (ANOVA) problem relating to two-way layouts. The problems of simultaneous testing and testing for ordered alternatives based on aligned rank order statistics are also considered. Various efficiency results pertaining to the proposed tests are studied.

Let us consider a two factor experiment comprising n blocks, each block containing $p(\geqq 2)$ plots receiving p different treatments. In accordance with the two-way ANOVA model, we express the yield X_{ij} of the plot receiving the jth treatment in the ith block as

$$X_{ij} = \mu + \alpha_i + \tau_j + \epsilon_{ij}, \qquad i = 1, \cdots, n, j = 1, \cdots, p; \tag{1.1}$$

where μ stands for the mean effect, $\alpha_1, \cdots, \alpha_n$ for the block effects (may or may not be stochastic), $\tau_1, \cdots, \tau_p$ for the treatment effects (assumed to be non-stochastic), and ϵ_{ij}'s are the residual error components. It is assumed that $\boldsymbol{\epsilon}_i = (\epsilon_{i1}, \cdots, \epsilon_{ip})$, $i = 1, \cdots, n$ are independent and identically distributed stochastic vectors having a continuous (joint) cumulative distribution function (cdf) $G(x_1, \cdots, x_p)$ which is symmetric in its p arguments; (this includes the conventional situation of independence and identity of distributions of all the np error components as a special case). We may set without any loss of generality $\sum_1^p \tau_j = 0$, and frame the null hypothesis of *no treatment effect* as

$$H_0 : \tau_1 = \cdots = \tau_p = 0. \tag{1.2}$$

The usual ANOVA test based on the variance-ratio criterion is valid only when G is a p-variate (totally symmetric) multinormal cdf. For arbitrarily continuous cdf $G(x_1, \cdots, x_p)$, intra-block rank tests are due to Friedman [7], Brown and Mood [3], and Sen [21]; generalizations of these tests to incomplete layouts are due to Durbin [6], Benard and Elteren [1], and Bhapkar [2]. Hodges and Lehmann [9] have pointed out that intra-block rank tests do not utilize the information contained in the interblock comparisons, and hence, are comparatively less efficient. They have suggested the use of ranking after alignment and also considered the Wilcoxon's and Kruskal-Wallis tests based on aligned observations whose asymptotic efficiency have been studied very recently by Mehra and Sarangi [14]: (After the first draft of the paper was submitted, the author came to know through the editor about the paper by Mehra and Sarangi [14], submitted

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earlier to the *Annals*. In view of this, the overlapping part is not considered here.) The object of the present investigation is to formulate briefly the theory of rank order tests based on observations after alignment and through a multivariate approach to justify the validity and efficacy of the proposed tests (which include the earlier works as special cases).

2. Preliminary notions. Let us define the aligned yields and errors by

$$(2.1)\quad Y_{ij} = \sum_{l=1}^{p} c_{jl}X_{il}, \qquad e_{ij} = \sum_{l=1}^{p} c_{jl}\epsilon_{il}, \qquad c_{jl} = \delta_{jl} - 1/p,$$

for $i = 1, \cdots, n, j, l = 1, \cdots, p$, where δ_{jl} is the Kronecker delta. By definition $(e_{i1}, \cdots, e_{ip})$, $i = 1, \cdots, n$, are n independent and identically distributed stochastic vectors having a continuous cdf $G^*(x_1, \cdots, x_p)$ which is also symmetric in its p arguments. Thus, if H_0 in (1.2) is true, $Y_{ij} = e_{ij}, j = 1, \cdots, p$, are interchangeable random variables for each $i(= 1, \cdots, n)$. On the other hand, if H_0 is not true, $Y_{ij} = \tau_j + e_{ij}, j = 1, \cdots, p$, are interchangeable only after shift in locations. Thus the test for the null hypothesis (1.2) reduces to that of testing the interchangeability of $(Y_{i1}, \cdots, Y_{ip})$, $(i = 1, \cdots, n)$ against shift alternatives.

