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The Sz.-Nagy-Foiaş model theory uses generalized restricted shifts as canonical models for contractions in Hilbert space. This paper considers a class of contractive and unitary perturbations of a generalized restricted shift acting on a Sz.-Nagy-Foiaş space generated by an analytic operator-valued function  $S(z)$  whose values are contractions on a separable Hilbert space. The spectra and characteristic functions of the perturbations are computed and related to the original operator. When the perturbation is unitary, a unitary equivalence to multiplication by  $e^{i\theta}$  on  $L^2(\mu)$ , for an operator-valued measure  $\mu$ , is given.

In [2], D. N. Clark studied the one-dimensional unitary perturbations of restricted shifts in  $H^2$ , i.e.  $S(z)$  a scalar inner function, and in [3], he announced results for the case where  $S(z)$  is an arbitrary scalar (characteristic) function. The general unitary perturbations are implicit in work of de Branges and Rovnyak [1], though in the context of the de Branges-Rovnyak model theory rather than the Sz.-Nagy-Foiaş. P. A. Fuhrmann [5] considered a class of completely nonunitary and unitary perturbations for the case of  $S(z)$  an inner function on a finite-dimensional space. In this case, the maps considered are always compact perturbations. Here we generalize results of [5] and [2]. We will follow the general outline of [5], and we correct a minor error occurring there so our description of the perturbations in the general case is actually as sharp as in the finite-dimensional case. As was pointed out in [5], these perturbations have applications to the theory of stability of linear control systems.

1. Preliminary results. For notation, let  $C$  and  $C_*$  be separable Hilbert spaces, and let  $L^2(C)$ ,  $L^2(C_*)$ ,  $H^2(C)$ ,  $H^2(C_*)$  denote the standard vector-valued Lebesgue and Hardy spaces defined on the unit circle. (See [6] or [8] for general references.) We will use " $t$ " to denote the argument of a function (vector or operator valued) defined on the unit circle, and for analytic functions, we will freely identify  $h(t)$  on the circle with its extension to the disc, denoted  $h(z)$ .  $S$  will denote a fixed purely contractive analytic operator-valued function from  $C$  to  $C_*$ , i.e.  $S(z): C \rightarrow C_*$ ,  $\|S(z)\| \leq 1$  for all  $|z| < 1$  and  $\|S(0)c\| < \|c\|$  for all  $c \in C$ , and let  $\Delta(t) = (I - S(t)^*S(t))^{1/2}$ . (Note that this denotes the unique positive root of the positive operator.) Let  $H =$

$H^2(C_*) \oplus \overline{\Delta L^2(C)}$ , where the second summand denotes the  $L^2(C)$  closure of  $\{\Delta(t)g(t) \mid g \in L^2(C)\}$ , and  $M = \{(S(z)f(z), \Delta(t)f(t)) \mid f \in H^2(C)\} \subset H$ . Then  $M$  is invariant under the (unilateral) shift  $U_+$  in  $H$  defined by  $U_+(f, g) = (zf(z), e^{it}g(t))$ , so  $K = H \ominus M$  is invariant under  $U_+^*$ , where  $U_+^*$  is the left-shift defined by  $U_+^*(f, g) = (z^{-1}(f(z) - f(0)), e^{-it}g(t))$ . We call  $K$  the Sz.-Nagy-Foiaş space generated by  $S$ . Let  $T$  denote the restricted right shift in  $K$ , i.e. the compression of  $U_+$  to  $K$ . Thus, for  $(f, g) \in K$ ,  $T(f, g) = P(zf, e^{it}g)$ , where  $P$  denotes projection onto  $K$ , and  $T^* = U_+^*|_K$ . Note that if  $S$  is inner, then  $\Delta(t) = 0$  a.e. and  $K = H^2(C) \ominus SH^2(C)$ .

Let  $\tilde{S}(z)$  be the analytic operator-valued function defined by  $\tilde{S}(z) = S(\bar{z})^*$ , i.e.  $\tilde{S}(t) = S(-t)^*: C_* \rightarrow C$ . Analogously to above, define  $\tilde{\Delta}(t): C_* \rightarrow C_*$  by  $\tilde{\Delta}(t) = (I - \tilde{S}(t)^*\tilde{S}(t))^{1/2}$ ,  $\tilde{H} = H^2(C) \oplus \overline{\tilde{\Delta}L^2(C_*)}$ ,  $\tilde{M} = \{(\tilde{S}f, \tilde{\Delta}f) \mid f \in H^2(C_*)\}$ ,  $\tilde{K} = \tilde{H} \ominus \tilde{M}$ , and  $\tilde{T} = \tilde{P}\tilde{U}_+|_{\tilde{K}}$ , where  $\tilde{U}_+$  is the unilateral shift in  $\tilde{H}$  and  $\tilde{P}$  is projection onto  $\tilde{K}$ . Note that  $\tilde{S}$  is inner if and only if  $S$  is inner. (We use “inner” in the sense of [6], i.e.  $S(t): C \rightarrow C_*$  is unitary a.e.; in the terminology of [8], this is called “inner from both sides”.) The following is an extension of [4, Theorem 2.1].

**THEOREM 1.1.** *The right shift  $T$  on  $K$  is unitarily equivalent to the left shift  $\tilde{T}^*$  on  $\tilde{K}$ .*

*Proof.* Let  $L = L^2(C_*) \oplus \overline{\Delta L^2(C)}$ ,  $\tilde{L} = L^2(C) \oplus \overline{\tilde{\Delta}L^2(C_*)}$ , and consider  $\tau: L \rightarrow \tilde{L}$  defined by

$$\tau(f, \Delta g) = e^{-it}(S(-t)^*f(-t) + \Delta^2(-t)g(-t), \tilde{\Delta}(t)(f(-t) - S(-t)g(-t))) .$$

Then

$$\begin{aligned} \|\tau(f, \Delta g)\|_{\tilde{L}}^2 &= \|S(-t)^*f(-t) + \Delta^2(-t)g(-t)\|_{L^2(C)}^2 \\ &\quad + \|\tilde{\Delta}(t)(f(-t) - S(-t)g(-t))\|_{L^2(C_*)}^2 \\ &= \|f(-t)\|^2 + (\|g(-t)\|^2 - \|S(-t)^*S(-t)g(-t)\|^2) \\ &= \|f(t)\|_{L^2(C_*)}^2 + \|\Delta(t)g(t)\|_{L^2(C)}^2 = \|(f, \Delta g)\|_L^2 , \end{aligned}$$

so  $\tau$  extends to an isometry from  $L$  to  $\tilde{L}$ . For  $f \in L^2(C)$ ,  $g \in L^2(C_*)$ ,  $(f, \Delta g) = \tau(\tau^*(f, \Delta g))$ , where  $\tau^*(f, \Delta g) = e^{-it}(S(t)f(-t) + \Delta^2(-t)g(-t), \Delta(t)(f(-t) - S(t)^*g(-t)))$ , so  $\tau$  is unitary.

