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On a Class of Degenerate Parabolic Equations of Kolmogorov Type

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1 Introduction

In this paper, we adapt the classical Levi parametrix method to construct a global fundamental solution to the following differential equation of Kolmogorov type:

$$\begin{aligned} Lu \equiv & \sum_{i,j=1}^{p_0} a_{ij}(z)\partial_{x_i x_j} u \\ & + \sum_{i=1}^{p_0} a_i(z)\partial_{x_i} u + \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j} u + c(z)u - \partial_t u = 0, \end{aligned} \tag{1.1}$$

where $z = (x, t) \in \mathbb{R}^N \times \mathbb{R}$ and $1 \le p_0 \le N$. By convenience, hereafter the term "Kolmogorov equation" will be shortened to KE. We assume the following hypotheses:

(H.1) the matrix $A_0 = (a_{ij})_{i,j=1,\dots,p_0}$ is symmetric and uniformly positive definite in $\mathbb{R}^{p_0}$: there exists a positive constant μ such that

$$\frac{|\eta|^2}{\mu} \le \sum_{i,j=1}^{p_0} a_{ij}(z)\eta_i\eta_j \le \mu|\eta|^2, \quad \forall \eta \in \mathbb{R}^{p_0},\ z \in \mathbb{R}^{N+1}; \tag{1.2}$$

(H.2) the matrix $B \equiv (b_{ij})$ has constant real entries and takes the following block

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form:

$$\begin{pmatrix} * & B_1 & 0 & \cdots & 0 \\ * & * & B_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & B_r \\ * & * & * & \cdots & * \end{pmatrix}, \tag{1.3}$$

where B_j is a $p_{j-1} \times p_j$ matrix of rank p_j, with

$$p_0 \geq p_1 \geq \cdots \geq p_r \geq 1, \qquad p_0 + p_1 + \cdots + p_r = N, \tag{1.4}$$

and the $*$-blocks are arbitrary.

The regularity hypotheses on the coefficients a_{ij}, a_i, c will be specified later: roughly speaking, we assume the Hölder continuity with respect to some homogeneous norm naturally induced by the equation.

The prototype of (1.1) is the following equation:

$$\partial_{x_1 x_1} u + x_1 \partial_{x_2} u - \partial_t u = 0, \qquad (x_1, x_2, t) \in \mathbb{R}^3, \tag{1.5}$$

whose fundamental solution was explicitly constructed by Kolmogorov [23]. In his celebrated paper [21], Hörmander generalized this result to *constant coefficients KEs*, that is, equations of the form (1.1), with constant a_{ij} and $a_i = c \equiv 0$ for $i = 1, \ldots, p_0$, satisfying the following condition:

$$\text{Ker}(A) \text{ does not contain nontrivial subspaces which are invariant for } B. \tag{1.6}$$

In (1.6), A denotes the $N \times N$ matrix

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{1.7}$$

We recall that, for constant coefficients equations, condition (1.6) is equivalent to the structural assumptions (H.1) and (H.2) which in turn are equivalent to the classical Hörmander condition:

$$\text{rank Lie}(X_1, \ldots, X_{p_0}, Y) = N + 1, \tag{1.8}$$

at any point of $\mathbb{R}^{N+1}$. In (1.8), $\mathrm{Lie}(X_1, \dots, X_{p_0}, Y)$ denotes the Lie algebra generated by the vector fields

$$X_i = \sum_{j=1}^{p_0} a_{ij} \partial_{x_j}, \quad i = 1, \dots, p_0, \qquad Y = \langle x, BD \rangle - \partial_t, \tag{1.9}$$

where $\langle \cdot, \cdot \rangle$ and D, respectively, denote the inner product and the gradient in $\mathbb{R}^N$. A proof of the equivalence of these conditions is given by Kupcov in [24, Theorem 3] and by Lanconelli and Polidoro in [26, Proposition A.1].

We recall that *constant coefficients* KEs have the remarkable property of being invariant with respect to the left translations in the law defined by

$$(x, t) \circ (\xi, \tau) = \big(\xi + E(\tau)x, t + \tau\big), \quad (x, t), (\xi, \tau) \in \mathbb{R}^N \times \mathbb{R}, \tag{1.10}$$

where

$$E(t) = e^{-tB^T}. \tag{1.11}$$

Moreover, we consider the family of dilations $(D(\lambda))_{\lambda > 0}$ on $\mathbb{R}^{N+1}$ defined by

$$D(\lambda) \equiv \big(D_0(\lambda), \lambda^2\big) = \mathrm{diag}\,\big(\lambda I_{p_0}, \lambda^3 I_{p_1}, \dots, \lambda^{2r+1} I_{p_r}, \lambda^2\big), \tag{1.12}$$

where I_{p_j} denotes the $p_j \times p_j$ identity matrix. It is known that if (and only if) all the $*$-blocks in (1.3) are zero matrices, then L is also homogeneous of degree two with respect to $(D(\lambda))$ in the sense that

$$L \circ D(\lambda) = \lambda^2 \big(D(\lambda) \circ L\big), \quad \forall \lambda > 0. \tag{1.13}$$

We remark explicitly that $\mathcal{G}_B \equiv (\mathbb{R}^{N+1}, \circ, D(\lambda))$ is a *homogeneous Lie group only determined by the matrix* B.

In some particular cases, variable coefficients KEs were first studied by Weber [36], Il'in [22], and Sonin [35] who used the parametrix method to construct a fundamental solution. Yet in these papers unnecessary restrictive conditions on the regularity of the coefficients are required. Assuming that the KE in (1.1) satisfies hypotheses (H.1) and (H.2) and that the $*$-blocks in (1.3) are zero matrices, the previous results were considerably generalized in a series of papers by Polidoro [31, 32, 33], by assuming a notion of regularity modeled on the homogeneous Lie group $\mathcal{G}_B$ (cf. Definitions 1.2 and 1.3 below). Some of the results of Polidoro were extended to nonhomogeneous KEs by Morbidelli [27]. We also refer to [25] for a survey of the most recent results about KEs.

In this paper, we aim to consider the general case of (1.1) satisfying (H.1) and (H.2) with arbitrary $*$-blocks.

The interest in obtaining results for the general class of KEs is not academic. It is well known that "homogeneous" KEs (i.e., KEs with null $*$-blocks in (1.3)) play a central role in the stochastic theory of diffusion processes. On the other hand, more general KEs have been recently considered for applications in mathematical finance. In the next section, we briefly recall some of the main motivations for studying KEs.