In this paper, we shall specifically consider the case $p > 2$. For $p = 2$, the problem reduces to that of the one sample location problem to which known solutions exist (cf. [22]). Let us arrange the $N(= np)$ observations $Y_{ij}, j = 1 \cdots, p, i = 1, \cdots, n$ in order of magnitude, and let R_{ij} be the rank of Y_{ij} for $j = 1, \cdots, p, i = 1, \cdots, n$. (Since, $e_{i1}, \cdots, e_{ip}$ are interchangeable random variables it follows that if their joint distribution has the rank >1 (or is absolutely continuous), ties among Y_{ij}'s may be neglected, in probability. In the sequel (cf. after (3.8)), it will be assumed that the rank of this distribution is >1.) For every N, we define a sequence $\mathbf{E}_N = (E_{N1}, \cdots, E_{NN})$ of rank functions where

$$(2.2)\qquad E_{N\alpha} = J_N(\alpha/N), \qquad 1 \leqq \alpha \leqq N;$$

the function J_N being defined in accordance with the Chernoff-Savage [4] conventions, and it satisfies the conditions (c.1)–(c.5) of Sen [19]. Thus, it is assumed that $\lim_{N\to\infty} J_N(u) = J(u)$ exists for all u: $0 < u < 1$ and is not a constant. Besides, among other conditions it is also assumed that $|J^{(r)}(u)| = |(d^r/du^r)J(u)| \leqq K[u(1 - u)]^{-r-\frac{1}{2}+\delta}(0 < u < 1)$, for some $\delta > 0$, $K < \infty$ and $r = 0, 1$. We define $Z_{N\alpha}^{(j)} = 1$, if the αth smallest observation among the N values of Y_{ij}'s is from the jth treatment and let $Z_{N\alpha}^{(j)} = 0$, otherwise, for $j = 1, \cdots, p$, $\alpha = 1, \cdots, N$. We shall be working then with the class of statistics $\mathbf{T}_N = (T_{N,1}, \cdots, T_{N,p})$ where

$$(2.3)\qquad T_{N,j} = (1/n) \sum_{\alpha=1}^{N} E_{N\alpha}Z_{N\alpha}^{(j)}, \qquad j = 1, \cdots, p$$

Our proposed test-statistic is a quadratic form in $\mathbf{T}_N$ and the same is formulated in Section 3.

3. The test-statistic and its rationality. Under the null hypothesis (1.2) the joint cdf of $(Y_{i1}, \cdots, Y_{ip})$ remains invariant under the $p!$ permutations of the coordinates among themselves, for each $i(= 1, \cdots, n)$. Thus, there exists a group $\mathcal{G}_n$ of $(p!)^n$ intra-block permutations which maps the sample space onto itself and leaves the distribution invariant (under H_0). Thus, conditioned on the n sets of ordered observations $(Y_{i(1)}, \cdots, Y_{i(p)})$, $i = 1, \cdots, n$, the conditional distribution of $(Y_{i1}, \cdots, Y_{ip})$, $i = 1, \cdots, n$, over the $(p!)^n$ intra-block permutations will be equally likely, each having the common permutational (conditional) probability $(p!)^n$. Let us denote this permutational (conditional) probability measure by $\mathcal{P}_n$. Then, by simple arguments, it follows that

$$E_{\mathcal{P}_n}(T_{N,j}) = \bar{E}_N = (1/N) \textstyle\sum_{\alpha=1}^{N} E_{N\alpha}, \qquad \text{for } j = 1, \cdots, p. \tag{3.1}$$

Let us also define

$$E_{NR_{i\cdot}} = (1/p) \textstyle\sum_{j=1}^{p} E_{NR_{ij}} \qquad \text{for } i = 1, \cdots, n; \tag{3.2}$$

$$\sigma^2(\mathcal{P}_n) = \{1/n(p-1)\} \textstyle\sum_{i=1}^{n} \sum_{j=1}^{p} \{E_{NR_{ij}} - E_{NR_{i\cdot}}\}^2. \tag{3.3}$$

Then by routine computations we have

$$\operatorname{Cov}_{\mathcal{P}_n}(T_{N,j}, T_{N,k}) = [(\delta_{jk}p - 1)/np]\sigma^2(\mathcal{P}_n) \quad \text{for } j, k = 1, \cdots, p. \tag{3.4}$$

Thus considering the quadratic form in $(T_{N,j}, j = 1, \cdots, p-1)$ with the inverse of the permutational covariance matrix as its discriminant and finally symmetrizing it we obtain the test-statistic

$$S_N = n \textstyle\sum_{j=1}^{p} \{T_{N,j} - \bar{E}_N\}^2/\sigma^2(\mathcal{P}_n). \tag{3.5}$$

For small values of n (and p) the exact permutation distribution of S_N can be computed (by reference to the $(p!)^n$ equally likely intra-block rank permutations), a task which becomes exceedingly laborious for large values of n or p. For this we consider the following large sample approach.