We can decompose  $L = K \oplus M \oplus K^2(C_*)$ , where

$$K^2(C_*) = \{(f, 0) \mid f \in L^2(C_*) \ominus H^2(C_*)\} ,$$

and similarly  $\tilde{L} = \tilde{K} \oplus \tilde{M} \oplus K^2(C)$ . It is easy to see that  $\tau(M) = K^2(C)$  and  $\tau(K^2(C_*)) = \tilde{M}$ , so therefore  $\tau(K) = \tilde{K}$ . Hence,  $\tau P = \tilde{P}\tau$  (here we consider the domains of  $P$  and  $\tilde{P}$  to be  $L$  and  $\tilde{L}$  respectively),

and  $\tau U = \tilde{U}^* \tau$ , where  $U$  and  $\tilde{U}$  are the bilateral shifts on  $L$  and  $\tilde{L}$ . Thus,  $\tau P U = \tilde{P} \tilde{U}^* \tau$ , which implies  $\tau T = \tilde{T}^* \tau$  on  $K$ . Therefore,  $\tau|_K$ , which we denote simply by  $\tau$ , is a unitary map from  $K$  to  $\tilde{K}$  satisfying the theorem.

It is now easy to derive an explicit formula for  $T$  which will be useful later on.

**COROLLARY.** For  $(f, \Delta g) \in K$ ,

$$T(f, \Delta g) = (zf(z) - S(z)Q(0), e^{it}\Delta(t)g(t) - \Delta(t)Q(0))$$

where  $Q(z)$  is the first component of  $\tau(f, \Delta g)$ .

*Proof.* This is obtained by computing

$$(\tau^* \tilde{T}^* \tau)(f, \Delta g).$$

If  $F = (f, g) \in K$  and  $\tau(F) = (Q, h)$ , denote by  $(\tau_1 F)(z)$  the  $C$ -valued function  $Q(z)$ . We derive several technical lemmas needed later on.

**LEMMA 1.2.** For  $|w| < 1, x \in C_*, y \in C$ , let

$$k_{w,x,y} = \left( \frac{I - S(z)S(w)^*}{1 - z\bar{w}} x, -\frac{\Delta(t)S(w)^*}{1 - e^{it}\bar{w}} x \right) + \left( \frac{S(z) - S(\bar{w})}{z - \bar{w}} y, \frac{\Delta(t)}{e^{it} - \bar{w}} y \right)$$

Then  $k_{w,x,y} \in K$  and

$$P((x/(1 - z\bar{w}), 0)) = k_{w,x,0}, \\ P\left(\left(\frac{S(z)}{e^{it} - \bar{w}} y, \frac{\Delta(t)}{e^{it} - \bar{w}} y\right)\right) = k_{w,0,y}.$$

*Proof.* Note, for  $(Sf, \Delta f) \in M$ ,

$$(k_{w,x,0}, (Sf, \Delta f)) = \left( \frac{I - S(z)S(w)^*}{1 - z\bar{w}} x, S(z)f(z) \right) - \left( \frac{\Delta(t)S(w)^*}{1 - e^{it}\bar{w}} x, f(t) \right) = 0$$

and hence  $k_{w,x,0} \in K$ .

Similarly

$$(k_{w,0,y}, (Sf, \Delta f)) = \left( \frac{S(z) - S(\bar{w})}{z - \bar{w}} y, S(z)f(z) \right) + \left( \frac{\Delta(t)^2}{e^{it} - \bar{w}} y, f(t) \right)$$

$$\begin{aligned}
&= \left( \frac{S(t)^* S(\bar{w})}{e^{it} - \bar{w}} y, f(t) \right) + \left( \frac{1}{e^{it} - \bar{w}} y, f(t) \right) \\
&= 0 + 0 = 0,
\end{aligned}$$

since  $f(t) \in H^2(C)$ . Hence  $k_{w,0,y} \in K$ .

Furthermore,

$$\begin{aligned}
&(x/(1 - z\bar{w}), 0) - k_{w,x,0} \\
&= \left( S(z) \frac{S(w)^*}{1 - z\bar{w}} x, A(t) \frac{S(w)^*}{1 - e^{it}\bar{w}} x \right) \in M
\end{aligned}$$

and

$$\begin{aligned}
&(S(t)y/(e^{it} - \bar{w}), A(t)y/(e^{it} - \bar{w})) - k_{w,0,y} \\
&= \left( \frac{S(\bar{w})}{e^{it} - \bar{w}} y, 0 \right) \in (K \oplus M)^\perp.
\end{aligned}$$

All the assertions of the lemma follow.

**LEMMA 1.3.** *If  $(f, g) = F \in K$ , then*

- (i)  $(F, k_{w,x,0})_K = (f(w), x)_{C^*}$
- (ii)  $(F, k_{w,0,y})_K = ((\tau_1 F)(w), y)_C$ .

*In particular*

- (iii) for  $x, y \in C_*$ ,  $\zeta$  and  $\eta$  in  $D$ ,

$$(k_{\zeta,x,0}, k_{\eta,y,0})_K = \left( \frac{I - S(\eta)S(\zeta)^*}{1 - \eta\bar{\zeta}} x, y \right)_{C^*}$$

- (iv) for  $x, y \in C$ ,

$$(k_{\zeta,0,x}, k_{\eta,0,y})_K = \left( \frac{I - S(\bar{\eta})^* S(\bar{\zeta})}{1 - \eta\bar{\zeta}} x, y \right)_C$$

and

- (v) for  $x \in C$ ,  $y \in C_*$ ,

$$(k_{\zeta,0,x}, k_{\eta,y,0})_K = \left( \frac{S(\eta) - S(\bar{\zeta})}{\eta - \bar{\zeta}} x, y \right)_{C^*}.$$

*Proof.* For  $(f, g) \in K$ ,  $x \in C_*$ , we have

$$\begin{aligned}
(f(w), x) &= \left( (f, g), \left( \frac{1}{1 - z\bar{w}} x, 0 \right) \right) \\
&= \left( (f, g), P \left( \frac{1}{1 - z\bar{w}} x, 0 \right) \right) \\
&= ((f, g), k_{w,x,0})
\end{aligned}$$

by Lemma 1.2, proving (i).