In order to state our main results, we recall the definition of homogeneous norm and B-Hölder continuity given by Polidoro [31].

Definition 1.1. Given a constant matrix B of the form (1.3) and $(D(\lambda))_{\lambda>0}$ defined as in (1.12), let $(q_j)_{j=1,\dots,N}$ be such that

$$D(\lambda) = \operatorname{diag}\left(\lambda^{q_1}, \lambda^{q_2}, \dots, \lambda^{q_N}, \lambda^2\right). \tag{1.14}$$

For every $z = (x,t) \in \mathbb{R}^{N+1}$, set

$$|x|_B = \sum_{j=1}^{N} \left|x_j\right|^{1/q_j}, \qquad \|z\|_B = |x|_B + |t|^{1/2}. \tag{1.15}$$

Clearly $\|\cdot\|_B$ is a norm on $\mathbb{R}^{N+1}$ homogeneous of degree one with respect to the dilations $(D(\lambda))$.

Definition 1.2. A function f is B-Hölder continuous of order $\alpha \in \,]0,1]$ on a domain Ω of $\mathbb{R}^{N+1}$, and $f \in C_B^{\alpha}(\Omega)$, if there exists a constant C such that

$$\left|f(z) - f(\zeta)\right| \le C\left\|\zeta^{-1} \circ z\right\|_B^{\alpha}, \quad \forall z, \zeta \in \Omega. \tag{1.16}$$

In (1.16), ζ^{-1} denotes the inverse of ζ in the law "$\circ$" in (1.10).

Next, we give the definition of solution to equation $Lu = f$.

Definition 1.3. A function u is a solution to the equation $Lu = f$ in a domain Ω of $\mathbb{R}^{N+1}$, if there exist the Euclidean derivatives $\partial_{x_i} u, \partial_{x_i x_j} u \in C(\Omega)$ for $i,j = 1,\dots,p_0$, the Lie[1] derivative $Yu \in C(\Omega)$, and equation

$$\sum_{i,j=1}^{p_0} a_{ij}(z)\partial_{x_i x_j} u(z) + \sum_{i=1}^{p_0} a_i(z)\partial_{x_i} u(z) + Yu(z) + c(z)u(z) = f(z) \tag{1.17}$$

is satisfied at any $z \in \Omega$.

[1]A function u is Lie differentiable with respect to the vector field Y in (1.9), at the point $z = (x,t)$, if $\lim_{\delta\to 0}(u(\gamma(\delta)) - u(\gamma(0)))/\delta \equiv Yu(z)$ exists and is finite, where γ denotes the integral curve of Y from z: $\gamma(\delta) = (E(-\delta)x, t-\delta)$, $\delta \in \mathbb{R}$. Clearly, if $u \in C^1$, then $Yu(x,t) = \langle x, BDu(x,t)\rangle - \partial_t u(x,t)$.

We are now in a position to state the following.

Theorem 1.4. Assume that L in (1.1) verifies hypotheses (H.1) and (H.2) and that the coefficients $a_{ij}, a_i, c \in C^{\alpha}_{B}(\mathbb{R}^{N+1})$ are bounded functions. Then there exists a fundamental solution Γ to L with the following properties:

(1) $\Gamma(\cdot,\zeta) \in L^1_{loc}(\mathbb{R}^{N+1}) \cap C(\mathbb{R}^{N+1} \setminus \{\zeta\})$ for every $\zeta \in \mathbb{R}^{N+1}$;

(2) $\Gamma(\cdot,\zeta)$ is a solution to $Lu = 0$ in $\mathbb{R}^{N+1} \setminus \{\zeta\}$ for every $\zeta \in \mathbb{R}^{N+1}$ (in the sense of Definition 1.3);

(3) let $g \in C(\mathbb{R}^N)$ such that

$$\left|g(x)\right| \le C_0 e^{C_0|x|^2}, \quad \forall x \in \mathbb{R}^N, \tag{1.18}$$

for some positive constant C_0; then there exists

$$\lim_{t\to\tau^+} \int_{\mathbb{R}^N} \Gamma(x,t,\xi,\tau) g(\xi) d\xi = g(x), \quad \forall x \in \mathbb{R}^N,\ \tau \in \mathbb{R}; \tag{1.19}$$

(4) let $g \in C(\mathbb{R}^N)$ verifying (1.18) and let f be a continuous function in the strip $S_{T_0,T_1} = \mathbb{R}^N \times \left]T_0, T_1\right[$, such that

$$\left|f(x,t)\right| \le C_1 e^{C_1|x|^2}, \quad \forall (x,t) \in S_{T_0,T_1} \tag{1.20}$$

and for any compact subset M of $\mathbb{R}^N$, there exists a positive constant C such that

$$\left|f(x,t) - f(y,t)\right| \le C|x-y|^{\beta}_{B}, \quad \forall x,y \in M,\ t \in \left]T_0, T_1\right[, \tag{1.21}$$

for some $\beta \in \left]0,1\right[$; then there exists $T \in \left]T_0, T_1\right]$ such that the function

$$u(x,t) = \int_{\mathbb{R}^N} \Gamma\big(x,t,\xi,T_0\big) g(\xi) d\xi - \int_{T_0}^{t} \int_{\mathbb{R}^N} \Gamma(x,t,\xi,\tau) f(\xi,\tau) d\xi\, d\tau \tag{1.22}$$

is a solution to the Cauchy problem

$$\begin{aligned} &Lu = f \quad \text{in } S_{T_0,T}, \\ &u\big(\cdot,T_0\big) = g \quad \text{in } \mathbb{R}^N; \end{aligned} \tag{1.23}$$

(5) if u is a solution to the Cauchy problem (1.23) with null f and g, and verifies estimate (1.20), then $u \equiv 0$ (see also Theorem 1.6 below); in particular, the function in (1.22) is the unique solution to problem (1.23) verifying estimate (1.20);

(6) the reproduction property holds:

$$\Gamma(x,t,\xi,\tau) = \int_{\mathbb{R}^N} \Gamma(x,t,y,s)\Gamma(y,s,\xi,\tau)dy, \quad \forall x,\xi \in \mathbb{R}^N,\ \tau < s < t; \tag{1.24}$$

(7) if $c(z) \equiv c$ is constant, then

$$\int_{\mathbb{R}^N} \Gamma(x,t,\xi,\tau)d\xi = e^{-c(t-\tau)}, \quad \forall x \in \mathbb{R}^N,\ \tau < t; \tag{1.25}$$