No matter whether H_0 in (1.2) holds or not, we define by $F_{[j]}(x)$ and $F_{[j,k]}(x, y)$ as the marginal cdf of Y_{ij} and (Y_{ij}, Y_{ik}), respectively, and let

$$H(x) = (1/p) \textstyle\sum_{j=1}^{p} F_{[j]}(x); \tag{3.6}$$

$$H^*(x, y) = \binom{p}{2}^{-1} \textstyle\sum_{1 \leqq j < k \leqq p} F_{[j,k]}(x, y). \tag{3.7}$$

Further, let

$$\sigma_0^2(\mathcal{P}_n) = (n(p-1))^{-1} \textstyle\sum_{i=1}^{n} \sum_{j=1}^{p} \{J(R_{ij}/N+1) - p^{-1} \sum_{l=1}^{p} J(R_{il}/N+1)\}^2,$$

and

$$\delta^2 = \textstyle\int_0^1 J^2(u)\, du - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} J[H(x)]J[H(y)]\, dH^*(x, y). \tag{3.8}$$

We also assume that if $p > 2$, the rank of the distribution of $\{J[H(Y_{ij})], j = 1, \cdots, p\}$ is >1; (for $p = 2$, the rank will be exactly equal to 1, and the results of [20] will apply).

THEOREM 3.1. *Under the assumed conditions, both $\sigma^2(\mathcal{P}_n)$ and $\sigma_0^2(\mathcal{P}_n)$ stochastically converge to δ^2, which is strictly positive.*

The proof of this theorem is a direct adaptation of Theorems 4.1 and 4.2 of [19], and hence is omitted.

THEOREM 3.2. *Under the conditions* (c.1)−(c.5) *of* [19], *the statistic* S_N *has asymptotically, in probability,* (*under* $\mathcal{P}_n$) *a chi-square distribution with* $p - 1$ *degrees of freedom* (d.f.).

PROOF. By virtue of (3.1), (3.3), (3.4), (3.5) and Theorem 3.1, it is sufficient to show that for any non-null $\boldsymbol{\delta}(\perp \mathbf{1}_p = (1, \cdots, 1))$, $n^{\frac{1}{2}}\boldsymbol{\delta}\cdot\mathbf{T}_N$ has asymptotically a normal distribution (under $\mathcal{P}_n$). Using the condition (c.2) of [19], we can write $n^{\frac{1}{2}}\boldsymbol{\delta}\cdot\mathbf{T}_N = n^{-\frac{1}{2}}\sum_{i=1}^{n}\{\sum_{j=1}^{p}\delta_j J(R_{ij}/N)\} + o_p(1)$.

Now, under $\mathcal{P}_n$, $U_{Ni} = \sum_{j=1}^{p}\delta_j J(R_{ij}/N+1)$, $i = 1, \cdots, n$ are all independent random variables, each having $p!$ equally likely realizations (obtained by intrablock rank permutations). It is also seen that under $\mathcal{P}_n$, $n^{-\frac{1}{2}}\sum_{i=1}^{n}U_{ni}$ has mean 0, variance $\sigma_0^2(\mathcal{P}_n)\cdot\sum_{j=1}^{p}\delta_j^2$ and $(1/n)\sum_{i=1}^{n}E_{\mathcal{P}_n}\{|U_{ni}|^{2+\delta'}\} < \infty$, $\delta' > 0$, ($\delta' \leqq \delta$), uniformly in $\mathbf{R}_i = (R_{i1}, \cdots, R_{ip})$, $i = 1, \cdots, n$ (by virtue of the growth condition (c.3) imposed on $J(u): 0 < u < 1$). The rest of the proof is then completed by Theorem 3.1 and the use of the Berry-Esseen theorem (cf. [13], p. 288) (which is also applicable for double sequence of random variables). QED.

By virtue of Theorem 3.2, we have the following test function:

$$\text{(3.9)} \qquad \phi(S_N) \text{ is } 1 \text{ or } 0 \text{ according as } S_N \text{ is } \geqq \text{ or } < S_{N,\epsilon},$$

where $S_{N,\epsilon}$ asymptotically equals to $\chi^2_{p-1,\epsilon}$, the $100\epsilon\%$ point of the right hand tail of a χ^2 variable with $(p-1)$ d.f.

