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ON A CLASS OF COMPLEX FUNCTIONAL EQUATIONS

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Abstract. We consider a class of complex functional equations that admit transcendental meromorphic solutions with relatively few distinct poles. The solutions are characterized and it is shown that they must also satisfy a functional equation of the certain simple form. The equations that are considered contains e.g. some delay equations and the generalized Schröder equation as special cases. The reasoning relies on the combination of Nevanlinna theory and algebraic field theory.

1. Introduction

In this article a meromorphic function means meromorphic in the whole complex plane. We shall assume that the reader is familiar to Nevanlinna theory, see e.g. [9]. Especially, a meromorphic function g is small relative to a meromorphic function f, if $T(r,g)=o(T(r,f))$, when $r\to\infty$ outside of a set of finite linear measure. This is denoted by writing $T(r,g)=S(r,f)$. Moreover, we use the conventional notions $\rho(f)$, $\mu(f)$, $\lambda(f)$ and $\overline{\lambda}(f)$ for the order, the lower order, the exponent of convergence of zeros and the exponent of convergence of distinct zeros of a meromorphic function f respectively.

In the collection [12] of research problems, Rubel asked what can be said about the differential algebraic meromorphic solutions of the equation

$$f(cz)=\frac{\sum_{j=0}^{n}a_j(z)(f(z))^j}{\sum_{j=0}^{m}b_j(z)(f(z))^j}, \tag{1.1}$$

where $n,m\in\mathbf{N}\cup\{0\}$, a_j's and b_j's are rational functions and $c\in\mathbf{C}\setminus\{0,1\}$. Ishizaki has considered the existence of meromorphic solutions of the linear equation

$$f(cz)=a(z)f(z)+b(z), \tag{1.2}$$

where a and b are meromorphic functions and c is a nonzero complex constant such that $|c|\neq 1$.

Theorem 1.1. (Ishizaki, [7]) *Suppose that $a(z)$ has neither a zero nor a pole at the origin, $b(z)$ does not have a pole at the origin, and suppose that $a(0)\neq c^j$, $j=0,1,2,\dots$. Then, the functional equation* (1.2) *has a meromorphic solution.*

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Ishizaki also partially answered Rubel's question by the following

Theorem 1.2. (Ishizaki, [7]) *Suppose that $a(z)$ and $b(z)$ are rational functions in* (1.2). *Then all transcendental meromorphic solutions of* (1.2) *are hypertranscendental i.e. do not satisfy any nontrivial algebraic differential equation with coefficients that are rational functions.*

We shall consider the question by Rubel assuming that there exist a meromorphic solution of (1.1) that has finitely many poles and the denominator of the right-hand side of (1.1) is not trivial i.e. $m \geq 1$. Then we shall conclude that the solution satisfies the first order linear differential equation with rational coefficients.

However, we shall also deal with more general functional equations than (1.1), and the conclusion concerning Rubel's question follows from those results. To begin with, we recall the famous Tumura–Clunie theorem.

Theorem 1.3. (Tumura and Clunie, [1], [14]) *If f is a nonconstant entire function, $\phi := be^g = f^n + a_{n-1}f^{n-1} + \cdots + a_0$, $b, a_0, a_1, \ldots, a_{n-1}$ are small meromorphic functions relative to f and g is an entire function, then $\phi = (f + a_{n-1}/n)^n$.*

There exist a number of generalizations of this theorem, in particular, Weissenborn proved the following

Theorem 1.4. (Weissenborn, [15]) *Let f be a meromorphic function and let ϕ be given by $\phi = f^n + a_{n-1}f^{n-1} + \cdots + a_0$, where $a_0, a_1, \ldots, a_{n-1}$ are small meromorphic functions relative to f. Then either*

$$\phi = \left(f + \frac{a_{n-1}}{n}\right)^n \tag{1.3}$$

or

$$T(r,f) \leq \overline{N}\left(r, \frac{1}{\phi}\right) + \overline{N}(r,f) + S(r,f). \tag{1.4}$$

We shall extend Tumura–Clunie theorem by using Weissenborn's theorem to concern rational functions in f with small coefficients, see Theorem 3.1 below. As an application, we shall prove reduction theorems for functional equations that admit meromorphic solutions with relatively few distinct poles only, see Theorems 3.4, 3.5 and 3.6. These kind of results improves in some sense the corresponding results in [4], [5], [10] and [13].

A typical example of a reduction theorem considered in the papers mentioned above is

Theorem 1.5. (Gundersen et al., [4]) *Let f be a transcendental meromorphic solution of the equation* (1.1), *where $|c| > 1$, $b_m(z) = 1$ and where a_j's and b_j's are meromorphic functions such that*

$$\max_{k,j}\{T(r,a_j), T(r,b_k)\} = S(r,f).$$

If $\overline{N}(r,f)+\overline{N}(r,1/f)=S(r,f)$, *then the equation (1.1) is either of the form*

$$f(cz)=a_n(z)(f(z))^n \text{ or } f(cz)=\frac{a_0(z)}{(f(z))^m}. \tag{1.5}$$

Another example is

Theorem 1.6. (Laine et al., [10]) *Assume* $c_1,c_2,\dots,c_n\in\mathbf{C}\setminus\{0\}$, $f(z)$ *is a transcendental meromorphic solution to the delay equation*

$$\sum_{\{J\}}\alpha_J(z)\Bigg(\prod_{j\in J}f(z+c_j)\Bigg)=R(z,f(z))=\frac{P(z,f(z))}{Q(z,f(z))},$$

where $\{J\}$ *denotes all subsets of the set* $\{1,2,\dots,n\}$, P *and* Q *are relatively prime polynomials in* f *over the field of rational functions, the coefficients* α_J *are rational functions and where* $q:=\deg_f Q>0$. *Then, if* $f(z)$ *has at most finitely many poles, it must be of the form*

$$f(z)=r(z)e^{g(z)}+s(z),$$

where $r(z)$ *and* $s(z)$ *are rational functions and* $g(z)$ *is a transcendental entire function satisfying a difference equation of the form either*

$$\sum_{j\in J}g(z+c_j)=(j_0-q)g(z)+d$$

or

$$\sum_{j\in J}g(z+c_j)=\sum_{j\in I}g(z+c_j)+d.$$

Here J *and* I *are nonempty disjoint subsets of* $\{1,2,\dots,n\}$, $j_0\in\{0,1,\dots,p\}$, $p:=\deg_f P$, *and* $d\in\mathbf{C}$.

This paper is organized as follows. The section 2 contains growth results that will be needed later in this paper. The reduction theorems are the main topic in the section 3 and they play the main role in this paper. The section 4 contains results of existence and value distribution related to the generalized Schröder equation (1.1). Also the question by Rubel is discussed in section 4.

2. Growth of meromorphic solutions

The proof of the following lemma is included in the proof of Lemma 4 in [2].

Lemma 2.1. *Let* f *be a transcendental meromorphic function and* $p(z)=a_kz^k+a_{k-1}z^{k-1}+\cdots+a_0$ *be a complex polynomial of degree* $k>0$. *For given* $0<\delta<|a_k|$, *let*

$$\begin{cases}\lambda=|a_k|+\delta\\ \mu=|a_k|-\delta.\end{cases}$$

Then, for given $\varepsilon>0$ *and for* r *large enough,*

$$\begin{aligned}(1-\varepsilon)T(\mu r^k,f) &\le T(r,f\circ p)\le(1+\varepsilon)T(\lambda r^k,f),\\ \overline{N}(\mu r^k,f)+\mathscr{O}(\log r) &\le \overline{N}(r,f\circ p)\le\overline{N}(\lambda r^k,f)+\mathscr{O}(\log r).\end{aligned}$$

The next lemma describes a well known method to prove estimate for the lower order of a meromorphic function.