For (ii) note that

$$(\tau_1 F)(t) = e^{-it}(S(-t)^* f(-t) + A(-t)g(-t))$$

is in  $H^2(C)$ . Hence

$$\begin{aligned} ((\tau_1 F)(w), y)_C &= \left( e^{-it}(S(-t)^* f(-t) + A(-t)g(-t)), \frac{1}{1 - e^{it}\bar{w}}y \right) \\ &= \left( S(t)^* f(t) + A(t)g(t), \frac{1}{e^{it} - \bar{w}}y \right) \\ &= \left( (f, g), \left( \frac{S(t)}{e^{it} - \bar{w}}y, \frac{A(t)}{e^{it} - \bar{w}}y \right) \right) \\ &= \left( (f, g), P \left( \frac{S(t)}{e^{it} - \bar{w}}y, \frac{A(t)}{e^{it} - \bar{w}}y \right) \right) \\ &= ((f, g), k_{w,0,y}) \end{aligned}$$

by Lemma 1.2, and (ii) follows.

A straight-forward computation gives

$$(\tau_1 k_{w,x,0})(z) = \frac{\tilde{S}(z) - \tilde{S}(\bar{w})}{z - \bar{w}}x$$

and

$$(\tau_1 k_{w,0,y})(z) = \frac{I - \tilde{S}(z)\tilde{S}(w)^*}{1 - z\bar{w}}y.$$

Hence (iii)-(v) follows from (i) and (ii).

We note that if  $F = (f, g) \in K$  is orthogonal to  $k_{w,x,y}$  for all  $w \in D, x \in C_*$  and  $y \in C$ , then  $f = 0$  and  $\tau_1 F = 0$ . From the formula for  $\tau_1$ , it follows that also  $g = 0$ , and hence  $F$  is the zero element of  $K$ . This implies that  $\{k_{w,x,y} \mid w \in D, x \in C_*, y \in C\}$  spans a dense subset of  $K$ . This fact will make the formulas (iii)-(v) useful for computations later on.

The next lemma follows from the corollary to Theorem 1.1 and direct computations.

- LEMMA 1.4. (i)  $Tk_{w,x,0} = \bar{w}^{-1}(k_{w,x,0} - k_{0,x,0}), w \neq 0$ .  
 (ii)  $Tk_{w,0,y} = \bar{w}k_{w,0,y} - k_{0,S(\bar{w})y,0}$ .  
 (iii)  $T^*k_{w,x,0} = \bar{w}k_{w,x,0} - k_{0,0,S(w)^*x}$ .  
 (iv)  $T^*k_{w,0,y} = \bar{w}^{-1}(k_{w,0,y} - k_{0,0,y}), w \neq 0$ .

We wish to distinguish two subspaces of  $K$  defined by

- $k_0$  = the closure of  $\{k_{0,x,0} \mid x \in C_*\}$
- $K_0$  = the closure of  $\{k_{0,0,y} \mid y \in C\}$ .

Let us simplify the notation for this special case by writing  $d_x$  for  $k_{0,z,0}$  and  $D_y$  for  $k_{0,0,y}$ .

LEMMA 1.5. *Let  $F = (f, g) \in K$ . Then*

(i)  $T^*F = (z^{-1}f(z), e^{-it}g(t))$  if and only if  $F \perp k_0$ .

(ii)  $TF = (zf(z), e^{it}g(t))$  if and only if  $F \perp K_0$ .

*Proof.* (i) holds if and only if  $f(0) = 0$  which holds if and only if  $F \perp k_0$  by Lemma 1.3 (i). By the corollary to Theorem 1.1, (ii) follows similarly, using Lemma 1.3 (ii).

LEMMA 1.6. *Let  $P_{k_0}$  and  $P_{K_0}$  denote the orthogonal projection onto  $k_0$  and  $K_0$  respectively. Then  $P_{k_0}F = d_x$ , where*

$$x = (I - S(0)S(0)^*)^{-1}f(0)$$

and  $P_{K_0}F = D_y$ , where  $y = (I - S(0)^*S(0))^{-1}(\tau_1 F)(0)$ . (Note since  $S(z)$  is a pure contractive function,  $x$  and  $y$  are well-defined for  $F$  in some dense subset of  $K$ .)

*Proof.* The map  $e_1$  densely defined by  $e_1: x \rightarrow d_{(I-S(0)S(0)^*)^{-1/2}x}$  is an isometry (using Lemma 1.3iii) of  $C_*$  into  $K$  with range equal to  $k_0$ , and with adjoint given by  $e_1^*: f \rightarrow (I - S(0)S(0)^*)^{-1/2}f(0)$  (using Lemma 1.3i). The formula for  $P_{k_0}$  follows by computing  $e_1 e_1^*$ . The formula for  $P_{K_0}$  follows similarly.

## 2. The perturbations.

DEFINITION 2.1. Let  $A: C \rightarrow C_*$  be a bounded linear map. We define  $Z(A)$  to be the unique bounded linear map on  $K$  such that

$$Z(A)F = \begin{cases} TF & \text{if } F \perp K_0 \\ d_{Ay} & \text{if } F = D_y. \end{cases}$$

REMARK 2.2. It follows that  $Z(A)^*$  is given by

$$Z(A)^*F = \begin{cases} T^*F & \text{if } F \perp k_0 \\ D_y & \text{if } F = d_x, \text{ where } y \\ & = (I - S(0)^*S(0))^{-1}A^*(I - S(0)S(0)^*)x. \end{cases}$$

We note that  $T = Z(-S(0))$  (by Lemma 1.3), and that  $Z(A)^*d_x = D_{A^*x}$  if and only if

$$(1) \quad AS(0)^*S(0) = S(0)S(0)^*A.$$

THEOREM 2.3. (i)  $Z(A)$  is a contraction if and only if

$$(2) \quad A^*(I - S(0)^*S(0))A \leq (I - S(0)S(0)^*).$$

(ii)  $Z(A)$  is unitary if and only if  $A = (I - S(0)S(0)^*)^{-1/2}V(I - S(0)^*S(0))^{1/2}$  for some unitary  $V$ .

(iii) If  $A$  satisfies condition (1), then  $Z(A)$  is a contraction if and only if  $\|A\| \leq 1$  and  $Z(A)$  is unitary if and only if  $A$  is unitary.