(8) let Γ^ε denote the fundamental solution to the constant coefficients KE

$$L^\varepsilon = (\mu+\varepsilon)\Delta_{\mathbb{R}^{p_0}} + \langle x, B\nabla\rangle - \partial_t, \tag{1.26}$$

where $\varepsilon > 0$, μ is as in (1.2), and $\Delta_{\mathbb{R}^{p_0}}$ denotes the Laplacian in the variables $x_1,\dots,x_{p_0}$; then for every positive ε and T, there exists a constant C, only dependent on μ, B, ε, and T, such that

$$\Gamma(z,\zeta) \le C\Gamma^\varepsilon(z,\zeta), \tag{1.27}$$

$$\left|\partial_{x_i}\Gamma(z,\zeta)\right| \le \frac{C}{\sqrt{t-\tau}}\Gamma^\varepsilon(z,\zeta), \tag{1.28}$$

$$\left|\partial_{x_i x_j}\Gamma(z,\zeta)\right| \le \frac{C}{t-\tau}\Gamma^\varepsilon(z,\zeta), \qquad \left|Y\Gamma(z,\zeta)\right| \le \frac{C}{t-\tau}\Gamma^\varepsilon(z,\zeta), \tag{1.29}$$

for any $i,j = 1,\dots,p_0$ and $z,\zeta \in \mathbb{R}^{N+1}$ with $0 < t-\tau < T$. □

Under the further hypothesis

(H.3) for every $i,j = 1,\dots,p_0$, there exist the derivatives $\partial_{x_i}a_{ij}, \partial_{x_i x_j}a_{ij}, \partial_{x_i}a_i \in C^\alpha_B(\mathbb{R}^{N+1})$ and they are bounded functions,

we define as usual the adjoint operator L^* of L:

$$L^*v = \sum_{i,j=1}^{p_0} a_{ij}\partial_{x_i x_j}v + \sum_{i=1}^{p_0} a_i^*\partial_{x_i}v - \langle x, B\nabla v\rangle + c^*v + \partial_t v, \tag{1.30}$$

where

$$a_i^* = -a_i + 2\sum_{j=1}^{p_0}\partial_{x_i}a_{ij}, \qquad c^* = c + \sum_{i,j=1}^{p_0}\partial_{x_i x_j}a_{ij} - \sum_{i=1}^{p_0}\partial_{x_i}a_i - \mathrm{tr}(B), \tag{1.31}$$

and we prove the following result.

Theorem 1.5. There exists a fundamental solution Γ^* of L^* verifying the dual properties in the statement of Theorem 1.4. Moreover, it holds that

$$\Gamma^*(z,\zeta) = \Gamma(\zeta, z), \quad \forall z, \zeta \in \mathbb{R}^{N+1},\ z \neq \zeta. \tag{1.32}$$

□

We close this section by stating a further uniqueness result.

Theorem 1.6. Assume that L in (1.1) verifies hypotheses (H.1), (H.2), and (H.3) and that the coefficients $a_{ij}, a_i, c \in C^{\alpha}_{B}(\mathbb{R}^{N+1})$ are bounded functions. If u is a solution to the Cauchy problem (1.23) with null f and g, such that

$$\int_{T_0}^{T} \int_{\mathbb{R}^N} \left|u(x,t)\right| e^{-C|x|^2}\, dx\, dt < +\infty \tag{1.33}$$

for some positive constant C, then $u \equiv 0$. □

The paper is organized as follows. In the next section, we present some motivation for studying KEs. In Section 3, we collect some preliminaries. In Section 4, we present the parametrix method for constructing a fundamental solution. In Section 5, we provide some potential estimates. Section 6 is devoted to the proofs of Theorems 1.4, 1.5, and 1.6.

2 Some motivation

In this section, we give some motivation for the study of KEs from probability, physics, and finance. The operator (1.5) is the lowest dimension version of the following degenerate parabolic operator in $\mathbb{R}^{N+1}$ with $N = 2n$:

$$L = \sum_{j=1}^{n} \partial^2_{x_j} + \sum_{j=1}^{n} x_j \partial_{x_{n+j}} - \partial_t. \tag{2.1}$$

Kolmogorov introduced (2.1) in [23] in order to describe the probability density of a system with $2n$ degree of freedom. The $2n$-dimensional space is the phase space, $(x_1, \dots, x_n)$ is the velocity, and $(x_{n+1}, \dots, x_{2n})$ is the position of the system. We also recall that (2.1) is a prototype for a family of evolution equations arising in the kinetic theory of gases that take the following general form:

$$Yu = \mathcal{J}(u). \tag{2.2}$$

Here $\mathbb{R}^{2n} \ni x \mapsto u(x,t) \in \mathbb{R}$ is the density of particles which have velocity $(x_1, \dots, x_n)$ and position $(x_{n+1}, \dots, x_{2n})$ at time t,

$$Yu \equiv \sum_{j=1}^{n} x_j \partial_{x_{n+j}} u + \partial_t u \tag{2.3}$$

is the so-called *total derivative of* u, and $\mathcal{J}(u)$ describes some kind of collision. This last term can take different form, either linear or nonlinear. For instance, in the usual Fokker-Planck equation, we have

$$\mathcal{J}(u) = - \sum_{i,j=1}^{n} \partial_{x_i} \left(a_{ij} \partial_{x_j} u + b_i u \right) + \sum_{i=1}^{n} a_i \partial_{x_i} u + cu, \tag{2.4}$$

where a_{ij}, a_i, b_i, and c are functions of (x,t). $\mathcal{J}(u)$ may also occur in nondivergence form and the coefficients may depend on $z \in \mathbb{R}^{2n+1}$ as well as on the solution u through some integral expressions. This kind of operator is studied as a simplified version of the Boltzmann collision operator. A description of wide classes of stochastic processes and kinetic models leading to equations of the previous type can be found in the classical monographies [9, 10, 15].