Lemma 2.2. *Let* $\phi : (r_0, \infty) \to (1, \infty)$*, where* $r_0 \geq 1$*, be a monotone increasing function. If for some real constant* $\alpha > 1$ *there exist a real number* $K > 1$ *such that* $\phi(\alpha r) \geq K\phi(r)$*, then*

$$\liminf_{r\to\infty} \frac{\log \phi(r)}{\log r} \geq \frac{\log K}{\log \alpha}. \tag{2.1}$$

Proof. Inductively, for any $m \in \mathbf{N}$,

$$\phi(\alpha^m r) \geq K^m \phi(r).$$

Let $r \in [r_0, \alpha r_0)$ and denote $s = \alpha^m r$. Then

$$m = \frac{\log s - \log r}{\log \alpha} > \frac{\log s - \log \alpha r_0}{\log \alpha}$$

and we obtain

$$\log \phi(s) \geq m \log K + \log \phi(r_0) > \frac{\log K}{\log \alpha}(\log s - \log \alpha r_0).$$

Dividing the above inequality by $\log s$ and letting $m \to \infty$ for each choice of $r \in [r_0, \alpha r_0)$ i.e. $s \to \infty$, we see that

$$\liminf_{s\to\infty} \frac{\log \phi(s)}{\log s} \geq \frac{\log K}{\log \alpha}. \qquad \square$$

We shall also need two lemmas that are generalizations of the results due Goldstein and Silvennoinen, see Theorem 10 in [2] and Theorems 2.1.2 and 2.1.5. in [13]. To prove the lemmas we need the following important result by Valiron and Mohon'ko.

Theorem 2.3. (Valiron–Mohon'ko, [9, Theorem 2.2.5 and Corollary 2.2.7]) *Let* f *be a meromorphic function. Then for all irreducible rational functions in* f*,*

$$R(z, f(z)) = \frac{\sum_{j=0}^{n} a_j(z)(f(z))^j}{\sum_{j=0}^{m} b_j(z)(f(z))^j}, \tag{2.2}$$

with meromorphic coefficients $a_j(z)$*,* $b_j(z)$*, the characteristic function of* $R(z, f(z))$ *satisfies*

$$T(r, R(z, f(z))) = dT(r, f) + \mathcal{O}(\Psi(r)), \quad r \to \infty, \tag{2.3}$$

where $d = \max\{n, m\}$ *and*

$$\Psi(r) = \max_{k,j}\{T(r, a_k), T(r, b_j)\}. \tag{2.4}$$

Lemma 2.4. *Suppose* f *is a transcendental meromorphic function. Let* $Q(z, f)$*,* $R(z, f)$ *be rational functions in* f *with small meromorphic coefficients relative to* f

such that $0 < q := \deg_f Q \le d := \deg_f R$ *and* $p(z) = p_k z^k + p_{k-1} z^{k-1} + \cdots + p_0 \in \mathbf{C}[z]$ *of degree* $k > 1$. *If* f *is a solution of the functional equation*

$$Q(z, f(p(z))) = R(z, f(z)), \tag{2.5}$$

then $d \ge qk$, *and for any* ε, $0 < \varepsilon < 1$, *there exist positive real constants* K_1 *and* K_2 *such that*

$$K_1(\log r)^{\alpha-\varepsilon} \le T(r,f) \le K_2(\log r)^{\alpha+\varepsilon}, \quad \alpha = \frac{\log d - \log q}{\log k}, \tag{2.6}$$

when r *is large enough.*

Proof. Let ε and δ be arbitrarily small positive real numbers. The use of Valiron–Mohon'ko Theorem, see Theorem 2.3, and Lemma 2.1 yields

$$q(1-\varepsilon)T(\mu r^k, f) \le qT(r, f \circ p) = dT(r,f) + S(r,f) \le d(1+\varepsilon)T(r,f),$$

where $\mu = |p_k| - \delta$ and r is large enough outside of a set of finite linear measure. As a consequence,

$$T(\mu r^k, f) \le \frac{d(1+\varepsilon)}{q(1-\varepsilon)} T(r,f)$$

outside of a set of finite linear measure. By the standard reasoning, see Lemma 1.1.1 in [9], the estimate

$$T(\mu r^k, f) \le \frac{d(1+\varepsilon)}{q(1-\varepsilon)} T(\sigma r,f)$$

holds for any real number $\sigma > 1$ when r is large enough. Denoting $t = \sigma r$ we obtain

$$T(\mu \sigma^{-k} t^k, f) \le \frac{d(1+\varepsilon)}{q(1-\varepsilon)} T(t,f).$$

By Lemma 3 in [3],

$$T(t,f) = \mathscr{O}((\log t)^{\alpha_1}), \quad t \to \infty, \tag{2.7}$$

where

$$\alpha_1 = \frac{\log(d/q) + \log((1+\varepsilon)/(1-\varepsilon))}{\log k}. \tag{2.8}$$

Since f is a transcendental function by assumption, letting ε be small enough in the equations (2.7) and (2.8) we see that $d \ge qk$.

To prove the first inequality in (2.6) we use again Valiron–Mohon'ko Theorem and Lemma 2.1 again to obtain

$$d(1-\varepsilon)T(r,f) \le dT(r,f) + S(r,f) = qT(r, f \circ p) \le q(1+\varepsilon)T(\lambda r^k, f) \tag{2.9}$$

where $\lambda = |p_k| + \delta$ and r is large enough outside of a set of finite linear measure. For arbitrary $\sigma > 1$, we deduce by using Lemma 1.1.1 in [9] that

$$T(\lambda(\sigma r)^k, f) \ge \frac{d(1-\varepsilon)}{q(1+\varepsilon)} T(r,f) \tag{2.10}$$

holds for all r large enough. Note that $\varepsilon > 0$ can be chosen so that $d(1-\varepsilon)/(q(1+\varepsilon)) > 1$, since $d \geq kq > q$. By Lemma 3 in [3] again, there exist a positive constant K such that

$$T(r,f) \geq K(\log r)^{\alpha_2}, \quad \text{where } \alpha_2 = \frac{\log(d/q) + \log((1-\varepsilon)/(1+\varepsilon))}{\log k}, \tag{2.11}$$

when r is large enough. □

Lemma 2.5. *Suppose f is a transcendental meromorphic solution of the equation* (2.5)*, where $p(z) = az + b$, $a, b \in \mathbf{C}$, $a \neq 0$ and $|a| \neq 1$. Then*

$$\mu(f) = \rho(f) = \frac{\log d - \log q}{\log |a|}. \tag{2.12}$$

Proof. The proof follows closely the proof of Theorem 3.4 in [4]. The use of Valiron–Mohon'ko theorem on each side of (2.5) gives

$$dT(r,f) = qT(r, f \circ p) + S(r,f). \tag{2.13}$$

Assume first $|a| < 1$. Let $\varepsilon > 0$ and $\delta > 0$ be such that $\lambda := |a| + \delta < 1$. Using Lemma 2.1 we see that

$$T(r,f) \leq \frac{q}{d}\frac{1+\varepsilon}{1-\varepsilon} T(\lambda r, f), \tag{2.14}$$

where r is large enough outside of a possible set of finite linear measure. By the equation (2.14) it is obvious that $q \geq d$. Applying Lemma 3.1 in [4] to the inequality (2.14) gives

$$\rho(f) \leq \frac{\log(q/d) + \log(1+\varepsilon) - \log(1-\varepsilon)}{-\log \lambda}. \tag{2.15}$$