4. Asymptotic efficiency of $\phi(S_N)$. For this study we shall consider a sequence of shift alternatives specified by

$$\text{(4.1)} \qquad K_N: \tau_j = \tau_{j,N} = N^{-\frac{1}{2}}\theta_j, \qquad j = 1, \cdots, p; \qquad \sum_1^p \theta_j = 0,$$

θ's being all real and finite. Thus, under $\{K_N\}$

$$\text{(4.2)} \qquad F_{[j]}(x) = F_{[j],N}(x) = F(x - N^{-\frac{1}{2}}\theta_j), \qquad j = 1, \cdots, p,$$

$$\text{(4.3)} \qquad F_{[j,k]}(x, y) = F_{[j,k],N}(x, y) = F^*(x - N^{-\frac{1}{2}}\theta_j, y - N^{-\frac{1}{2}}\theta_k),$$

for $j \neq k = 1, \cdots, p$, where F and F^* are some continuous cdf's. It also follows from (3.6) and (3.7) that $H(x)$ and $H^*(x, y)$ then converges to $F(x)$ and $F^*(x, y)$, respectively, as $N \to \infty$. We assume that $F(x)$ is absolutely continuous and it satisfies the conditions of Lemma 7.2 of Puri [16]. Let then

$$\text{(4.4)} \quad B(F) = \int_{-\infty}^{\infty}(d/dx)J[F(x)]\,dF(x);$$

$$\text{(4.5)} \qquad A^2 = \int_0^1 J^2(u)\,du - [\int_0^1 J(u)\,du]^2,$$

$$\text{(4.6)} \qquad \rho_J = A^{-2}[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J[F(x)]J[F(y)]\,dF^*(x, y) - \{\int_0^1 J(u)\,du\}^2].$$

Then on expressing $T_{N,j}$ $(j = 1, \cdots, p)$ in the Chernoff-Savage integral form (cf. [4], p. 973), adapting the same line of proof as in Theorem 5.1 of [19] and avoiding the details, we arrive at the following.

THEOREM 4.1. *Under the sequence of alternatives* $\{K_N\}$ *in* (4.1), $[n^{\frac{1}{2}}(T_{N,j} - \bar{E}_N), j = 1, \cdots, p]$ *has (jointly) asymptotically a multinormal distribution with a mean vector* $[p^{-\frac{1}{2}}\theta_j B(F), j = 1, \cdots, p]$ *and a covariance matrix with the elements* $(1/p)$. $(\delta_{jk}p - 1)A^2(1 - \rho_J), j, k = 1, \cdots, p$.

Again it follows from (3.8), (4.1), (4.2), (4.3) and (4.6) (through some simplifications that) under $\{K_N\}$, δ^2, defined by (3.8), converges to $A^2(1 - \rho_J)$, as $N \rightarrow \infty$. Hence from Theorem 3.1 we obtain that

$$\sigma^2(\mathcal{P}_n) \rightarrow_P A^2(1 - \rho_J), \quad \text{under } \{K_n\}. \tag{4.7}$$

From (3.5), (4.7) and Theorem 4.1, we readily obtain the following.

THEOREM 4.2. *Under the sequence of alternatives* $\{K_N\}$ *in* (4.1) *and subject to the conditions of Theorem* 1 *of Chernoff and Savage* [4] *and of Lemma* 7.2 *of Puri* [16], S_N, *defined by* (3.5), *has asymptotically a noncentral chi-square distribution with* $(p - 1)$ d.f. *and the noncentrality parameter*

$$\Delta_S = [(1/p)\textstyle\sum_{j=1}^{p}\theta_j^2]\{[B(F)]^2/A^2(1 - \rho_J)\}, \tag{4.8}$$

where $B(F)$, A^2 *and* ρ_J *are defined by* (4.4), (4.5) *and* (4.6). *Thus, under* H_0 *in* (1.2), S_N *has asymptotically (unconditionally) a chi-square distribution with* $p - 1$ d.f.