*Proof.* (i) Since  $Z(A)$  maps  $K_0^\perp$  isometrically onto  $k_0^\perp$  and sends  $K_0$  onto  $k_0$ ,  $Z(A)$  is a contraction if and only if it is contractive on  $K_0$ . By Lemma 1.3, this holds precisely when  $\|Ay\|^2 - \|S(0)^*Ay\|^2 \leq \|y\|^2 - \|S(0)y\|^2$  for all  $y \in C$ , but this is clearly equivalent to (2).

(ii) As above,  $Z(A)$  is isometric precisely when equality holds in (2). By [5, Theorem 1.7(i)], this holds if and only if  $A = (I - S(0)S(0)^*)^{-1/2}V(I - S(0)^*S(0))^{1/2}$  for some isometry  $V$ . By Lemma 1.3,  $Z(A)^*$  is isometric if and only if  $(I - S(0)S(0)^*) = (I - S(0)S(0)^*)^{1/2}VV^*(I - S(0)S(0)^*)^{1/2}$ , which holds if and only if  $VV^* = I$ , so  $V$  must be unitary.

(iii) If (1) holds, then (2) reduces to  $(A^*A)(I - S(0)^*S(0)) \leq (I - S(0)^*S(0))$ , which, using (1) again, holds if and only if  $A^*A \leq I$ , i.e.  $\|A\| \leq 1$ . In the second case, (1) implies that  $A = V$ .

REMARK 2.4. In [5], it was claimed that (1) was a necessary condition for  $Z(A)$  to be a contraction. Clearly, if  $A$  is bounded, then  $Z(\lambda A)$  will be a contraction for all sufficiently small scalars  $\lambda$ . There is also an error in Theorem 1.7(iii) in [5], which states that if  $P$  and  $Q$  are unitarily equivalent strictly positive operators and  $X$  is a solution of  $P \geq X^*QX$  and  $Q \geq XPX^*$ , then  $X$  is a contraction such that  $XP = QX$ . The matrices

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad Q = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad X = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

provide a counterexample.

3. Characteristic functions and spectra. The Sz.-Nagy-Foiaş model theory for contractions assigns to each contraction  $T$  on a Hilbert space  $H$  the triple  $\{\mathcal{D}_T, \mathcal{D}_{T^*}, \Theta_T(\lambda)\}$  where  $D_T = (I - T^*T)^{1/2}$ ,  $D_{T^*} = (I - TT^*)^{1/2}$ ,  $\mathcal{D}_T = \overline{D_T H}$ ,  $\mathcal{D}_{T^*} = \overline{D_{T^*} H}$ , and  $\Theta_T(\lambda) = [-T + \lambda D_{T^*}(I - \lambda T^*)^{-1}D_T]$  is an analytic operator-valued function whose values are contractions from  $\mathcal{D}_T$  to  $\mathcal{D}_{T^*}$ , the defect spaces of  $T$ . (This holds since  $TD_T = D_{T^*}T$ .) We call this triple the characteristic function of  $T$ , and if  $T$  is completely nonunitary (c.n.u.), i.e. there is no reducing subspace on which  $T$  is unitary, then  $T$  is unitarily equivalent to the adjoint of the restricted shift on the Sz.-Nagy-Foiaş space generated by its characteristic function [8, p. 248]. In most



cases, one is unable to get any "concrete" information from this representation for a specific operator because of computational difficulties involved in simplifying the form of the characteristic function. However, if  $A$  satisfies (1), then we can apply Fuhrmann's proof [5, p. 169-172] verbatim to get the following two theorems.

**THEOREM 3.1.** *If  $A$  is a strict contraction satisfying (1), then  $Z(A)$  is a c.n.u. contraction on  $K$  with characteristic function  $\{K_0, k_0, \Theta_{Z(A)}(z)\}$ , where  $\Theta_{Z(A)}(z)$  is given by  $\Theta_{Z(A)}(z)D_y = d_{G(z)y}$  where*

$$\begin{aligned} & (I - S(0)S(0)^*)^{1/2}G(z)(I - S(0)^*S(0))^{-1/2} \\ & = (I - AA^*)^{1/2}(I - \Gamma(z)A^*)^{-1}(\Gamma(z) - A)(I - A^*A)^{-1/2} \end{aligned}$$

and

$$\Gamma(z) = (I - S(0)S(0)^*)^{1/2}(I - S(z)S(0)^*)^{-1}(S(z) - S(0))(I - S(0)^*S(0))^{-1/2}.$$

Note that the above are matrix fractional linear transformations.

We call an open arc  $\gamma$  of the unit circle regular for  $S(z)$  if  $S(z)$  has analytic continuation over  $\gamma$  and for all  $\lambda \in \gamma$ ,  $S(\lambda)$  is unitary. Let  $\sigma(T)$  and  $\sigma(Z(A))$  denote the spectrum of  $T$  and  $Z(A)$  respectively. Recall [8, Theorem VI, 4.1] that  $\sigma(T) = \{|z| < 1 \mid S(z) \text{ is not boundedly invertible}\} \cup \{|\lambda| = 1 \mid \lambda \text{ lies on no regular arc of } S\}$ .

**THEOREM 3.2.** *Under the assumptions of 3.1, (i)  $\sigma(Z(A)) = \{|\lambda| = 1 \mid \lambda \text{ lies on no regular arc of } S\} \cup \{|z| < 1 \mid (\Gamma(z) - A) \text{ is not boundedly invertible}\}$ .*

(ii)  $Z(A)^*$  and  $Z(A)^{**}$  both converge to zero in the strong operator topology if and only if  $S(Z)$  is inner.

**REMARK 3.3.** (a) If  $A$  is a strict contraction not satisfying (1),  $\Theta_{Z(A)}(z)$  as defined above still has an interpretation as a characteristic function. (b) Note that  $W \equiv Z(0)$  is the completely nonunitary partial isometry with initial space  $K_0$  and final space  $k_0$  and agreeing with  $T$  on  $K_0$ . With this choice of  $A$ , (1) is satisfied and Theorem 3.1 says  $\Theta_{Z(0)}(z)D_y = d_{(I - S(0)S(0)^*)^{-1/2}\Gamma(z)(I - S(0)^*S(0))^{1/2}y}$ . It is not difficult to see that  $\Theta_{Z(0)}(z)$  coincides (see [8] page 192 for definition) with  $\Gamma(z): C \rightarrow C_*$ . Hence the partial isometry  $W$  can be represented as the compressed right shift  $T'$  on the Sz.-Nagy-Foiaş space  $K'$  associated with  $\Gamma(z)$  rather than with  $S(z)$ . (c) For  $A$  a contraction from  $C$  to  $C_*$ , let  $Z'(A)$  be the associated perturbation of  $T'$  in  $K'$ . Since  $\Gamma(0) = 0$ , (1) is satisfied for any  $A$ . In particular, for  $A = -S(0)$ , Theorem 3.1 gives  $\Theta_{Z'(-S(0))}(z)D'_y = d'_{S(z)y}$ , and hence  $\Theta_{Z'(-S(0))}(z)$  coincides with  $S(z)$ . Hence the operator  $T$  on  $K$  is unitarily equivalent to