Letting $\delta \to 0$ and $\varepsilon \to 0$ we obtain

$$\rho(f) \leq \frac{\log d - \log q}{\log |a|}. \tag{2.16}$$

To prove $\mu(f) \geq \frac{\log d - \log q}{\log |a|}$, we assume $q > d$, since the case $q = d$ is trivial by the inequality (2.16) and the fact that $0 \leq \mu(f) \leq \rho(f)$. Let now $\varepsilon > 0$ be such that $q(1-\varepsilon)/(d(1+\varepsilon)) > 1$ and $\delta > 0$ be such that $\nu = |a| - \delta > 0$. Applying again Lemma 2.1 to the equation (2.13) we get

$$dT(r,f) \geq q\frac{1-\varepsilon}{1+\varepsilon} T(\nu r, f),$$

where r is large enough outside of a possible set of finite linear measure. Using again the standard reasoning to get rid of the exceptional set, we obtain that for any fixed $\sigma > 1$ there exist $R > 0$ such that

$$T\left(\frac{\sigma}{\nu} r, f\right) \geq \frac{q(1-\varepsilon)}{d(1+\varepsilon)} T(r,f), \quad r \geq R.$$

By Lemma 2.2,

$$\mu(f) \geq \frac{\log\left(\frac{q(1-\varepsilon)}{d(1+\varepsilon)}\right)}{\log\frac{\sigma}{\nu}} = \frac{\log(q/d) + \log(1-\varepsilon) - \log(1+\varepsilon)}{\log\sigma - \log(|a| - \delta)}.$$

Letting $\sigma \to 1$, $\varepsilon \to 0$ and $\delta \to 0$ we obtain

$$\mu(f) \geq \frac{\log d - \log q}{\log|a|}. \tag{2.17}$$

Assume next $|a| > 1$. Let $\varepsilon > 0$ and $\delta > 0$ be such that $|a| - \delta > 1$. From the equation (2.5) we get

$$\frac{(1+\varepsilon)d}{(1-\varepsilon)q}T(\gamma r, f) \geq T(r, f),$$

where $\gamma = \frac{1}{|a|-\delta} < 1$ and r is large enough outside of a set of finite linear measure. By this inequality it is obvious that we must have $d \geq q$. Again, the use of Lemma 3.1 in [4] gives

$$\rho(f) \leq \frac{\log\frac{d}{q} + \log(1+\varepsilon) - \log(1-\varepsilon)}{-\log\gamma} \to \frac{\log d - \log q}{\log|a|},$$

when we let $\varepsilon \to 0$ and $\delta \to 0$.

To prove the estimate (2.17) in the case $|a| > 1$, we note that we have the inequality (2.14) with $\lambda = |a| + \delta > 1$. Getting rid of the exceptional set again, we get for any $\sigma > 1$ such $R > 0$ that

$$\frac{(1-\varepsilon)d}{(1+\varepsilon)q}T(r, f) \leq T(\sigma\lambda r, f), \quad r \geq R.$$

Again, we may assume $d > q$ and thus there exist $\varepsilon > 0$ such that $(1-\varepsilon)d > (1+\varepsilon)q$ and the use of Lemma 2.2 gives

$$\mu(f) \geq \frac{\log d - \log q + \log(1-\varepsilon) - \log(1+\varepsilon)}{\log\sigma + \log(|a| + \delta)}.$$

We let $\delta \to 0$, $\varepsilon \to 0$ and $\sigma \to 1$ to obtain the estimate (2.17). □

3. Distribution of poles

To express the bounds for the growth of the integrated counting function and the characteristic function of a meromorphic function we shall use the notion of a comparison function $\Delta(r)$, that is a positive monotone increasing real function on $(1, \infty)$. We shall denote

$$\mathscr{L}_\Delta = \{h \text{ meromorphic} \,|\, T(r, h) = \mathscr{O}(\Delta(r)),\ r \to \infty\}. \tag{3.1}$$

When $\mathscr{L}_\Delta$ is equipped with the pointwise multiplication and summation it shall become a field. For example, if $\Delta(r) = \log r$, then $\mathscr{L}_\Delta = \mathbf{C}(z)$ i.e. the field of complex rational functions.

We begin by a version of the famous Tumura–Clunie theorem, see Theorem 1.3.

Theorem 3.1. *Let f and g be transcendental meromorphic functions such that*

$$\overline{N}(r,g)+\overline{N}(r,f)=\mathscr{O}(\Delta(r))=S(r,f),\quad r\to\infty. \tag{3.2}$$

Let

$$\begin{aligned}A(z,f)&:=f^n+a_{n-1}f^{n-1}+\cdots+a_1f+a_0\quad\text{and}\\B(z,f)&:=f^m+b_{m-1}f^{m-1}+\cdots+b_1f+b_0,\end{aligned}$$

where m,n are nonnegative integers and the coefficients a_j and b_k are meromorphic functions such that

$$\Psi(r):=\max_{j,k}\{T(r,a_j),T(r,b_k)\}=S(r,f). \tag{3.3}$$

If $A(z,f)$ and $B(z,f)$ are relatively prime as polynomials in f over the field $\mathscr{L}_\Psi$ and $m\geq 1$, then the functional equation

$$g=\frac{A(z,f)}{B(z,f)} \tag{3.4}$$

must reduce to the form

$$g=\frac{A(z,f)}{(f-s)^m},\quad s=-\frac{b_{m-1}}{m}. \tag{3.5}$$

Furthermore, $\overline{N}(r,f-s)+\overline{N}(r,1/(f-s))=\mathscr{O}(\Delta(r))+\mathscr{O}(\Psi(r))$, when $r\to\infty$.

To prove this theorem, we need the following variant of Lemma 2.3 in [10]. The proof can be worked out just by a small modification of the proof of the original lemma.