Now, if the error component ϵ_{ij} in (1.1) has a finite variance σ_ϵ^2 and if $(\epsilon_{ij}, \epsilon_{ik})$ have the correlation ρ_ϵ for all $j \neq k = 1, \cdots, p$, it is well-known that the classical ANOVA test (actually $(p - 1)\mathfrak{F}_{p-1,(n-1)(p-1)}$) has asymptotically a noncentral χ^2 distribution with $(p - 1)$ d.f. and the noncentrality parameter

$$\Delta_F = [(1/p)\textstyle\sum_{j=1}^{p}\theta_j^2]/\sigma_\epsilon^2(1 - \rho_\epsilon). \tag{4.9}$$

Hence from Theorem 4.2 and (4.9), we arrive at the following.

THEOREM 4.3. *For the sequence of alternatives* $\{K_N\}$ *in* (4.1), *the asymptotic relative efficiency* (ARE) *of the* S_N*-test with respect to the classical* ANOVA *test is given by*

$$e(\{S_J\}, \{\mathfrak{F}\}) = \sigma_\epsilon^2(1 - \rho_\epsilon)[B(F)]^2/A^2(1 - \rho_J). \tag{4.10}$$

Now, it follows from (2.1) that the variance of e_{ij} is nothing but the variance of the marginal cdf $F(x)$ and is equal to

$$\sigma_e^2 = \sigma_\epsilon^2(p - 1)(1 - \rho_\epsilon)/p. \tag{4.11}$$

Hence, we may rewrite (4.10) as

$$e(\{S_N\}, \{\mathfrak{F}\}) = \{p/[(p - 1)(1 - \rho_J)]\}\{\sigma_e^2[B(F)]^2/A^2\}, \tag{4.12}$$

which is independent of ρ_ϵ. Let us then consider the following two lemmas.

LEMMA 4.4. *If the* cdf $G(x_1, \cdots, x_p)$ *of* $\mathbf{X} = (X_1, \cdots, X_p)$ *is symmetric in its p arguments and the univariate marginal* cdf $F(x)$ *(corresponding to G) is non-degenerate, then* ρ_J, *defined by* (4.6), *is* $\geqq -1/(p - 1)$.

PROOF. We define the random variable $Z = \sum_{j=1}^{p} J[F(X_j)]$, which has the variance

$$\sigma_Z^2 = pA^2[1 + (p-1)\rho_J] \geqq 0 \tag{4.13}$$

where $A^2 = \int_{-\infty}^{\infty} J^2[F(x)]\, dF(x) - [\int_{-\infty}^{\infty} J[F(x)]\, dF(x)]^2 > 0$ since $J(F)$ is not a constant $(0 < F < 1)$ and F is non-degenerate. Hence, $\rho_J \geqq -1/(p-1)$.

LEMMA 4.5. *If $G(x_1, \cdots, x_p)$ is a totally symmetric continuous p-variate $(p > 2)$ singular distribution on the $(p-1)$-flat $\sum_{j=1}^{p} x_j = 0$, while there is no lower dimensional space containing the scatter of the points of G, then*

$$\rho_J = -1/(p-1)$$

iff $J[F(x)]$ is a linear function of x with probability one.

PROOF. $\sigma_Z^2 = 0$ iff $Z = \sum_{j=1}^{p} J[F(X_i)]$ is a constant, with Probability 1 i.e., $\sum_{j=1}^{p} J[F(X_j)]$ is a constant for all $\mathbf{X}$ such that $\sum_{j=1}^{p} X_j = 0$, with Probability 1. The *if* part of the proof is obvious. To prove the *only if* part, suppose, $J[F(X_j)]$ is not a linear function of X_j. Then $\sum_{j=1}^{p} J[F(X_j)] =$ constant along with $\sum_{j=1}^{p} X_j = 0$ (with Probability 1), implies that the scatter of the cdf G is really contained in a $(p-2)$-dimensional hyper-space, which contradicts the hypothesis that the rank is $p-1$. Hence the lemma. (Note that for $p = 2$, $J[F(x)]$ may not be linear in x, viz., $J[F(x)] = a + bx^3$.)

By virtue of Lemmas 4.4 and 4.5, we have from (4.12)

$$e(\{S_N\}, \{\mathfrak{F}\}) \geqq \sigma_e^2[B(F)]^2/A^2, \tag{4.14}$$

where the equality sign holds iff $J[F(x)]$ is a linear function of x, with Probability 1.