$Z'(-S(0))$  on  $K'$ . It is then clear that the formula above for  $\Theta_{Z(A)}$  ( $A$  not necessarily satisfying (1)), interpreted for  $D_y$  and  $d_z$  in  $K'$ , gives the characteristic function for  $Z'(A)$ . In this sense, it is perhaps more natural to study perturbations of  $Z'(-S(0))$  on  $K'$  rather than of  $T$  on  $K$ . It is now seen that (1) is the condition that  $Z(A)$  and  $Z'(A)$  be unitarily equivalent.

**4. Unitary perturbations.** Since the characteristic function of a unitary map is zero, the above method fails totally when  $Z(A)$  is unitary. However, when  $A$  satisfies (1) we can still get spectral information about  $Z(A)$  by adapting techniques of D. N. Clark [2] to a more general setting. We begin with two technical lemmas.

**LEMMA 4.1.** *If  $A$  is unitary and satisfies (1), then  $\alpha = \alpha_A = -(I + S(0)A^*)(S(0)^* + A^*)^{-1}$  is unitary from  $C$  to  $C_*$ .*

*Proof.*  $\alpha$  is a priori defined on some dense set  $D_1 \subset C$  since  $S(0)^*$  is a pure contraction and  $A^*$  is unitary. (1) implies that

$$(I + AS(0)^*)(I + S(0)A^*)(S(0)^* + A^*)^{-1} = (S(0) + A).$$

Thus, for  $x \in D_1$ ,  $(\alpha x, \alpha x) = ((S(0)^* + A^*)^{-1}x, (S(0) + A)x) = (x, x)$ , so  $\alpha$  can be extended to an isometry on  $C$ . Similarly,  $\alpha^*$  is an isometry on  $C_*$  so  $\alpha$  is unitary.

Note that in fact, (1) is also necessary for  $\alpha$  to be unitary, and  $\alpha_A$  determines  $A$  by  $A^* = -(\alpha + S(0))^{-1}(\alpha S(0)^* + I)$ . Also, we have  $(\alpha + S(0))^{-1} = -(S(0)^* + A^*)(I - S(0)S(0)^*)^{-1}$ ; our  $\alpha$  corresponds to  $-\alpha$  used in [2].

**LEMMA 4.2.** *For  $F = (f, g) \in K$ ,*

- (i)  $(Z(A) - T)(F) = k_{0, x, 0}$  where  $x = -(\alpha^* + S(0)^*)^{-1}(\tau_1 F)(0)$
- (ii)  $(Z(A)^* - T)(F) = k_{0, 0, y}$  where  $y = -(\alpha + S(0))^{-1}f(0)$

*Proof.* Since  $Z(A) = T(I - P_{K_0}) + Z(A)P_{K_0}$ , we obtain

$$\begin{aligned} (Z(A) - T)(F) &= d_{(S(0)+A)(I-S(0)^*S(0))^{-1}(\tau_1 F)(0)} \\ &= d_x, \end{aligned}$$

with  $x$  as in (i).

(ii) follows similarly.

For  $|z| < 1$ , define  $\varphi(z): C_* \rightarrow C_*$  by  $\varphi(z) = (I - S(z)\alpha^*)(I + S(z)\alpha^*)^{-1}$ . Then straight-forward calculation gives

$$(3) \quad \varphi(\zeta) + \varphi(\eta)^* = 2(I + S(\zeta)\alpha^*)^{-1}(I - S(\zeta)S(\eta)^*)(I + \alpha S(\eta)^*)^{-1}$$

and hence (let  $z = \zeta = \eta$ )  $\varphi(z)$  has nonnegative real part for  $|z| < 1$ . By the operator-valued version of the Herglotz theorem, there exists

a non-negative operator-valued measure  $\mu$  on  $[0, 2\pi]$  such that  $\varphi(z) = \int_0^{2\pi} (e^{i\theta} + z)(e^{i\theta} - z)^{-1} d\mu(\theta)$ .

Thus

$$(4) \quad \varphi(\zeta) + \varphi(\eta)^* = 2 \int (1 - \zeta\bar{\eta})(1 - e^{-i\theta}\zeta)^{-1}(1 - e^{i\theta}\eta)^{-1} d\mu(\theta).$$

Comparing (3) and (4) yields

$$(5) \quad \frac{I - S(\zeta)S(\eta)^*}{1 - \zeta\bar{\eta}} = \int \frac{I + S(\zeta)\alpha^*}{1 - e^{-i\theta}\zeta} d\mu(\theta) \frac{I + \alpha S(\eta)^*}{1 - e^{i\theta}\bar{\eta}}.$$

Similar computations give

$$(6) \quad \begin{aligned} \frac{S(\zeta) - S(\bar{\eta})}{\zeta - \bar{\eta}} &= -\frac{1}{2}(I + S(\zeta)\alpha^*) \left( \frac{\varphi(\zeta) - \varphi(\bar{\eta})}{\zeta - \bar{\eta}} \right) (I + S(\bar{\eta})\alpha^*)\alpha \\ &= -\int \frac{I + S(\zeta)\alpha^*}{1 - e^{-i\theta}\zeta} d\mu(\theta) \frac{I + S(\bar{\eta})\alpha^*}{e^{i\theta} - \bar{\eta}} \alpha \end{aligned}$$

and

$$(7) \quad \frac{I - \tilde{S}(\zeta)\tilde{S}(\eta)^*}{1 - \zeta\bar{\eta}} = \int_0^{2\pi} \alpha^* \frac{I + \alpha S(\bar{\zeta})^*}{e^{-i\theta} - \zeta} d\mu(\theta) \frac{I + S(\bar{\eta})\alpha^*}{e^{i\theta} - \bar{\eta}} \alpha.$$

We define Hilbert space  $L^2(\mu)$  as in Shulman [7]. For  $f = x_1\chi_{E_1} + \dots + x_n\chi_{E_n}$  a simple  $C_*$ -valued function, where  $\chi_{E_1}, \dots, \chi_{E_n}$  are characteristic functions of disjoint Borel sets and  $x_1, \dots, x_n$  are corresponding elements of  $C_*$  define

$$\|f\|_\mu^2 = \int (d\mu(t)f(t), f(t)) = (\mu(E_1)x_1, x_1) + \dots + (\mu(E_n)x_n, x_n).$$