Lemma 3.2. *Let $a_0,a_1,\dots,a_n,b_0,b_1,\dots,b_m,f$ be meromorphic functions and denote $\Psi(r):=\max_{j,k}\{T(r,a_j),T(r,b_k)\}$. Let also*

$$\begin{aligned}A(z,f)&:=a_nf^n+a_{n-1}f^{n-1}+\cdots+a_0\quad\text{and}\\B(z,f)&:=b_mf^m+b_{m-1}f^{m-1}+\cdots+b_0.\end{aligned}$$

If A and B have no common factor of positive degree in f over the field $\mathscr{L}_\Psi$, then

$$\overline{N}(r,1/B(z,f))\leq\overline{N}(r,A(z,f)/B(z,f))+\mathscr{O}(\Psi(r)),\quad r\to\infty. \tag{3.6}$$

Proof of Theorem 3.1. Using Lemma 3.2 and the assumptions, we obtain

$$\begin{aligned}\overline{N}(r,1/B(z,f))&\leq\overline{N}(r,A(z,f)/B(z,f))+\mathscr{O}(\Psi(r))\\&=\overline{N}(r,g)+\mathscr{O}(\Psi(r))\\&=\mathscr{O}(\Delta(r))+\mathscr{O}(\Psi(r))=S(r,f),\quad r\to\infty.\end{aligned} \tag{3.7}$$

If $B(z,f)\not\equiv(f+\frac{b_{m-1}}{m})^m$, then the application of Theorem 1.4, the estimate (3.7) and the assumptions yields

$$T(r,f)\leq\overline{N}(r,1/B(z,f))+\overline{N}(r,f)+S(r,f)=S(r,f), \tag{3.8}$$

that is a contradiction. Hence, we must have $B(z,f) = (f + \frac{b_{m-1}}{m})^m$. Denoting $s := -\frac{b_{m-1}}{m}$, using the assumptions and the estimate (3.7) we obtain

$$\overline{N}(r, f-s) + \overline{N}(r, 1/(f-s)) = \mathscr{O}(\Delta(r)) + \mathscr{O}(\Psi(r)), \quad r \to \infty. \qquad \square$$

In the sequel we shall consider polynomials

$$\begin{aligned} A(z,y) &:= y^n + a_{n-1}y^{n-1} + \cdots + a_1 y + a_0, \\ B(z,y) &:= y^m + b_{m-1}y^{m-1} + \cdots + b_1 y + b_0 \quad \text{and} \\ C(z,y) &:= c_q y^q + c_{q-1}y^{q-1} + \cdots + c_1 y + c_0, \end{aligned}$$

where n, m and q are nonnegative integers and the coefficients a_i, b_j and c_k are rational functions. We also assume $A(z,y)$ and $B(z,y)$ are relatively prime as polynomials in y over the field of rational functions. We shall denote $d := \max\{m, n\}$ and consider the functional equation

$$C(z, f(p(z))) = \frac{A(z, f(z))}{B(z, f(z))}, \tag{3.9}$$

where $p(z)$ is a complex polynomial to be specified later. We shall investigate the equation (3.9) in the case when a transcendental meromorphic solution f has relatively few distinct poles compared to its growth. We shall use a result that is of algebraic nature and so let us take a short look at differential algebra before proceeding.

As we noted earlier $\mathscr{L}_\Delta$ is a field. In fact, for an arbitrary function $g \in \mathscr{L}_\Delta$, we have $g' \in \mathscr{L}_\Delta$. That means that $\mathscr{L}_\Delta$ is a differential field. Another important property of $\mathscr{L}_\Delta$ is that it is relatively algebraically closed subfield of the field of meromorphic functions, i.e. if a meromorphic function satisfies an algebraic equation over the field $\mathscr{L}_\Delta$, it must be an element of it. For the proof of this fact, see [6].

For meromorphic functions $g_1, g_2, \dots, g_n$ that are algebraically independent over the field $\mathscr{L}_\Delta$, the notion $\mathscr{L}_\Delta(g_1, g_2, \dots, g_n)$ means all rational functions in $g_1, g_2, \dots, g_n$ with coefficients lying in the field $\mathscr{L}_\Delta$. Note that $\mathscr{L}_\Delta(g_1, g_2, \dots, g_n)$ is a field and if $g_1', g_2', \dots, g_n' \in \mathscr{L}_\Delta(g_1, g_2, \dots, g_n)$, it becomes a differential field.

Now we are ready to state the algebraic result mentioned above. Originally it is due Kolchin, but the proof of a more general setting can also be found in [11, Lemma 4.6].

Lemma 3.3. (Kolchin, [8]) *Let $a_1, a_2, \dots, a_n \in \mathscr{L}_\Delta$ and suppose that meromorphic functions $A_1, A_2, \dots, A_n$ are algebraically independent over the field $\mathscr{L}_\Delta$ and that they form a solution of the system of differential equations*

$$y' = a_i y, \quad i = 1, 2, \dots, n. \tag{3.10}$$

If a nontrivial meromorphic function f is such that $f'/f \in \mathscr{L}_\Delta$ and f is algebraic over the differential field $\mathscr{L}_\Delta(A_1, A_2, \dots, A_n)$, then the minimal polynomial of f over the field $\mathscr{L}_\Delta(A_1, A_2, \dots, A_n)$ is of the form

$$y^K - pA_1^{q_1}A_2^{q_2}\cdots A_n^{q_n}, \tag{3.11}$$

where $K \in \mathbf{N}$, $p \in \mathscr{L}_\Delta$ *and* $q_k \in \mathbf{Z}$ *for* $k = 1, 2, \dots, n$.

First we look at the case when p is a linear polynomial in (3.9).

Theorem 3.4. *Suppose that in the equation* (3.9) *we have* $m > 0$, $d \neq q$, $p(z) = az + b$, $a, b \in \mathbf{C}$, $a \neq 0$ *and* $|a| \neq 1$. *Assume* f *is a transcendental meromorphic function such that*

$$\overline{\lambda}(1/f) < \frac{\log d - \log q}{\log |a|} =: \alpha. \tag{3.12}$$

If f *is a solution of the functional equation* (3.9), *then* f *satisfies the functional equation*

$$c(z)(f(p(z)) - s(p(z)))^L = \frac{1}{(f(z) - s(z))^K}, \quad s = -\frac{b_{m-1}}{m} \in \mathbf{C}(z), \tag{3.13}$$

where K *and* L *are positive integers such that* $m = r_m K$, $n = r_n K$ *and* $q = r_m L$, *where* r_m, r_n *are nonnegative integers satisfying* $r_n \leq r_m$, *and* c *is a rational function. Furthermore,*

$$\max\{\overline{\lambda}(f - s), \overline{\lambda}(1/(f - s))\} < \alpha. \tag{3.14}$$

Proof. By assumption, there exists a positive real constant $\beta < \alpha$ such that $\overline{N}(r, f) = \mathscr{O}(r^\beta)$, when $r \to \infty$. Let $\varepsilon > 0$ and $\lambda := |a| + \varepsilon$. Taking advantage of Lemma 2.1, we obtain

$$\overline{N}(r, f \circ p) \leq \overline{N}(\lambda r, f) + \mathscr{O}(\log r) \leq \mathscr{O}((\lambda r)^\beta) + \mathscr{O}(\log r) = \mathscr{O}(r^\beta), \tag{3.15}$$

when $r \to \infty$. Let $\sigma > 0$ be such that $\beta + \sigma < \alpha$. Lemma 2.5 yields then

$$\frac{r^\beta}{T(r, f)} \leq \frac{r^\beta}{r^{\alpha - \sigma}} = r^{\beta + \sigma - \alpha} \to 0, \quad r \to \infty. \tag{3.16}$$

Combining (3.15) and (3.16) gives

$$\begin{aligned} \overline{N}(r, C(z, f \circ p)) + \overline{N}(r, f) &\leq \overline{N}(r, f \circ p) + \mathscr{O}(\log r) + \mathscr{O}(r^\beta) \\ &\leq \mathscr{O}(r^\beta) = S(r, f), \end{aligned}$$

when $r \to \infty$. Now we have shown that the assumption (3.2) of Theorem 3.1 holds and thus we may conclude that the equation (3.9) must be of the form

$$C(z, f \circ p) = \frac{A(z, f)}{(f - s)^m}, \quad s = -\frac{b_{m-1}}{m}, \tag{3.17}$$

and

$$\overline{N}(r, f - s) + \overline{N}(r, 1/(f - s)) = \mathscr{O}(r^\beta), \quad r \to \infty. \tag{3.18}$$