We shall now consider two specific types of rank order tests, namely, the normal scores and the rank-sum tests and study the resulting ARE. Let $\Phi(x)$ denote the standardized normal cdf, $\phi(x)$ the corresponding density function, and let $\mathcal{A}_{N\alpha}$ be the expected value of the αth smallest observation of a sample of size N drawn from the cdf $\Phi(x)$, for $\alpha = 1, \cdots, N$. For the normal scores $(\mathcal{L}_{N-})$ test, we use

$$E_{N\alpha} = \mathcal{A}_{N\alpha} \quad \text{for all} \quad \alpha = 1, \cdots, N, \tag{4.15}$$

and the corresponding ARE reduces to

$$e(\{\mathcal{L}_N\}, \{\mathfrak{F}\}) = [p/(p-1)(1-\rho_\phi)] \cdot \{\sigma_e^2[\int_{-\infty}^{\infty} (f^2(x)/\phi(\Phi^{-1}[F(x)]))\, dx]^2\} \tag{4.16}$$

where $f(x) = F'(x)$ and ρ_ϕ is the value of ρ_J in (4.6) when $J(F(x)) = \Phi^{-1}[F(x)]$. Consequently from (4.14) and (4.16), we have

$$e(\{\mathcal{L}_N\}, \{\mathfrak{F}\}) \geqq \sigma_e^2[\int_{-\infty}^{\infty} (f^2(x)/\phi(\Phi^{-1}[F(x)]))\, dx]^2, \tag{4.17}$$

where the equality sign holds iff $\Phi^{-1}[F(x)]$ is a linear function of x i.e., $F(x) = \Phi(ax + b)$ (with real and finite a, b) is also normal. Since σ_e^2 is the variance of the cdf $F(x)$, using the well-known result of Chernoff and Savage [4] on the efficiency of the normal scores tests, we obtain that the right hand side of (4.17) is $\geqq 1$, where the equality sign holds iff $F(x)$ is a normal cdf. Since, e_{ij} is a linear function of $(\epsilon_{i1}, \cdots, \epsilon_{ip})$, this means that $F(x)$ is normal iff the cdf $G(x_1, \cdots, x_p)$ of $(\epsilon_{i1}, \cdots, \epsilon_{ip})$ is also multivariate normal. *Thus, referred to the model* (1.1), *the normal scores* ($\mathcal{L}_N$-*test*) *is asymptotically at least as efficient as the classical* ANOVA *test*; *they are asymptotically equally efficient* iff *the errors are normally distributed.*

Let us then consider the rank-sum (H_N-) test for which $E_{N\alpha} = \alpha/(N + 1)$ for $\alpha = 1, \cdots, N$. In this case, (4.12) reduces to

$$(4.18) \quad e(\{H_N\}, \{\mathfrak{F}\}) = [p/(p - 1)(1 - \rho_g)]\{12\sigma_e^2[\textstyle\int_{-\infty}^{\infty} f^2(x)\,dx]^2\},$$

where ρ_g is the grade correlation of any two variates of $F(x_1, \cdots, x_p)$, (for definition see Hoeffding [10], p. 318). Now, from (4.14) and (4.18), we have

$$(4.19) \qquad e(\{H_N\}, \{\mathfrak{F}\}) \geqq 12\sigma_e^2[\textstyle\int_{-\infty}^{\infty} f^2(x)\,dx]^2.$$

Proceeding then precisely on the same line as in Hodges and Lehmann [8], it is easily seen that

$$(4.20) \qquad \inf_{F\varepsilon\mathfrak{F}_0} 12\sigma_e^2[\textstyle\int_{-\infty}^{\infty} f^2(x)\,dx]^2 = 0.864,$$

where $\mathfrak{F}_0$ is the family of all univariate continuous cdf's. Hence, the lower bound (0.864) to the efficiency of the rank-sum test (for one criterion ANOVA problem) with respect to the classical ANOVA test, proved by Hodges and Lehmann [8], also remains valid for the aligned rank-sum test in two-way layout (even under a somewhat more general situation where the errors are symmetric dependent). However, it is to be noted that by virtue of (4.14), the strict inequality in (4.19) holds unless F is a uniform cdf. But for a uniform cdf, the lower bound in (4.20) is not attained. Hence, $e(\{H_N\}, \{\mathcal{F}\})$ can not attain its lower bound 0.864. Further it is conjectured that there can not be any cdf $G(\boldsymbol{\varepsilon})$ of $\boldsymbol{\varepsilon}_i$ (in (1.1)) for which the corresponding cdf $G^*(\mathbf{e})$ of $\mathbf{e}_i$ (in (2.1)) has univariate marginals all uniform. If the errors ϵ_{ij}'s have normal distribution, it is well known that $\rho_g = (6/\pi)\text{Sin}^{-1}(\rho/2)$, (where ρ is the product moment correlation), and hence, (4.18) reduces to

$$(4.21) \quad (3/\pi)\cdot(p/(p - 1)[1 + (6/\pi)\ \text{Sin}^{-1}\ (\tfrac{1}{2}(p - 1))]) = e_p \quad \text{(say)}.$$