This does not depend on the representation of  $f(t)$  in terms of characteristic functions. Let  $\mathcal{A} = \{f(t): [0, 2\pi] \rightarrow C_* \mid f \text{ is Borel measurable, } \int \|f(t)\|^2 d(u(t)x, x) < \infty \text{ for all } x \in C_*, \text{ the range of } f(t) \text{ is contained in a finite dimensional subspace of } C_*\}$ . For  $f \in \mathcal{A}$  let  $e_1, e_2, \dots, e_k$  be a basis for the smallest subspace which contains the range of  $f(t)$ , and define

$$\alpha(f, t) = (\mu(t)e_1, e_1) + \dots + (\mu(t)e_k, e_k).$$

The definition is independent of the choice of basis for this subspace, and  $\|f\|_\mu^2 \leq \int \|f(t)\|^2 d\alpha(f, t)$  whenever  $f$  is a simple function. For  $f \in \mathcal{A}$ , there is a sequence of simple functions  $\{f_n(t)\}$  such that the range of  $f_n(t)$  is contained in the range of  $f(t)$  for  $n = 1, 2, \dots$ , and such that  $\int \|f_n(t) - f(t)\|^2 d\alpha(f, t) \rightarrow 0$  as  $n \rightarrow \infty$ . We can define  $\|f(t)\|_\mu^2$  unambiguously as

$$\|f\|_\mu^2 = \lim_{n \rightarrow \infty} \|f_n\|_\mu^2.$$

By  $L^2(\mu)$  is meant the Hilbert space completion of the inner product space of equivalence classes of functions with finite-dimensional range in  $\mu$ -norm. The definition of  $L^2(\mu)$  is such that explicit formulas can be written only for an element associated with the equivalence class of an element of  $\mathcal{A}$ . This, however, causes no difficulties for our purposes. It is clear, for example, that the transformation  $h(t) \rightarrow e^{it}h(t)$  is unitary in  $L^2(\mu)$ , with spectrum equal to  $\text{supp}(\mu)$  (the complement of the largest open set on which  $\mu$  is zero).

We are now in position to define a unitary transformation of  $K$  onto  $L^2(\mu)$  which transforms the operator  $Z(A)$  on  $K$  to the operator of multiplication on  $e^{it}$  on  $L^2(\mu)$ .

**THEOREM 4.3.** *Define  $V$  on elements in  $K$  of the form  $k_{\zeta,x,y}$  by*

$$V(k_{\zeta,x,y}) = \frac{I + \alpha S(\zeta)^*}{1 - e^{it}\bar{\zeta}}x - \frac{I + S(\bar{\zeta})\alpha^*}{e^{it} - \bar{\zeta}}\alpha y.$$

*Then  $V$  is well-defined and extends uniquely to a unitary transformation (also  $V$ ) of  $K$  onto  $L^2(\mu)$  such that  $VZ(A) = e^{it}V$ .*

*Proof.* We first check that  $V$  is an isometry on those vectors where it is defined. Note, for  $x, y \in C_*$ ,

$$\begin{aligned} (k_{\eta,y,0}, k_{\zeta,x,0})_K &= \left( \frac{I - S(\zeta)S(\eta)^*}{1 - \bar{\eta}\zeta}y, x \right)_{C_*} \\ &= \left( \int \frac{I + S(\zeta)\alpha^*}{1 - e^{-it}\zeta}d\mu(t) \frac{I + \alpha S(\eta)^*}{1 - e^{it}\eta}y, x \right)_{C_*} \text{ by (5)} \\ &= \left( \frac{I + \alpha S(\eta)^*}{1 - e^{it}\bar{\eta}}y, \frac{I + \alpha S(\zeta)^*}{1 - e^{it}\bar{\zeta}}x \right)_{L^2(\mu)} \\ &= (Vk_{\eta,y,0}, Vk_{\zeta,x,0})_{L^2(\mu)}. \end{aligned}$$

Also, for  $x, y \in C$ ,

$$\begin{aligned} (k_{\eta,0,y}, k_{\zeta,0,x})_K &= \left( \frac{I - S(\bar{\zeta})^*S(\bar{\eta})}{1 - \bar{\eta}\zeta}y, x \right)_C \\ &= \left( \int \alpha^* \frac{I + \alpha S(\bar{\zeta})^*}{e^{-it} - \zeta}d\mu(t) \frac{I + S(\bar{\eta})\alpha^*}{e^{it} - \bar{\eta}}\alpha y, x \right)_C \\ &= \left( \frac{I + S(\bar{\eta})\alpha^*}{e^{it} - \bar{\eta}}\alpha y, \frac{I + S(\bar{\zeta})\alpha^*}{e^{it} - \bar{\zeta}}\alpha x \right)_{L^2(\mu)} \\ &= (Vk_{\eta,0,y}, Vk_{\zeta,0,x})_{L^2(\mu)} \end{aligned}$$

and finally, for  $x \in C_*$  and  $y \in C$ ,

$$\begin{aligned}
 (k_{\eta,0,y}, k_{\zeta,x,0})_K &= \left( \frac{S(\zeta) - S(\bar{\eta})}{\zeta - \bar{\eta}} y, x \right)_{C_*} \\
 &= \left( - \int \frac{I + S(\zeta)\alpha^*}{1 - e^{-it}\zeta} d\mu(t) \frac{I + S(\bar{\eta})\alpha^*}{e^{it} - \bar{\eta}} \alpha y, x \right)_{C_*} \text{ by (6)} \\
 &= (Vk_{\eta,0,y}, Vk_{\zeta,x,0})_{L^2(\mu)}.
 \end{aligned}$$

Hence  $V$  is isometric (and hence also well-defined) on its domain. Since elements of the form  $k_{\eta,x,y}$  span a dense set in  $K$ ,  $V$  extends by linearity and continuity to be an isometry of  $K$  into  $L^2(\mu)$ . Since the range of  $V$  contains all elements of the form  $x/(1 - e^{it}\bar{w})$  and  $x/(e^{it} - \bar{w})$  for  $x \in C_*$  and  $|w| < 1$ , it follows that  $V$  is onto  $L^2(\mu)$ .