Denoting $h := f - s$ and using the lemma of the logarithmic derivative (note that $\rho(f) = \alpha < \infty$ by Lemma 2.5) we obtain

$$\begin{aligned} T(r, h'/h) &= T(r, (f-s)'/(f-s)) \\ &\le \overline{N}(r, f-s) + \overline{N}(r, 1/(f-s)) + m(r, (f-s)'/(f-s)) \\ &= \mathscr{O}(r^\beta) + \mathscr{O}(\log r), \quad r \to \infty. \end{aligned}$$

This means that h'/h is an element of the differential field

$$\mathscr{L} := \{g \text{ meromorphic} \,|\, T(r,g) = \mathscr{O}(r^\beta)\}.$$

Using (3.18) we obtain similarly as in (3.15) the estimates

$$\overline{N}(r, h \circ p) \le \mathscr{O}(r^\beta), \quad r \to \infty$$

and

$$\overline{N}(r, \frac{1}{h \circ p}) \le \mathscr{O}(r^\beta), \quad r \to \infty.$$

By using the estimates above and the lemma of a logarithmic derivative gives

$$\begin{aligned} T(r, D(h \circ p)/h \circ p) &= m(r, D(h \circ p)/h \circ p) + \overline{N}(r, 1/h \circ p) + \overline{N}(r, h \circ p) \\ &= \mathscr{O}(r^\beta), \quad r \to \infty. \end{aligned}$$

Thus we have shown that the logarithmic derivative of $h \circ p$ is an element of the field $\mathscr{L}$.

Next we prove that f must satisfy the equation of the form (3.13). Since h satisfies

$$C(z, (h+s) \circ p) = \frac{A(z, h+s)}{h^m} \tag{3.19}$$

we see that h is algebraic over the differential field $\mathscr{L}(h \circ p)$. It is easy to see that the polynomials y^m and $A^*(z,y) := A(z, y+s)$ cannot have a common factor in $\mathscr{L}[y]$ that is of positive degree in y. Indeed, if they had, y would divide $A^*(z,y)$, and there would exist $M(z,y) \in \mathscr{L}[y]$ such that $A^*(z,y) = M(z,y)y$. As a consequence, $M(z,y) \in \mathbf{C}(z)[y]$ and $A(z,y) = A^*(z, y-s) = M(z, y-s)(y-s)$, which would mean that $A(z,y)$ and $B(z,y)$ had a common factor $y - s$, that is excluded by the assumptions. We shall write below

$$\begin{aligned} A(z, y+s) = A^*(z,y) = y^n + a_{n-1}^* y^{n-1} + \cdots + a_0^* \quad \text{and} \\ C(z, y + s \circ p) = C^*(z,y) = c_q^* y^q + c_{q-1}^* y^{q-1} + \cdots + c_0^*. \end{aligned} \tag{3.20}$$

where $a_0^*, a_1^*, \dots, a_{n-1}^*, c_0^*, c_1^*, \dots, c_q^* \in \mathbf{C}(z)$.

The use of Lemma 3.3 states that the minimal polynomial of h over the field $\mathscr{L}(h \circ p)$ is of the form

$$y^K - r(h \circ p)^L, \quad r \in \mathscr{L}, \quad L \in \mathbf{Z} \setminus \{0\}, \quad K \in \mathbf{N}. \tag{3.21}$$

It is easy to see that the minimal polynomial of $h \circ p$ over the field $\mathscr{L}(h)$ is either of the form $y^{|L|} - h^K/r$ or $y^{|L|} - r/h^K$. Since h satisfies the equation (3.19) over the field $\mathscr{L}(h \circ p)$, we must have $d \geq K$. By the same equation, $h \circ p$ satisfies an equation of degree q over the field $\mathscr{L}(h)$, and thus $q \geq |L|$.

By the division algorithm, there exist nonnegative integers r_m, r_n, s_m and s_n such that $\max\{s_m, s_n\} < K$, $m = r_m K + s_m$ and $n = r_n K + s_n$. Since $h^K = r(h \circ p)^L$, we may write the equation (3.19) in the form

$$C^*(z, h \circ p) h^{s_m} r^{r_m} (h \circ p)^{L r_m} = \sum_{j=0}^{K-1} h^j \left(\sum_{k=0}^{r_n} a^*_{kK+j} r^k (h \circ p)^{Lk} \right), \tag{3.22}$$

where $a^*_k = 0$ whenever $k > n$ and $a^*_n = 1$. The function h cannot satisfy a nontrivial algebraic equation over the field $\mathscr{L}(h \circ p)$ of lower degree than K and so

$$C^*(z, h \circ p) r^{r_m} (h \circ p)^{L r_m} = \sum_{k=0}^{r_n} a^*_{kK+s_m} r^k (h \circ p)^{Lk} \tag{3.23}$$

and

$$\sum_{k=0}^{r_n} a^*_{kK+j} r^k (h \circ p)^{Lk} = 0, \quad j = 0, 1, \ldots, K-1,\ j \neq s_m. \tag{3.24}$$

Since $h \circ p$ is transcendental over the field $\mathscr{L}$, the equation (3.24) yields

$$a^*_{kK+j} = 0, \quad k = 0, 1, \ldots, r_n, \quad j = 0, 1, \ldots, K-1,\ j \neq s_m. \tag{3.25}$$

Since $a^*_0 = A(z, s) \neq 0$, a^*_0 must appear in the equation (3.23) and thus we must have $kK + s_m = 0$ for some $k \in \{0, 1, 2, \ldots, r_n\}$ which is possible only if $k = 0$ and $s_m = 0$. By the same reason, $L < 0$, since otherwise the left-hand side of (3.23) would be divisible by a monomial as a polynomial over the field $\mathscr{L}$, but it is impossible since the right-hand side of (3.23) is not divisible by a monomial. Further, $a^*_{r_n K + s_n} = a_n = 1$ and thus $r_n K + s_n = r_n K + s_m$ and so $s_n = s_m = 0$.

Rewriting (3.23) in the form

$$C^*(z, h \circ p) r^{r_m} (h \circ p)^{|L|(r_n - r_m)} = \sum_{k=0}^{r_n} a^*_{kK} r^k (h \circ p)^{(r_n - k)|L|}, \tag{3.26}$$

we see that $r_n \leq r_m$, since the right-hand side of the equation above is not divisible by a monomial as a polynomial over $\mathscr{L}$. Multiplying the both sides of the equation above by $(h \circ p)^{|L|(r_m - r_n)}$ and comparing the degrees and the leading terms, we see that $q = |L| r_m$ and $r^{r_m} = a^*_0 / c_q$. Since r is meromorphic, it must be a rational function. □

In the case when p is not linear, we have the following

Theorem 3.5. *Suppose $p(z)$ is a nonconstant complex polynomial of degree $k > 1$ and $d > kq$. Assume f is a transcendental meromorphic solution of the functional equation* (3.9)*, where $m \geq 1$. If β is a real number such that $1 \leq \beta < \alpha := \frac{\log d - \log q}{\log k}$, then*

$$\overline{N}(r, f) \neq \mathscr{O}((\log r)^\beta), \quad r \to \infty. \tag{3.27}$$