It may be noted that for $p = 2$, e_p is equal to $3/\pi$, (the efficiency of the usual rank sum tests by Wilcoxon or Kruskal and Wallis for one criterion variance analysis), while for $p > 2$, e_p is strictly greater than $3/\pi$. It may also be noted that the ARE of Friedman's [7] test with respect to the $\mathcal{F}$-test is equal to $3p/\{(p + 1)\pi\}$, which is strictly less than $3/\pi$. Since, the comparisons of these ARE's are already tabulated in [14], to avoid repetitions, the details are omitted. Note that the tabulated values of e_p in [14], though related to mutually independent errors, are shown here to be true also for intra-block symmetric-dependent errors.

5. Some additional remarks on aligned rank order tests in two-way layouts. We may consider the more general case where the jth treatment is applied to $m_j(\geqq 1)$ plots within each block, for $j = 1, \cdots, p$. Let $M = \sum_{j=1}^{p} m_j$ and $N' = nM$. Thus after alignment (ignoring treatment), to the N' observations N'

rank scores $\{E'_{N,\alpha} = J_{N'}(\alpha/N'), 1 \leqq \alpha \leqq N'\}$ will correspond. The average of the nm_j rank scores for the jth treatment is denoted by $T_{N',j}$, $j = 1, \cdots, p$. The (pooled) within block mean square of the rank scores $(\sigma^2(\mathcal{P}_n))$ is defined as in (3.3) with p replaced by M. The permutational-invariance of Section 3 also holds with p replaced by M. This yields that $E_{\mathcal{P}_n}(T_{N',j}) = (1/N') \sum_{\alpha=1}^{N'} E_{N',\alpha} = \bar{E}_{N'}$, $j = 1, \cdots, p$, and

$$(5.1) \quad \operatorname{Cov}_{\mathcal{P}_n}\{T_{N',j}, T_{N',k}\} = \sigma^2(\mathcal{P}_n)(\delta_{jk}M - m_k)/(Mm_kn), \quad j, k = 1, \cdots, p.$$

This leads to the test statistic

$$(5.2) \qquad S_{N',M} = [n/\sigma^2(\mathcal{P}_n)] \sum_{j=1}^{p} m_j[T_{N',j} - \bar{E}_{N'}]^2.$$

It can be shown similarly that under H_0 in (1.2), $S_{N',M}$ has asymptotically, in probability, (under $\mathcal{P}_n$) a chi-square distribution with $p - 1$ d.f., and it follows by straightforward extensions of Theorems 4.1 and 4.2 that for the sequence of alternatives in (4.1) (with N replaced by N') $S_{N',M}$ has asymptotically a non-central chi-square distribution with $p - 1$ d.f. and non-centrality parameter

$$(5.3) \quad \Delta_M = \{M^{-1} \sum_{j=1}^{p} m_j(\theta_j - M^{-1} \sum_{k=1}^{p} m_k\theta_k)^2\}\{B(F)^2/A^2(1 - \rho_J)\},$$

where $F(x)$ is the marginal cdf of an aligned error component. Comparison with the parametric ANOVA-test again leads to the ARE considered in (4.10) and (4.12). Hence, the bounds obtained in Section 4 also remain true in this more general case (with p replaced by M).