It remains to show  $VZ(A) = e^{it}V$ . By Lemmas 1.4 and 4.2,

$$\begin{aligned}
 Z(A)k_{w,x,0} &= \bar{w}^{-1}k_{w,x,0} - \bar{w}^{-1}k_{0,x,0} \\
 &\quad + \bar{w}^{-1}k_{0,(\alpha^*+S(0)^*)^{-1}(S(0)^*-S(w)^*)x,0} \\
 &= \bar{w}^{-1}(k_{w,x,0} - k_{0,(\alpha^*+S(0)^*)^{-1}(\alpha^*+S(w))x,0})
 \end{aligned}$$

and hence

$$\begin{aligned}
 VZ(A)k_{w,x,0} &= \bar{w}^{-1}(1 - e^{it}\bar{w})^{-1}(I + \alpha S(w)^*)x - \bar{w}^{-1}(I + \alpha S(w)^*)x \\
 &= \bar{w}^{-1}[(1 - e^{it}\bar{w})^{-1} - 1](I + \alpha S(w)^*)x \\
 &= e^{it} \frac{I + \alpha S(w)^*}{1 - e^{it}\bar{w}} x = e^{it}Vk_{w,x,0}.
 \end{aligned}$$

Similarly

$$\begin{aligned}
 Z(A)k_{w,0,y} &= \bar{w}k_{w,0,y} - k_{0,S(\bar{w})y,0} \\
 &\quad - k_{0,(\alpha^*+S(0)^*)^{-1}(I-S(0)^*S(\bar{w}))y,0} \\
 &= \bar{w}k_{w,0,y} - k_{0,(\alpha^*+S(0)^*)^{-1}(I+\alpha^*S(\bar{w}))y,0}.
 \end{aligned}$$

So

$$\begin{aligned}
 VZ(A)k_{w,0,y} &= -\bar{w}(e^{it} - \bar{w})^{-1}(I + S(\bar{w})\alpha^*)\alpha y - (I + S(\bar{w})\alpha^*)\alpha y \\
 &= -e^{it} \frac{I + S(\bar{w})\alpha^*}{e^{it} - \bar{w}} \alpha y \\
 &= e^{it}Vk_{w,0,y}.
 \end{aligned}$$

The theorem follows.

We note the following inversion formula for  $V$ .

**THEOREM 4.4.** *Let  $V^*: L^2 \rightarrow K$  be defined, for  $F$  in  $\mathcal{A}$ , by  $V^*F = (W_1F, W_2F)$  where  $(W_1F)(z) = (I + S(z)\alpha^*) \int (1 - e^{-it}z) d\mu(t)F(t)$  and  $(W_2F)(t) = \lim_{r \rightarrow 1} (I - S(re^{it})^*S(re^{it}))^{-1/2}$ .*

$$\begin{aligned}
 &(\alpha^* + S(re^{it}))^* \int (e^{i(t-\theta)} - r)^{-1} d\mu(\theta) F(\theta) \\
 &\quad - \int (S(re^{it})^* - S(re^{it})^* S(re^{it}) \alpha^*) (1 - re^{i(t-\theta)})^{-1} d\mu(\theta) F(\theta).
 \end{aligned}$$

Then  $V^*$  is the adjoint of  $V$  defined in Theorem 4.3.

*Proof.* To obtain  $W_1$ , rewrite equation (5) substituting  $z$  for  $\zeta$  and noting that

$$\begin{aligned}
 V k_{\gamma, z, 0} &= \frac{I + \alpha S(\bar{\eta})^*}{1 - e^{it} \bar{\eta}} x \text{ to obtain} \\
 \frac{I - S(z) S(\bar{\eta})^*}{1 - \zeta \bar{\eta}} x &= \int \frac{I + S(z) \alpha^*}{1 - e^{-it} z} d\mu(t) (V k_{\gamma, z, 0})(t).
 \end{aligned}$$

Similarly, using equation (6),

$$\frac{S(z) - S(\bar{\eta})}{z - \bar{\eta}} y = \int \frac{I + S(z) \alpha^*}{1 - e^{-it} z} d\mu(t) (V k_{\gamma, 0, y})(t).$$

This proves the correctness of the formula for  $W_1$  for all  $F$  of the form  $V k_{\gamma, x, y}$ , and hence by approximation for all  $F \in \mathcal{A}$ . To obtain the formula for  $W_2$ , we first find a formula for  $(\tau_1 V^* F)(z)$ . By an argument dual to that above, we find

$$(\tau_1 V^* F)(z) = -\alpha^* (I + \alpha S(\bar{z})^*) \int (e^{-it} - z)^{-1} d\mu(t) F(t).$$

The formula for  $W_2$  is then obtained by using the explicit formulas for  $\tau$  and  $\tau^*$  in Theorem 1.1.

**THEOREM 4.5.** *Let  $A$  be unitary and satisfy (1). Then  $\sigma(Z(A)) = \{|\lambda| = 1 | \lambda \text{ lies on no regular arc of } S\} \cup \{|\lambda| = 1 | \lambda \text{ lies on a regular arc of } S \text{ but } (I + S(\lambda)\alpha^*) \text{ is not boundedly invertible}\}$ .*

*Proof.* Since  $Z(A)$  has a representation as multiplication by  $e^{i\theta}$  on  $L^2(\mu)$ , we have  $\sigma(Z(A)) = \text{supp}(\mu)$ , the complement of the largest open set on which  $\mu$  is zero. By the integral representation of  $\varphi$ , we see that the complement of  $\text{supp}(\mu)$  is the set of  $\lambda$  at which  $\varphi(z)$  has analytic continuation with  $\text{Re } \varphi(\lambda) = 0$ . Since  $\varphi(z) = (I - S(z)\alpha^*)(I + S(z)\alpha^*)^{-1}$ , we have  $(I + \varphi(z)) = 2(I + S(z)\alpha^*)^{-1}$  and  $S(z) = (I - \varphi(z))(I + \varphi(z))^{-1}\alpha$ .

Now, suppose  $\varphi(z)$  has continuation at  $\lambda$  and  $\text{Re } \varphi(\lambda) = 0$ . Then  $(I + \varphi(\lambda))$  is boundedly invertible, and hence  $(I + \varphi(z))^{-1}$  extends to an analytic function in a neighborhood of  $\lambda$ . Thus,  $S(z)$  has analytic continuation at  $\lambda$  and  $(I + S(\lambda)\alpha^*)$  is boundedly invertible; since  $\text{Re } \varphi(\lambda) = 0$ ,  $S(\lambda)$  is unitary. Conversely, suppose  $S(z)$  has analytic

continuation at  $\lambda$ ,  $(I + S(\lambda)\alpha^*)$  is boundedly invertible, and  $S(\lambda)$  is unitary. Then  $(I + S(z)\alpha^*)^{-1}$  is analytic in some neighborhood of  $\lambda$ , so  $\varphi(z)$  has analytic continuation at  $\lambda$ ; since  $S(\lambda)$  is unitary,  $\operatorname{Re} \varphi(\lambda) = 0$ . By taking complements, the theorem now follows.