Linear KEs also arise in mathematical finance in some generalization of the celebrated Black-Scholes model [8]. Consider a "stock" whose price S_t is given by the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{2.5}$$

where μ and σ are positive constants and W_t is a Wiener process. Also consider a "bond" whose price B_t only depends on a constant interest rate r:

$$B_t = B_0 e^{tr}. \tag{2.6}$$

Finally, consider a "European option" which is a contract which gives the *right* (but not the *obligation*) to buy the stock at a given "exercise price" E and at a given "expiry time" T. The problem studied in [8] is to find a fair price of the option contract. Under some assumptions on the financial market, Black and Scholes show that the price of the option, as a function of the time and of the stock price $V(t, S_t)$, is the solution of the following partial differential equation:

$$-rV + \frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = 0 \tag{2.7}$$

in the domain $(S, t) \in \mathbb{R}^+ \times]0, T[$, with the *final condition*

$$V(T, S_T) = \max(S_T - E, 0). \tag{2.8}$$

In the last decades, the Black-Scholes theory has been developed by many authors and mathematical models involving KEs have appeared in the study of the so-called path-dependent contingent claims (see, e.g., [1, 4, 5, 37]). *Asian options* are options whose exercise price is not fixed as a given constant E, but depends on some average of the history of the stock price. In this case, the value of the option at the expiry time T is (for a geometric average option)

$$V(S_T, M_T) = \max\left(S_T - e^{M_T/T}, 0\right), \quad M_t = \int_0^t \log(S_\tau)\,d\tau. \tag{2.9}$$

If we suppose by simplicity that the interest rate is $r = 0$, the Black-Scholes method leads to the following degenerate equation:

$$S^2\partial_S^2 V + (\log S)\partial_M V + \partial_t V = 0, \quad S, t > 0, \ M \in \mathbb{R}, \tag{2.10}$$

which can be reduced to the KE (1.5) by means of an elementary change of variables (see [6, page 479]). A numerical study of the solution of the Cauchy problem related to (2.10) is also proposed in [6].

A more recent motivation from finance comes from the model by Hobson and Rogers [20]. In the Black-Scholes theory, the hypothesis that the volatility σ in the stochastic differential equation (2.5) is constant contrasts with the empirical observations. Aiming to overcome this problem, many authors proposed different models based on a stochastic volatility (see [16] for a survey). However, the presence of a second Wiener process leads to some difficulties in the arbitrage argument underlying the Black-Scholes theory. The model proposed by Hobson and Rogers for European options assumes that the volatility only depends on the difference between the present stock price and the past price. This simple model seems to capture the features observed in the market and avoid the problems related to the use of many sources of randomness.

As in the study of Asian options, in the Hobson-Rogers model for European options, the value of the option $V(t, S_t, M_t)$ is supposed to depend on the time t, on the price of the stock S_t, on some average M_t, and must satisfy the differential equation

$$\frac{1}{2}\sigma^2(S - M)\left(\partial_S^2 V - \partial_S V\right) + (S - M)\partial_M V + \partial_t V = 0 \tag{2.11}$$

that is a *nonhomogeneous KE* with Hölder continuous coefficients. In the recent paper [14], the Cauchy problem related to (2.11) has been studied numerically. In [13] the stability and the rate of convergence of different numerical methods for solving (2.11) are tested. The numerical schemes proposed in these papers rely on the approximation of the directional derivative Y by the finite difference $-(u(x,y,t) - u(x,y+\delta x, t-\delta))/\delta$, hence this method, which is respectful of the non-Euclidean geometry of the Lie group, seems to provide a good approximation of the solution.

Nonhomogeneous KEs also arise in the theory of bonds and interest rates and are considered in the study of the possible realization of Heath-Jarrow-Morton [19] models in terms of a finite-dimensional Markov diffusion (see, e.g., [7, 34]).

Recently, in [12] Corielli and one of the authors investigated the parametric approximation of risk neutral transition densities in the option valuation: more precisely they considered the approximation and estimation of general probability density functions in terms of fundamental solutions of suitable PDEs with constant coefficients. Expansions of this kind seem a natural tool for obtaining approximate solution for valuation problems while controlling the approximation error. However they are still unknown in the financial literature.

Finally, we recall that KEs with *nonlinear total derivative term* of the form

$$\Delta_x u + \partial_y g(u) - \partial_t u = f, \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n,\ y, t \in \mathbb{R}, \tag{2.12}$$

have been considered for convection-diffusion models (cf. [17, 28]), for pricing models of options with memory feedback (cf. [30]), and for mathematical models for utility functional and decision making (cf. [2, 3, 11, 29]). The linearized equation of (2.12)

$$g'(u)\partial_y v - \partial_t v = -\Delta_x v, \tag{2.13}$$

if $g'(u)$ is different from zero and smooth enough, can be reduced to the form (1.1) with $N = n + 2$ and

$$A = \begin{pmatrix} 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{2.14}$$

3 Preliminaries

In this section, we recall some known results for constant coefficients KEs, that is, equations of the form

$$\sum_{i,j=1}^{p_0} a_{ij}\partial_{x_i x_j} u + \langle x, BDu\rangle - \partial_t = 0, \tag{3.1}$$

with constant a_{ij}'s and satisfying hypotheses (H.1) and (H.2). Moreover, we prove some preliminary results.

First we recall the explicit expression of the fundamental solution to (3.1). We set

$$\mathcal{C}(t) = \int_0^t E(s)AE^T(s)ds, \quad t \in \mathbb{R}, \tag{3.2}$$

where $E(\cdot)$ is as in (1.11). It is known (see, e.g., [26]) that (H.1) and (H.2) are equivalent to the condition

$$\mathcal{C}(t) > 0, \quad \forall t > 0. \tag{3.3}$$

If (3.3) holds, then a fundamental solution to (3.1) is given by

$$\Gamma(x, t, \xi, \tau) = \Gamma\big(x - E(t-\tau)\xi, t-\tau\big), \tag{3.4}$$

where $\Gamma(x, t) = 0$ if $t \le 0$ and

$$\Gamma(x, t) = \frac{(4\pi)^{-N/2}}{\sqrt{\det \mathcal{C}(t)}} \exp\left(-\frac{1}{4}\langle \mathcal{C}^{-1}(t)x, x\rangle - t\,\mathrm{tr}(B)\right) \quad \text{if } t > 0. \tag{3.5}$$