Secondly, we consider the problem of testing H_0 in (1.2) against the ordered alternative $H_1 : \tau_1 \leqq \cdots \leqq \tau_p$ (where at least one strict inequality holds). When the errors ϵ_{ij}'s are independent and identically distributed, asymptotically distribution-free rank tests for this problem have been considered by Hollander [12] and Doksum [5], and their generalizations are due to Puri and Sen [17]. The aligned rank order statistics considered in this paper provide very simple tests which are valid even in the more general case where $(\epsilon_{i1}, \cdots, \epsilon_{ip})$ are exchangeable, for each $i = 1, \cdots, n$. Define

$$(5.4) \quad T_N^* = (12n)^{\frac{1}{2}} \sum_{j=1}^{p} (j - \tfrac{1}{2}(p + 1))T_{N,j}/\{\sigma^2(\mathcal{P}_n)p(p^2 - 1)\}^{\frac{1}{2}}.$$

Using the proof of Theorem 3.2, it follows that under $\mathcal{P}_n$, T_N^* has asymptotically, in probability, a standard normal distribution, and the test procedure consists in rejecting H_0 in favour of H_1 when T_N^* exceeds the upper $100\epsilon\%$ point of a standard normal distribution. Again, using Theorem 4.1, it follows that under (4.1) (and subject to H_1), T_N^* has asymptotically a normal distribution with unit variance and mean $p^{-1}[B(F) \sum_{j=1}^{p} (j - (p + 1)/2)\theta_j][\sigma^2(\mathcal{P}_n)(p^2 - 1)/12]^{-\frac{1}{2}}$. The classical parametric test for this problem is based on the treatment means $\{X_{\cdot j} - X_{\cdot k}, j < k\}$ and is considered in detail in [5], [12], [17]. Again, on noting that even for symmetric dependent errors, $n^{\frac{1}{2}}(X_{\cdot j} - X_{\cdot\cdot})$, $j = 1, \cdots, p$, are asymptotically normally distributed having the common correlation coefficient $-1/(p - 1)$, the ARE of T_N^* with respect to the parametric test can be shown to be given by (4.10) and (4.12). Hence, the same efficiency bounds are also applicable in this case.

Finally, we consider the problem of simultaneous testing all paired differences $\tau_j - \tau_k$, $j \neq k = 1, \cdots, p$. The existing rank procedures include (i) treatments versus control multiple comparison sign test (cf. [23]), (ii) simultaneous sign-test (cf. [15]), (iii) simultaneous rank-sum test based on intra-block ranks (cf. [15]) and (iv) treatments versus control multiple comparison signed rank test (cf. [15], [11]). The sign-tests are usually of considerable low efficiency. Also the rank-sum test, like Friedman's [7] χ_r^2-test, does not utilize the information on inter-block comparisons, and hence, appears to lose some efficiency. The treatment versus control procedures depend on the other hand on the choice of a control when a natural one may not exist. A procedure free from all these drawbacks is sketched below. Define $\mathbf{T}_N$ and $\sigma^2(\mathcal{P}_n)$ as in (2.3) and (3.3) respectively, and let

$$W_N = \max_{1 \leq j<k \leq p} [n^{\frac{1}{2}} |T_{N,j} - T_{N,k}|/\sigma(\mathcal{P}_n)]. \tag{5.5}$$

Also, let $R_{p,\epsilon}$ be the upper $100\epsilon\%$ point of the distribution of the sample range of a sample of size p from a standard normal distribution. Since, under $\mathcal{P}_n$, $n^{\frac{1}{2}}(T_{N,j} - \bar{E}_N)$, $j = 1, \cdots, p$ are equicorrelated and asymptotically, in probability, normally distributed random variables (cf. Theorem 3.2), we readily arrive at the following.

THEOREM 5.1. *Under $\mathcal{P}_n$, the upper $100\epsilon\%$ point of the permutation distribution of W_n converges, in probability, to $R_{p,\epsilon}$.*

Again, using Theorems 3.1 and 4.1, it can be shown that the upper $100_\epsilon\%$ point of the null (unconditional) distribution of W_N also converges to $R_{p,\epsilon}$ as $n \to \infty$. Hence, we have the following simultaneous test procedure.

In the model (1.1), regard those pairs (τ_j, τ_k) $j \neq k$ to be significantly different from each other for which

$$n^{\frac{1}{2}} |T_{N,j} - T_{N,k}| \geq \sigma(\mathcal{P}_n)R_{p,\epsilon} \quad \text{for} \quad j \neq k = 1, \cdots, p. \tag{5.6}$$

By a straightforward extension of Theorem 4.2, it can be shown that the ARE of the simultaneous test in (5.6) with respect to Tukey's T-method (cf. [18], p. 74) is again given by (4.10) and (4.12). Consequently, all the efficiency results of Section 4 also hold for the above simultaneous test procedure.

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