Since  $(I + S(\lambda)\alpha^*) = [(I + S(0)A^*) - S(\lambda)(S(0)^* + A^*)](I + S(0)A^*)^{-1}$ , we see that  $(I + S(\lambda)\alpha^*)$  is boundedly invertible if and only if  $B(\lambda) = -[(I + S(0)A^*) - S(\lambda)(S(0)^* + A^*)]$  is boundedly invertible. With  $\Gamma$  as in Theorem 3.1, we have, since  $A$  satisfies (1),  $(\Gamma(\lambda) - A) = (I - S(0)S(0)^*)^{1/2}(I - S(\lambda)S(0)^*)^{-1}B(\lambda)A(I - S(0)^*S(0))^{-1/2}$ . Thus,  $(\Gamma(\lambda) - A)$  is invertible, but not necessarily boundedly, if and only if  $B(\lambda)$  is invertible. Since boundedness follows immediately in the finite-dimensional case, we have the following generalization of [5, Theorem 3.6] to the case of general analytic contractions  $S(z)$ .

**COROLLARY 4.6.** *If  $A$  is unitary on  $C$ ,  $C$  finite-dimensional, and  $A$  satisfies (1), then  $\sigma(Z(A)) = \{|\lambda| = 1 \mid \lambda \text{ lies on no regular arc of } S\} \cup \{|\lambda| = 1 \mid \lambda \text{ lies on a regular arc for } S \text{ but } (\Gamma(\lambda) - A) \text{ is not invertible}\}$ .*

In the finite-dimensional case,  $Z(A)$  is a compact perturbation of  $T$ . Hence by the known spectral behavior of  $T$  and Weyl's theorem,  $\{|\lambda| = 1 \mid \lambda \text{ lies on a regular arc for } S \text{ but } \Gamma(\lambda) - A \text{ is not invertible}\}$  must be eigenvalues for  $Z(A)$ .

We can also adapt Fuhrmann's calculations [5, page 174] to determine eigenvalues in our more general setting.

**THEOREM 4.7.** *If  $A$  is unitary and satisfies (1), and  $\lambda$  lies on a regular arc for  $S$ , then  $\lambda$  is an eigenvalue for  $Z(A)$  if and only if the range of  $\Gamma(\lambda) - A$  is not dense in  $C_*$ .*

**REMARK 4.8** If  $A$  does not satisfy (1), all of the above results apply to  $Z'(A)$ , as in Remark 3.3. Also, we still have from Theorem 2.3 that  $A = (I - S(0)S(0)^*)^{-1/2}V(I - S(0)^*S(0))^{1/2}$  for some unitary  $V$ . This implies that  $\tilde{\alpha}_A = \tilde{\alpha} = (A^* + S(0)^*)^{-1}(I + A^*S(0))$  is unitary. (Note that if  $A$  satisfies (1), then  $\tilde{\alpha} = \alpha$  used above.) In this case, the results of §4 still hold with  $\tilde{\alpha}$  in place of  $\alpha$ .

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# Pacific Journal of Mathematics

Vol. 63, No. 2

April, 1976

Joseph Anthony Ball and Arthur R. Lubin, <i>On a class of contractive perturbations of restricted shifts</i> .....	309
Joseph Becker and William C. Brown, <i>On extending higher derivations generated by cup products to the integral closure</i> .....	325
Andreas Blass, <i>Exact functors and measurable cardinals</i> .....	335
Joseph Eugene Collison, <i>A variance property for arithmetic functions</i> .....	347
Craig McCormack Cordes, <i>Quadratic forms over nonformally real fields with a finite number of quaternion algebras</i> .....	357
Freddy Delbaen, <i>Weakly compact sets in <math>H^1</math></i> .....	367
G. D. Dikshit, <i>Absolute Nörlund summability factors for Fourier series</i> .....	371
Edward Richard Fadell, <i>Nielsen numbers as a homotopy type invariant</i> .....	381
Josip Globevnik, <i>Analytic extensions of vector-valued functions</i> .....	389
Robert Gold, <i>Genera in normal extensions</i> .....	397
Solomon Wolf Golomb, <i>Formulas for the next prime</i> .....	401
Robert L. Griess, Jr., <i>The splitting of extensions of <math>SL(3, 3)</math> by the vector space <math>F_3^3</math></i> .....	405
Thomas Alan Keagy, <i>Matrix transformations and absolute summability</i> .....	411
Kazuo Kishi, <i>Analytic maps of the open unit disk onto a Gleason part</i> .....	417
Kwangil Koh, Jiang Luh and Mohan S. Putcha, <i>On the associativity and commutativity of algebras over commutative rings</i> .....	423
James C. Lillo, <i>Asymptotic behavior of solutions of retarded differential difference equations</i> .....	431
John Alan MacBain, <i>Local and global bifurcation from normal eigenvalues</i> .....	445
Anna Maria Mantero, <i>Sets of uniqueness and multiplicity for <math>L^p</math></i> .....	467
J. F. McClendon, <i>Embedding metric families</i> .....	481
L. Robbiano and Giuseppe Valla, <i>Primary powers of a prime ideal</i> .....	491
Wolfgang Ruess, <i>Generalized inductive limit topologies and barrelledness properties</i> .....	499
Judith D. Sally, <i>Bounds for numbers of generators of Cohen-Macaulay ideals</i> .....	517
Helga Schirmer, <i>Mappings of polyhedra with prescribed fixed points and fixed point indices</i> .....	521
Cho Wei Sit, <i>Quotients of complete multipartite graphs</i> .....	531
S. Sznajder and Zbigniew Zielezny, <i>Solvability of convolution equations in <math>\mathfrak{K}'_p</math>, <math>p &gt; 1</math></i> .....	539
Mitchell Herbert Taibleson, <i>The existence of natural field structures for finite dimensional vector spaces over local fields</i> .....	545
William Yslas Vélez, <i>A characterization of completely regular fields</i> .....	553
P. S. Venkatesan, <i>On right unipotent semigroups</i> .....	555
Kenneth S. Williams, <i>A rational octic reciprocity law</i> .....	563
Robert Ross Wilson, <i>Lattice orderings on the real field</i> .....	571
Harvey Eli Wolff, <i>V-localizations and V-monads. II</i> .....	579