We remark that $\Gamma(\cdot,\cdot)$ is a C^∞ function outside the diagonal of $\mathbb{R}^{N+1} \times \mathbb{R}^{N+1}$ and satisfies the usual properties (1.24) and (1.25) (with $c = 0$). If all the $*$-blocks in (1.3) are zero matrices, then Γ is also $D(\lambda)$-homogeneous:

$$\Gamma\big(D(\lambda)z\big) = \lambda^{-Q}\Gamma(z), \quad \forall z \in \mathbb{R}^{N+1} \setminus \{0\},\ \lambda > 0, \tag{3.6}$$

where

$$Q = p_0 + 3p_1 + \cdots + (2r+1)p_r \tag{3.7}$$

is the so-called *homogeneous dimension of* $\mathbb{R}^N$ with respect to the dilations group in $\mathbb{R}^N$:

$$D_0(\lambda) = \text{diag}\left(\lambda I_{p_0}, \lambda^3 I_{p_1}, \dots, \lambda^{2r+1} I_{p_r}\right). \tag{3.8}$$

Next we prove some estimates for the fundamental solution to constant coefficients KEs which generalize some result in [31, Section 2]. Given B in the form (1.3), we denote by B_0 the matrix obtained by substituting the $*$-blocks with null blocks and we set $E_0(t) = e^{-tB_0^T}$, $t \in \mathbb{R}$. Moreover, for $t \in \mathbb{R}$ and $\zeta \in \mathbb{R}^{N+1}$, we set

$$\mathcal{C}_\zeta(t) = \int_0^t E(s)A(\zeta)E^T(s)ds, \qquad \mathcal{C}_{\zeta,0}(t) = \int_0^t E_0(s)A(\zeta)E_0^T(s)ds. \tag{3.9}$$

In the following statements, we also denote by $\mathcal{C}$ the matrix in (3.2) with $A \equiv \left(\begin{smallmatrix} I_{p_0} & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and

$$\mathcal{C}_0(t) = \int_0^t E_0(s) \begin{pmatrix} I_{p_0} & 0 \\ 0 & 0 \end{pmatrix} E_0^T(s)ds, \tag{3.10}$$

where I_{p_0} denotes the identity matrix in $\mathbb{R}^{p_0}$. Hypothesis (1.2) yields an immediate comparison between the quadratic forms associated to $\mathcal{C}_\zeta$ and $\mathcal{C}$:

$$\mu^{-1}\mathcal{C}(t) \le \mathcal{C}_\zeta(t) \le \mu\mathcal{C}(t) \tag{3.11}$$

for any $t \in \mathbb{R}^+$ and $\zeta \in \mathbb{R}^{N+1}$. Since $\mathcal{C}_\zeta(t)$, $t > 0$, is symmetric and positive definite, analogous estimates hold for $\mathcal{C}_\zeta^{-1}$, $\mathcal{C}_{\zeta,0}$, and $\mathcal{C}_{\zeta,0}^{-1}$ in terms of $\mathcal{C}^{-1}$, $\mathcal{C}_0$, and $\mathcal{C}_0^{-1}$, respectively.

We now denote, respectively, by Γ^+ and Γ^- the fundamental solutions of the operators

$$L^+ = \mu\Delta_{\mathbb{R}^{p_0}} + \langle x, B\nabla\rangle - \partial_t, \qquad L^- = \frac{1}{\mu}\Delta_{\mathbb{R}^{p_0}} + \langle x, B\nabla\rangle - \partial_t. \tag{3.12}$$

Moreover, for fixed $w \in \mathbb{R}^{N+1}$, we denote by Z_w the fundamental solution to the frozen Kolmogorov operator

$$L_w = \sum_{i,j=1}^{p_0} a_{ij}(w)\partial_{x_i x_j} + \langle x, BDu\rangle - \partial_t. \tag{3.13}$$

An explicit expression of Γ^+, Γ^-, and Γ_w is given by (3.4) and (3.5).

Proposition 3.1. For every $z, \zeta, w \in \mathbb{R}^{N+1}$ with $z \neq \zeta$, it holds that

$$\frac{1}{\mu^N}\Gamma^-(z,\zeta) \le Z_w(z,\zeta) \le \mu^N\Gamma^+(z,\zeta). \tag{3.14}$$

□

Proof. We only prove the second inequality. We first note that, by (3.11), we have

$$\det \mathcal{C}_w(t) \geq \mu^{-N} \det \mathcal{C}(t), \quad \forall t > 0, \tag{3.15}$$

$$\exp\left(-\frac{1}{4}\langle \mathcal{C}_w^{-1}(t)\omega, \omega\rangle\right) \leq \exp\left(-\frac{1}{4\mu}\langle \mathcal{C}^{-1}(t)\omega, \omega\rangle\right) \quad \forall t > 0,\ \omega \in \mathbb{R}^N. \tag{3.16}$$

Given $z, \zeta \in \mathbb{R}^{N+1}$, for convenience, we set $s = t - \tau$, $\omega = x - E(s)\xi$ and $c_N = (4\pi)^{-N/2}$. Then we have

$$\begin{aligned} Z_w(z,\zeta) &= \frac{c_N e^{-s \operatorname{tr} B}}{\sqrt{\det \mathcal{C}_w(s)}} \exp\left(-\frac{1}{4}\langle \mathcal{C}_w^{-1}(s)\omega, \omega\rangle\right) \\ &\leq \mu^{N/2} \frac{c_N e^{-s \operatorname{tr} B}}{\sqrt{\det \mathcal{C}(s)}} \exp\left(-\frac{1}{4\mu}\langle \mathcal{C}^{-1}(s)\omega, \omega\rangle\right) \quad \text{(by (3.15) and (3.16))} \\ &= \mu^N \Gamma^+(z,\zeta). \end{aligned} \tag{3.17}$$

■

The next lemma provides an asymptotic comparison near 0 of $\mathcal{C}_\zeta$ and $\mathcal{C}_{\zeta,0}$.

Lemma 3.2. There exist two positive constants C_0 and t_0, only dependent on μ in (1.2) and the matrix B, such that

$$\big(1 - C_0 t\big)\mathcal{C}_{\zeta,0}(t) \leq \mathcal{C}_\zeta(t) \leq \big(1 + C_0 t\big)\mathcal{C}_{\zeta,0}(t) \tag{3.18}$$

for any $\zeta \in \mathbb{R}^{N+1}$ and $t \in [0, t_0]$. □

Lemma 3.2 can be proved following the arguments in [26], handling with care the dependence of the coefficients on ζ. The proof will be omitted.