Proof. Assume there exist a real number β, $1 \le \beta < \alpha$, such that $\overline{N}(r,f) = \mathscr{O}((\log r)^\beta)$, when $r \to \infty$. Let $\varepsilon > 0$ and $\lambda := |a_k| + \varepsilon$, where a_k is the leading coefficient of the polynomial p. Using Lemma 2.1 and the assumptions, we obtain

$$\begin{aligned}\overline{N}(r, f\circ p) &\le \overline{N}(\lambda r^k, f) + \mathscr{O}(\log r) \le \mathscr{O}((\log\lambda + k\log r)^\beta) + \mathscr{O}(\log r)\\ &\le \mathscr{O}((\log r)^\beta), \quad r\to\infty.\end{aligned} \tag{3.28}$$

Let $\sigma > 0$ be such that $\beta + \sigma < \alpha$. By Lemma 2.4 there exist a real number $K > 0$ for sufficiently large values of r such that

$$\frac{(\log r)^\beta}{T(r,f)} \le \frac{(\log r)^\beta}{K(\log r)^{\alpha-\sigma}} = K^{-1}(\log r)^{\beta+\sigma-\alpha} \to 0, \tag{3.29}$$

when we let $r \to \infty$. Combining (3.28) and (3.29) gives

$$\begin{aligned}\overline{N}(r, C(z, f\circ p)) + \overline{N}(r,f) &\le \overline{N}(r, f\circ p) + \mathscr{O}(\log r) + \mathscr{O}((\log r))^\beta\\ &\le \mathscr{O}((\log r)^\beta) = S(r,f), \quad r\to\infty.\end{aligned}$$

Thus, the assumption (3.2) of Theorem 3.1 holds and we may conclude that the equation (3.9) must be of the form

$$C(z, f\circ p) = \frac{A(z,f)}{(f-s)^m}, \quad \text{where } s = -\frac{b_{m-1}}{m}, \tag{3.30}$$

and

$$\overline{N}(r, f-s) + \overline{N}(r, 1/(f-s)) = \mathscr{O}((\log r)^\beta), \quad r\to\infty. \tag{3.31}$$

Denoting $h := f - s$ and using the lemma of the logarithmic derivative (note that $\rho(f) = 0$ by Lemma 2.4) we obtain

$$\begin{aligned}T(r, h'/h) &= T(r, (f-s)'/(f-s))\\ &\le \overline{N}(r, f-s) + \overline{N}(r, 1/(f-s)) + m(r, (f-s)'/(f-s))\\ &= \mathscr{O}((\log r)^\beta) + \mathscr{O}(\log r), \quad r\to\infty.\end{aligned}$$

This means that h'/h is an element of the differential field

$$\mathscr{L} := \{g \text{ meromorphic} \mid T(r,g) = \mathscr{O}((\log r)^\beta)\}.$$

Using (3.31) we obtain similarly as in (3.28) the estimates

$$\overline{N}(r, h\circ p) + \overline{N}(r, \frac{1}{h\circ p}) \le \mathscr{O}((\log r)^\beta), \quad r\to\infty.$$

By using the estimates above and the lemma of the logarithmic derivative gives

$$\begin{aligned}T(r, D(h\circ p)/h\circ p) &= m(r, D(h\circ p)/h\circ p) + \overline{N}(r, 1/h\circ p) + \overline{N}(r, h\circ p)\\ &= \mathscr{O}((\log r)^\beta), \quad r\to\infty.\end{aligned}$$

Thus we have shown that the logarithmic derivative of $h\circ p$ is an element of the field $\mathscr{L}$.

Again, we may apply Lemma 3.3 to show that h satisfies the equation of the form

$$h^K = r(h \circ p)^L, \quad r \in \mathscr{L}, \quad L \in \mathbf{Z}, \quad K \in \mathbf{N}. \tag{3.32}$$

Now we may proceed similarly as in the proof of Theorem 3.4 starting from the equation (3.21) to the end of that proof, where we concluded that r must actually be a rational function.

Denoting $g = h'/h$ and taking the logarithmic derivatives on both sides of the equation (3.32), we see that g satisfies

$$p'(z) \cdot Lg(p(z)) = Kg(z) - \frac{r'(z)}{r(z)}. \tag{3.33}$$

The meromorphic solution of this equation must be a rational function, see [13, Lemma 5.1.1], and thus by Hadamard factorization, $h = S\exp(P)$, where S is a rational function and P is a polynomial. But $\rho(f) = \rho(h) = 0$ by Lemma 2.4, and so P must be a constant and f rational, that is a contradiction. □

To give an idea of generalizing Theorem 3.4, we consider the case when the solution has finitely many poles only but the functional equation is more complicated.

Theorem 3.6. *Let $p_1, p_2, \ldots, p_r$ be distinct nontrivial complex polynomials and $C(z, y_1, \ldots, y_r)$ a polynomial in variables $y_1, y_2, \ldots, y_r$ with rational coefficients. Assume f is a transcendental meromorphic function of finite order of growth such that it has only finitely many poles and the functions $f(p_1), f(p_2), \ldots, f(p_r)$ are algebraically independent over the field $\mathbf{C}(z)$. If f satisfies the functional equation*

$$C(z, f(p_1(z)), f(p_2(z)), \ldots, f(p_r(z))) = \frac{A(z, f(z))}{B(z, f(z))}, \tag{3.34}$$

where $m = \deg_f B \geq 1$, then f must satisfy the functional equation

$$c(z)\prod_{j=1}^{r}(f(p_j(z)) - s(p_j(z)))^{q_j} = \frac{1}{(f(z) - s(z))^K}, \quad s = -\frac{b_{m-1}}{m} \in \mathbf{C}(z), \tag{3.35}$$

where $c \in \mathbf{C}(z)$, $K \in \mathbf{N}$, $q_1, \ldots, q_r \in \mathbf{N} \cup \{0\}$, and the function $f - s$ has at most finitely many zeros and poles.

Proof. First we apply Theorem 3.1 to conclude that $B(z, f) = (f - s)^m$, where $s = -b_{m-1}/m$, and $h := f - s$ has at most finitely many zeros and poles. By assumption, h is of finite order, and so h'/h is a rational function. Consequently, the functions

$$h(p_1), h(p_2), \ldots, h(p_r) \tag{3.36}$$

have at most finitely many zeros and poles, and they are of finite order of growth. This means that the logarithmic derivatives of the functions (3.36) are rational functions. Moreover, h satisfies

$$C(z, h(p_1) + s(p_1), h(p_2) + s(p_2), \ldots, h(p_r) + s(p_r)) = \frac{A(z, h + s)}{h^m}, \tag{3.37}$$

which means that h is algebraic over the differential field

$$\mathscr{L} := \mathbf{C}(z)(h(p_1), h(p_2), \ldots, h(p_r)).$$

Applying Lemma 3.3, we deduce that the minimal polynomial of h over $\mathscr{L}$ is of the form

$$y^K - c(z)\prod_{j=1}^{r} h(p_j)^{q_j},\ c \in \mathbf{C}(z),\ q_j \in \mathbf{Z},\ K \in \mathbf{N}. \tag{3.38}$$