Remark 3.3. As an immediate consequence of (3.11) and Lemma 3.2, for some positive t_1, we have

$$\frac{1}{2\mu}\mathcal{C}_0(t) \leq \frac{1}{2}\mathcal{C}_{\zeta,0}(t) \leq \mathcal{C}_\zeta(t) \leq 2\mathcal{C}_{\zeta,0}(t) \leq 2\mu\mathcal{C}_0(t), \tag{3.19}$$

$$(2\mu)^{-N} \det \mathcal{C}_0(1) \leq 2^{-N} \det \mathcal{C}_{\zeta,0}(1) \leq \frac{\det \mathcal{C}_\zeta(t)}{t^Q} \leq 2^N \det \mathcal{C}_{\zeta,0}(1) \leq (2\mu)^N \det \mathcal{C}_0(1), \tag{3.20}$$

for any $\zeta \in \mathbb{R}^{N+1}$ and $t \in [0, t_1]$. Analogous estimates also hold for $\mathcal{C}_\zeta^{-1}$.

Lemma 3.4. For every $T > 0$, there exists a positive constant C, only dependent on μ, B, and T, such that

$$\left|\left(\mathcal{C}_w^{-1}(t)y\right)_i\right| \le C\frac{\left|D_0\left(\frac{1}{\sqrt{t}}\right)y\right|}{\sqrt{t}}, \tag{3.21}$$

$$\left|\left(\mathcal{C}_w^{-1}(t)\right)_{ij}\right| \le \frac{C}{t}, \tag{3.22}$$

for every $i, j = 1, \dots, p_0$, $t \in]0, T]$, $w \in \mathbb{R}^{N+1}$, and $y \in \mathbb{R}^N$. □

Proof. We only show (3.21) since the proof of (3.22) is analogous. Let t_1 be as in Remark 3.3: we first consider the case $t \in]0, t_1]$. We recall that $(D_0(\lambda)y)_i = \lambda y_i$ for $i = 1, \dots, p_0$ and

$$\mathcal{C}_{w,0}^{-1}(t) = D_0\left(\frac{1}{\sqrt{t}}\right)\mathcal{C}_{w,0}^{-1}(1)D_0\left(\frac{1}{\sqrt{t}}\right), \tag{3.23}$$

see [26]. Then we have

$$\begin{aligned}
\left|\left(\mathcal{C}_w^{-1}(t)y\right)_i\right| &\le \left|\left((\mathcal{C}_w^{-1}(t) - \mathcal{C}_{w,0}^{-1}(t))y\right)_i\right| + \left|\left(\mathcal{C}_{w,0}^{-1}(t)y\right)_i\right| \\
&= \frac{1}{\sqrt{t}}\left|\left(D_0(\sqrt{t})\left(\mathcal{C}_w^{-1}(t) - \mathcal{C}_{w,0}^{-1}(t)\right)D_0(\sqrt{t})D_0\left(\frac{1}{\sqrt{t}}\right)y\right)_i\right| \\
&\quad + \frac{1}{\sqrt{t}}\left|\left(\mathcal{C}_{w,0}^{-1}(1)D_0\left(\frac{1}{\sqrt{t}}\right)y\right)_i\right| \\
&\equiv I_1 + I_2.
\end{aligned} \tag{3.24}$$

In order to estimate I_1, we note that

$$\begin{aligned}
&\left\|D_0(\sqrt{t})\left(\mathcal{C}_w^{-1}(t) - \mathcal{C}_{w,0}^{-1}(t)\right)D_0(\sqrt{t})\right\| \\
&\quad = \sup_{|\xi|=1}\left|\left\langle\left(\mathcal{C}_w^{-1}(t) - \mathcal{C}_{w,0}^{-1}(t)\right)D_0(\sqrt{t})\xi, D_0(\sqrt{t})\xi\right\rangle\right| \\
&\quad \le \sup_{|\xi|=1}\left|\left\langle\mathcal{C}_{w,0}^{-1}(t)D_0(\sqrt{t})\xi, D_0(\sqrt{t})\xi\right\rangle\right| \quad \text{(by Remark 3.3 since } 0 < t \le t_1) \\
&\quad = \sup_{|\xi|=1}\left|\left\langle\mathcal{C}_{w,0}^{-1}(1)\xi, \xi\right\rangle\right| \le \mu\left\|\mathcal{C}_0^{-1}(1)\right\| \quad \text{(by (3.23) and Remark 3.3).}
\end{aligned} \tag{3.25}$$

Hence, we infer

$$I_1 \le \frac{\mu}{\sqrt{t}}\left\|\mathcal{C}_0^{-1}(1)\right\|\left|D_0\left(\frac{1}{\sqrt{t}}\right)y\right|. \tag{3.26}$$

On the other hand, again by Remark 3.3, we have

$$I_2 \le \frac{\left\|\mathcal{C}^{-1}_{w,0}(1)\right\|}{\sqrt{t}}\left|D_0\left(\frac{1}{\sqrt{t}}\right)y\right| \le \frac{\mu}{\sqrt{t}}\left\|\mathcal{C}^{-1}_{0}(1)\right\|\left|D_0\left(\frac{1}{\sqrt{t}}\right)y\right|. \tag{3.27}$$

The proof of the case $t \in [t_1, T]$ is easier:

$$\begin{aligned}\left|\left(\mathcal{C}^{-1}_{w}(t)y\right)_i\right| &= \frac{1}{\sqrt{t}}\left|\left(D_0(\sqrt{t})\mathcal{C}^{-1}_{w}(t)D_0(\sqrt{t})D_0\left(\frac{1}{\sqrt{t}}\right)y\right)_i\right| \\ &\le \frac{\mu}{\sqrt{t}}\sup_{t_0\le t\le T}\left\|D_0(\sqrt{t})\mathcal{C}^{-1}(t)D_0(\sqrt{t})\right\|\left|D_0\left(\frac{1}{\sqrt{t}}\right)y\right| \quad (\text{by } (3.11)).\end{aligned} \tag{3.28}$$

■

In the next statement, $Z(z,\zeta)$ denotes the parametrix of L, that is, the fundamental solution, with pole at ζ, to the constant coefficients Kolmogorov operator

$$L_\zeta = \sum_{i,j=1}^{p_0} a_{ij}(\zeta)\partial_{x_i x_j} + \langle x, B\nabla\rangle - \partial_t. \tag{3.29}$$

Moreover, Γ^ε, $\varepsilon > 0$, denotes the fundamental solution to the constant coefficients KE (1.26).