Obviously, the left-hand side of the equation (3.37) is transcendental over $\mathbf{C}(z)$. The irreducibility of the right-hand side of the equation (3.37) can be seen in the same manner as in the proof of Theorem 3.5 and we shall use the notion $A^*(z,y) = A(z, y+s)$ as in (3.20) and

$$C^*(z, y_1, y_2, \ldots, y_r) = C(z, y_1 + s\circ p_1, \ldots, y_r + s\circ p_r).$$

There exist nonnegative integers r_m, r_n, s_m, s_n such that $\max\{s_m, s_n\} < K$ and

$$m = r_m K + s_m \text{ and } n = r_n K + s_n.$$

Since h is a root of the polynomial (3.38), we may write the equation (3.37) in the form

$$\begin{aligned} &C^*(z, h(p_1), h(p_2), \ldots, h(p_r)) h^{s_m} c^{r_m} \left((h(p_1))^{q_1}\cdots(h(p_r))^{q_r}\right)^{r_m} \\ &= \sum_{j=0}^{K-1} h^j \left(\sum_{k=0}^{r_n} a^*_{Kk+j} c^k \left((h(p_1))^{q_1}\cdots(h(p_r))^{q_r}\right)^k\right). \end{aligned} \tag{3.39}$$

Since h cannot satisfy a nontrivial algebraic equation over the field $\mathbf{C}(z, h\circ p_1, \ldots, h\circ p_r)$ of lower degree than K, we must have

$$\begin{aligned} &C^*(z, h(p_1), h(p_2), \ldots, h(p_r)) c^{r_m} \left((h(p_1))^{q_1}\cdots(h(p_r))^{q_r}\right)^{r_m} \\ &= \sum_{k=0}^{r_n} a^*_{Kk+s_m} c^k \left((h(p_1))^{q_1}\cdots(h(p_r))^{q_r}\right)^k \end{aligned} \tag{3.40}$$

and

$$\sum_{k=0}^{r_n} a^*_{Kk+j} c^k \left((h(p_1))^{q_1}\cdots(h(p_r))^{q_r}\right)^k = 0, \quad j = 0, 1, \ldots, K-1, \quad j \neq s_m. \tag{3.41}$$

Since the functions $h(p_1), h(p_2), \ldots, h(p_r)$ are algebraically independent over the field of rational functions, the equations (3.40) and (3.41) must be polynomial identities in $h(p_1), h(p_2), \ldots, h(p_r)$. Hence, $a^*_{Kk+j} = 0$, $j = 0, 1, \ldots, K-1$, $j \neq s_m$ and $k = 0, 1, \ldots, r_n$. Since $A^*(z,y)$ cannot be divisible by y, the coefficient $a^*_0 \neq 0$, and thus $s_m = 0$ and the right-hand side of (3.40) is not divisible by any of the functions $h(p_1), h(p_2), \ldots, h(p_r)$. Thus all powers $q_1, q_2, \ldots, q_r$ must be nonpositive integers. □

Example 3.7. The equation

$$(f(p(z)))^2 = \frac{1}{f(z)} \tag{3.42}$$

admits the solution $f(z) = \exp(z^2)$ when $p(z) = \frac{i}{\sqrt{2}}z$. If $p(z) = z + z_0$, where $\exp(z_0) = -1/2$, it admits the solution $f(z) = \exp(\exp(z))$. The latter solution shows that the assumption of the solution being of finite order is not necessary in Theorem 3.6.

4. The generalized Schröder equation

Theorem 3.6 gives a partial answer to a question posed by Rubel ([12]) concerning the generalized Schröder equation (1.1), where the coefficients are rational functions. It is known that meromorphic solutions of (1.1) are of finite order, see Lemma 2.5 (proved also in [4].) Theorem 3.6 shows that all meromorphic solutions with at most finitely many poles (e.g. entire solutions) satisfy the first order linear differential equation with rational coefficients since they are of the form $f(z) = r(z)\exp(g(z)) + s(z)$, where r and s are rational functions and g is a polynomial.

We give a converse result to Theorem 3.4 in the case when $b = 0$.

Theorem 4.1. *Let a be a nonzero complex constant such that $|a| \neq 1$ and let q and m be distinct positive integers. Suppose f is a transcendental meromorphic solution of the equation*

$$c(z)(f(az))^q = \frac{1}{(f(z))^m}, \tag{4.1}$$

where c is a rational function. Then either

(i) *$f(z) = r(z)\exp(\alpha z^\rho)$, where r is a rational function, $\alpha \in \mathbf{C} \setminus \{0\}$ and $\rho = \rho(f) = \frac{\log m - \log q}{\log|a|}$, or*

(ii) *$\max\{\overline{\lambda}(f), \overline{\lambda}(1/f)\} = 0$ and $\lambda(f) = \lambda(1/f) = \rho(f) = \frac{\log m - \log q}{\log|a|}$. Furthermore, if $|a| > 1$, then m is divisible by q and otherwise q is divisible by m.*

Proof. Note first that $\rho := \rho(f) = \frac{\log m - \log q}{\log|a|} > 0$ by Lemma 2.5 and the assumption $m \neq q$. Assume first f has at most finitely many zeros and poles. Then, by Hadamard factorization, there exist a rational function $r(z)$ and a polynomial $\Lambda(z) = \sum_{j=1}^{\rho} \lambda_j z^j$, $\lambda_\rho \neq 0$, such that $f(z) = r(z)\exp(\Lambda(z))$. Substituting this representation of f into the equation (4.1), we see that

$$q\sum_{j=1}^{\rho} \lambda_j (az)^j + m\sum_{j=1}^{\rho} \lambda_j z^j = 0,$$

and so

$$\lambda_j(qa^j + m) = 0, \quad j = 1, \dots, \rho.$$

Since $\lambda_\rho \neq 0$, we have $a^\rho = -m/q$. If $a^k = -m/q$ for some $k \in \{1, 2, \ldots, \rho - 1\}$, then $a^{\rho-k} = 1$, that is excluded by the assumption $|a| \neq 1$. Thus $\lambda_j = 0$ for $k = 1, 2, \ldots, \rho - 1$.

Assume next f has infinitely many zeros. First, we look at the case when $|a| > 1$. Choose a finite zero z_0 of f that is of the smallest modulus such that $|z_0| > R$, where $R := \max\{1,\, |z_k| \,|\, c(z_k) = 0 \text{ or } c(z_k) = \infty\}$.

Denote by Z the number of zeros and by P the number of poles of f inside the disk about zero of radius R. Further, denote the multiplicity of the zero z_0 by μ. By iterating the equation (4.1), we see that f has a zero sequence of the form

$$a^{2j} z_0, \quad j = 0, 1, 2, \ldots, \tag{4.2}$$

and corresponding multiplicities are given by

$$t^{2j}\mu, \quad t = m/q, \quad j = 0, 1, 2, \ldots. \tag{4.3}$$

Similarly, there exist a pole sequence of f which is of the form

$$a^{2j+1} z_0, \quad j = 0, 1, 2, \ldots \tag{4.4}$$

with corresponding multiplicities

$$t^{2j+1}\mu, \quad j = 0, 1, 2, \ldots. \tag{4.5}$$

If we had assumed that f had infinitely many poles instead of infinitely many zeros, then (4.2) would be the sequence of poles and (4.4) would be the sequence of zeros. We immediately see that t must be a positive integer i.e. q must divide m.

Iterating the equation (4.1) we see that f has a zero sequence $a^{-2j}z_0$, $j = 0, 1, 2, \ldots$, which tends to origin unless the coefficient c has a pole or zero at some point $a^{-k}z_0$, $k \in \mathbf{N}$. Of course, the infinite zero sequence cannot have a finite limit, and so we conclude that every zero sequence of f is initialized by a zero or a pole of the coefficient c. The same conclusion is true for a pole sequence also. In other words, outside of the disk containing all the zeros and the poles of c, f has at most finitely many zero sequences that are of the form (4.2), and finitely many pole sequences that are of the form (4.4).

In a disk about the origin of radius $r > |z_0|$ there exist a nonnegative integer k such that $|a|^{2k}|z_0| \leq r < |a|^{2(k+1)}|z_0|$. By the discussion above, f has at most finitely many zero and pole sequences, say at most q pieces all together, and so f has at most qk zeros and $q(k-1)$ poles in the annulus $\{z |\, R < |z| \leq r\}$, and

$$\frac{\log r - \log|z_0|}{2\log|a|} - 1 < k \leq \frac{\log r - \log|z_0|}{2\log|a|}. \tag{4.6}$$

Using the estimate (4.6), we get

$$\log \overline{n}(r, 1/f) = \log(qk + Z) \leq \log\left(q\frac{\log r - \log|z_0|}{2\log|a|} + Z\right). \tag{4.7}$$

Dividing the above estimate by $\log r$ and letting $r \to \infty$ we see that

$$\overline{\lambda}(f) = \limsup_{r\to\infty} \frac{\log \overline{n}(r, 1/f)}{\log r} = 0. \tag{4.8}$$

Similarly we obtain $\overline{\lambda}(1/f) = 0$.

We continue by counting the number of zeros according to their multiplicities. By (4.3), the number of zeros in the annulus $\{z \mid R < |z| \leq r\}$ can be calculated by using the formula for geometric sum and so

$$n(r, 1/f) \geq \mu \frac{(t^2)^{k+1} - 1}{t^2 - 1} + K > \frac{(t^2)^{k+1}(1 - t^{-2k-2})}{t^2 - 1}, \tag{4.9}$$

where the nonnegative integer K is the number of zeros according to their multiplicities inside the disk $\{z \mid |z| \leq R\}$. Using (4.6) and (4.9) gives

$$\begin{aligned} \overset{+}{\log}\, n(r, 1/f) &> \overset{+}{\log}\, (t^2)^{k+1} + \overset{+}{\log}\, (1 - t^{-2k-2}) - \overset{+}{\log}\, (t^2 - 1) \\ &\geq \left(\frac{\log r - \log |z_0|}{\log |a|} \right) \overset{+}{\log}\, t - \overset{+}{\log}\, (t^2 - 1). \end{aligned}$$

Hence by Lemma 2.5,

$$\lambda(f) = \limsup_{r \to \infty} \frac{\overset{+}{\log}\, n(r, 1/f)}{\log r} \geq \frac{\log t}{\log |a|} = \rho(f), \tag{4.10}$$

and so $\lambda(f) = \rho(f)$. In the similar way as we calculated $\lambda(f)$ we are able to obtain $\lambda(1/f) = \rho(f)$.

Assume next $|a| < 1$ and denote $w = 1/a$. We may look at the equation

$$c(wz)(f(wz))^m = \frac{1}{(f(z))^q}, \tag{4.11}$$

and we may repeat the similar reasoning as above. □

Theorem 4.2. *Let $p(z) = az$, where a is a nonzero complex constant such that $|a| \neq 1$. Suppose f is a transcendental meromorphic function such that it has infinitely many poles but $\overline{\lambda}(1/f) < \rho(f)$. If f is a solution of the equation* (3.9)*, where $m \geq 1$ and $d \neq q$, then f is hypertranscendental.*

Proof. Using Theorem 3.4 we see that f satisfies the equation

$$c(z)\left(f(az) - s(az)\right)^L = \frac{1}{\left(f(z) - s(z)\right)^K}, \quad s = -\frac{b_{m-1}}{m}, \quad K, L \in \mathbf{N}. \tag{4.12}$$

Denoting $g = (f - s)'/(f - s)$ we note that g satisfies

$$\frac{c'(z)}{c(z)} + a \cdot Lg(az) = -Kg(z). \tag{4.13}$$

Since f has infinitely many poles, g is a transcendental meromorphic function, and by Theorem 1.2, g is hypertranscendental, which means that f is also hypertranscendental. □

Finally, we consider the question of existence of the solutions.

Theorem 4.3. *Let $a \in \mathbf{C} \setminus \{0\}$ such that $|a| > 1$ and let $c(z)$ be a meromorphic function such that it does not have a zero or a pole at the origin. Then, the equation*

$$c(z)f(az) = \frac{1}{(f(z))^m}, \quad m \in \mathbf{N}, \tag{4.14}$$

admits a meromorphic solution, provided that $m \neq -a^{j+1}$, $j = 0, 1, 2, \ldots$.

Proof. The equation

$$ay(az) = -my(z) - \frac{c'(z)}{c(z)}, \tag{4.15}$$

admits a meromorphic solution $y(z)$ by Theorem 1.1. Assume first y has a pole at the origin and look at the series expansions of $y(z)$ and $y(az)$ at the origin,

$$\begin{aligned} y(z;0) &= \alpha_{-\mu} z^{-\mu} + \alpha_{-\mu+1} z^{-\mu+1} + \cdots \\ y(az;0) &= \alpha_{-\mu}(az)^{-\mu} + \alpha_{-\mu+1}(az)^{-\mu+1} + \cdots , \end{aligned}$$

where α_j's are complex constants and μ is a positive integer. Substituting these expansions into the equation (4.15) we see that

$$m = -a^{1-\mu}, \tag{4.16}$$

which is impossible by the fact that $m \in \mathbf{N}$.

Hence y is analytic at the origin and thus there exist a meromorphic function f in some disk $|z| < R$, where $R > 0$, such that, $f'/f = y$. Thus integrating both sides of the equation (4.15) shows that f satisfies the equation (4.14). By [4], Proposition 2.2, f can be meromorphically continued over the whole complex plane. □

Example 4.4. The equation

$$(z-1)f(2z) = \frac{1}{(f(z))^2}. \tag{4.17}$$

admits a meromorphic solution by Theorem 4.3. We shall show that it must be hypertranscendental.

The meromorphic function $g = f'/f$ satisfies

$$2g(2z) = -2g(z) - \frac{1}{z-1}. \tag{4.18}$$

By this equation, $z = 1$ is a pole of $g(z)$ or $g(2z)$. Assume first $g(z)$ has a pole at $z = 1$ and $g(2z)$ doesn't. Since g is a logarithmic derivative of f, the series expansion at $z = 1$ has the form

$$g(z;1) = \frac{a_{-1}}{z-1} + a_0 + a_1(z-1) + \cdots , \tag{4.19}$$

where a_{-1} is an integer. Substituting this expansion into the equation (4.18) we obtain $a_{-1} = -1/2$ that is a contradiction. Hence, also $g(2z)$ has a pole at $z = 1$. Iterating the equation (4.18) we see that $g(4z)$ has a pole at $z = 1$, $g(8z)$ has a pole at $z = 1$ etc. So g has infinitely many poles and it is transcendental. By Theorem 1.2, g as well as f are hypertranscendental.

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