Proposition 3.5. Given $\varepsilon > 0$ and a polynomial function p, there exists a constant C, only dependent on ε, μ, B, and p, such that, if we set $\eta = |D_0((t-\tau)^{-1/2})(x - E(t-\tau)\xi)|$, then we have

$$\left|p(\eta)\right| Z_w(z,\zeta) \le C\Gamma^\varepsilon(z,\zeta), \tag{3.30}$$

for any $z, \zeta, w \in \mathbb{R}^{N+1}$. □

Proof. For convenience, we set $s = t-\tau$ and $\omega = x - E(s)\xi$. By Lemma 3.2, we may consider $t_0 > 0$ such that (3.18) holds and

$$\left(1 - C_0 t_0\right)^2 \ge \frac{\mu + \frac{\varepsilon}{2}}{\mu+\varepsilon}, \tag{3.31}$$

where C_0 is the constant in (3.18). We first prove (3.30) for $s \in [0, t_0]$. Then, by (3.11), we have

$$\begin{aligned}
\left|p(|\eta|)\right| Z_w(z,\zeta) &\leq \frac{c_N \mu^{N/2} e^{-s\,\mathrm{tr}\,B}}{\sqrt{\det \mathcal{C}(s)}} \left|p(|\eta|)\right| \exp\left(-\frac{1}{4}\left\langle \mathcal{C}_w^{-1}(s)\omega,\omega\right\rangle\right) \\
&\leq \frac{c_N \mu^{N/2} e^{-s\,\mathrm{tr}\,B}}{\sqrt{\det \mathcal{C}(s)}} \left|p(|\eta|)\right| \exp\left(-\frac{(1-C_0 t_0)}{4\mu}\left\langle \mathcal{C}_0^{-1}(1)\eta,\eta\right\rangle\right) \\
&\qquad\qquad \text{(by Lemma 3.2 and (3.11))} \\
&\leq \frac{C_1 e^{-s\,\mathrm{tr}\,B}}{\sqrt{\det \mathcal{C}(s)}} \exp\left(-\frac{(1-C_0 t_0)}{4\left(\mu+\frac{\varepsilon}{2}\right)}\left\langle \mathcal{C}_0^{-1}(1)\eta,\eta\right\rangle\right) \\
&\leq \frac{C_1 e^{-s\,\mathrm{tr}\,B}}{\sqrt{\det \mathcal{C}(s)}} \exp\left(-\frac{(1-C_0 t_0)^2}{4\left(\mu+\frac{\varepsilon}{2}\right)}\left\langle \mathcal{C}^{-1}(s)\omega,\omega\right\rangle\right) \\
&\qquad\qquad \text{(by Lemma 3.2 applied to the matrix } \mathcal{C}) \\
&\leq C\Gamma^{\varepsilon}(z,\zeta) \quad \text{(by (3.31))}.
\end{aligned} \tag{3.32}$$

We next consider $s \geq t_0$. In this case, by Proposition 3.1, we have

$$\left|p(|\eta|)\right| Z(z,\zeta) \leq C_1 \left|p(|\omega|)\right| \Gamma^{+}(z,\zeta) \tag{3.33}$$

and the thesis follows by a standard argument. ■

Next we prove some estimates for the derivatives of $Z_w(z,\zeta)$.

Proposition 3.6. For every $\varepsilon > 0$ and $T > 0$, there exists a positive constant C, only dependent on μ, B, ε, and T, such that

$$\left|\partial_{x_i} Z_w(z,\zeta)\right| \leq \frac{C}{\sqrt{t-\tau}}\Gamma^{\varepsilon}(z,\zeta), \qquad \left|\partial_{x_i x_j} Z_w(z,\zeta)\right| \leq \frac{C}{t-\tau}\Gamma^{\varepsilon}(z,\zeta), \tag{3.34}$$

for every $z, \zeta, w \in \mathbb{R}^{N+1}$ such that $0 < t-\tau < T$ and every $i, j = 1, \ldots, p_0$. □

Proof. We put again $s = t - \tau$ and $\omega = x - E(s)\xi$. Then, for $i = 1, \ldots, p_0$, we have

$$\begin{aligned}
\left|\partial_{x_i} Z_w(z,\zeta)\right| &= \frac{1}{2}\left|\left(\mathcal{C}_w^{-1}(s)\omega\right)_i\right| Z_w(z,\zeta) \\
&\leq \frac{C}{\sqrt{s}}\left|D_0\left(\frac{1}{\sqrt{s}}\right)\omega\right| Z_w(z,\zeta) \quad \text{(by (3.21))}
\end{aligned} \tag{3.35}$$

and the first estimate follows by Proposition 3.5. The proof of the second estimate is analogous. ■

4 The parametrix method

In this section, we describe the Levi parametrix method to construct a fundamental solution Γ for the KE (1.1). Throughout this section, we assume that L in (1.1) verifies hypotheses (H.1) and (H.2) and that the coefficients $a_{ij}, a_i, c \in C_B^\alpha(\mathbb{R}^{N+1})$ are bounded functions. We remind that Z_w denotes the fundamental solution to the "frozen" Kolmogorov operator

$$L_w = \sum_{i,j=1}^{p_0} a_{ij}(w)\partial_{x_i x_j} + \langle x, BDu\rangle - \partial_t, \tag{4.1}$$

and $Z(z,\zeta) = Z_\zeta(z,\zeta)$ is the so-called parametrix. Hereafter $z = (x,t)$ and $\zeta = (\xi,\tau)$. According to Levi's method, we look for the fundamental solution Γ in the form

$$\Gamma(z,\zeta) = Z(z,\zeta) + J(z,\zeta). \tag{4.2}$$

The function J is unknown and supposed to be of the form

$$J(z,\zeta) = \int_{S_{\tau,t}} Z(z,w)\Phi(w,\zeta)dw, \quad S_{\tau,t} = \mathbb{R}^N \times]\tau,t[, \tag{4.3}$$

where Φ has to be determined by imposing that Γ is solution to L:

$$0 = L\Gamma(z,\zeta) = LZ(z,\zeta) + LJ(z,\zeta), \quad z \neq \zeta. \tag{4.4}$$

Assuming that J can be differentiated under the integral sign, we get

$$LJ(z,\zeta) = \int_{S_{\tau,t}} LZ(z,w)\Phi(w,\zeta)dw - \Phi(z,\zeta), \tag{4.5}$$

hence (4.4) yields

$$\Phi(z,\zeta) = LZ(z,\zeta) + \int_{S_{\tau,t}} LZ(z,w)\Phi(w,\zeta)dw. \tag{4.6}$$

Thus we obtain an integral equation whose solution Φ can be determined by the successive approximation method:

$$\Phi(z,\zeta) = \sum_{k=1}^{+\infty} (LZ)_k(z,\zeta), \tag{4.7}$$

where

$$\begin{aligned}(LZ)_1(z,\zeta) &= LZ(z,\zeta),\\ (LZ)_{k+1}(z,\zeta) &= \int_{S_{\tau,t}} LZ(z,w)(LZ)_k(w,\zeta)dw.\end{aligned} \tag{4.8}$$

The previous arguments are made rigorous by the following propositions.

Proposition 4.1. There exists $k_0 \in \mathbb{N}$ such that, for every $T > 0$ and $\zeta \in \mathbb{R}^{N+1}$, the series

$$\sum_{k=k_0}^{+\infty} (LZ)_k(\cdot,\zeta) \tag{4.9}$$

converges uniformly in the strip $S_{\tau,T} \equiv \{(x,t) \in \mathbb{R}^{N+1} \mid \tau < t < T\}$. Moreover, the function $\Phi(\cdot,\zeta)$ defined by (4.7) solves the integral equation (4.6) in $S_{\tau,T}$ and satisfies the following estimate: for any $\varepsilon > 0$, there exists a positive constant C such that

$$\left|\Phi(z,\zeta)\right| \le C\frac{\Gamma^{\varepsilon}(z,\zeta)}{(t-\tau)^{1-\alpha/2}}, \quad \forall z \in S_{\tau,T}. \tag{4.10}$$

□

Proposition 4.2. For every $\zeta \in \mathbb{R}^{N+1}$, the function $J(\cdot,\zeta)$ defined by (4.3) solves (4.5) in $\mathbb{R}^{N+1} \setminus \{\zeta\}$ in the sense of Definition 1.3. □

The remainder of this section is devoted to the proof of Proposition 4.1. The proof of Proposition 4.2 is more involved since it requires the study of some singular integrals which will be made in the next section. Then Proposition 4.2 will be a direct consequence of the results in Section 5 and Lemma 6.1.

Lemma 4.3. For every $\varepsilon > 0$ and $T > 0$, there exists a positive constant C, only dependent on ε, T, μ, and B, such that

$$\left|(LZ)_k(z,\zeta)\right| \le \frac{M_k}{(t-\tau)^{1-\alpha k/2}}\Gamma^{\varepsilon}(z,\zeta), \tag{4.11}$$

for any $k \in \mathbb{N}$ and $z,\zeta \in \mathbb{R}^{N+1}$ with $0 < t-\tau \le T$, where

$$M_k = C^k \frac{\Gamma_E^k\left(\frac{\alpha}{2}\right)}{\Gamma_E\left(\frac{\alpha k}{2}\right)}, \tag{4.12}$$

and Γ_E the Euler Gamma function. As a consequence, there exists $k_0 \in \mathbb{N}$ such that the function $(LZ)_k(\cdot,\zeta)$ is bounded for $k \ge k_0$ in $S_{\tau,T}$. □

Proof. We use the notations of Proposition 3.5 and we prove estimate (4.11) by an inductive argument. For $z \neq \zeta$, we have

$$\left|LZ(z,\zeta)\right| \leq \left|\sum_{i,j=1}^{p_0} \left(a_{ij}(z) - a_{ij}(\zeta)\right)\partial_{x_i x_j} Z(z,\zeta)\right| + \left|\sum_{i=1}^{p_0} a_i(z)\partial_{x_i} Z(z,\zeta)\right| + \left|c(z)\right|Z(z,\zeta). \tag{4.13}$$

By assumption $a_{ij} \in C^{\alpha}_{B}(\mathbb{R}^{N+1})$ so that

$$\left|a_{ij}(z) - a_{ij}(\zeta)\right| \leq C_1 \left\|\zeta^{-1} \circ z\right\|_B^{\alpha} = C_1 (t-\tau)^{\alpha/2} \left\|(\eta, 1)\right\|_B^{\alpha}. \tag{4.14}$$

Hence, by Proposition 3.6, we infer

$$\left|\sum_{i,j=1}^{p_0} \left(a_{ij}(z) - a_{ij}(\zeta)\right)\partial_{x_i x_j} Z(z,\zeta)\right| \leq C_2 \left\|(\eta,1)\right\|_B^{\alpha} \frac{\Gamma^{\varepsilon/2}(z,\zeta)}{(t-\tau)^{1-\alpha/2}}, \tag{4.15}$$

and, since the coefficients are bounded functions,

$$\left|\sum_{i=1}^{p_0} a_i(z)\partial_{x_i} Z(z,\zeta)\right| \leq C_3 \frac{\Gamma^{\varepsilon/2}(z,\zeta)}{\sqrt{t-\tau}}. \tag{4.16}$$

By Proposition 3.1, we have

$$\left|c(z)Z(z,\zeta)\right| \leq C_4 \Gamma^{\varepsilon}(z,\zeta). \tag{4.17}$$

Therefore, (4.11) for $k = 1$ easily follows from the above estimates and Proposition 3.5.

We now assume that (4.11) holds for k and prove it for $k+1$. We have

$$\begin{aligned}
&\left|(LZ)_{k+1}(z,\zeta)\right| \\
&\quad = \left|\int_{S_{\tau,t}} LZ(z,w)(LZ)_k(w,\zeta)\,dw\right| \\
&\quad \leq \int_{\tau}^{t} \frac{M_1}{(t-s)^{1-\alpha/2}} \frac{M_k}{(s-\tau)^{1-k\alpha/2}} \int_{\mathbb{R}^N} \Gamma^{\varepsilon}(x,t,y,s)\Gamma^{\varepsilon}(y,s,\xi,\tau)\,dy\,ds \\
&\qquad\qquad \text{(by the inductive hypothesis and denoting } (y,s) = w) \\
&\quad = \Gamma^{\varepsilon}(z,\zeta) \int_{\tau}^{t} \frac{M_1}{(t-s)^{1-\alpha/2}} \frac{M_k}{(s-\tau)^{1-k\alpha/2}}\,ds \\
&\qquad\qquad \text{(by the reproduction property (1.24) for } \Gamma^{\varepsilon}),
\end{aligned} \tag{4.18}$$

and the thesis follows by the well-known properties of the Euler Gamma function.

The boundedness of $(LZ)_k$, for $k \geq k_0$ suitably large, directly follows from (4.11) and the explicit expression of Γ^ε. Indeed, by (3.20) of Remark 3.3, we have

$$\left|(LZ)_k(z,\zeta)\right| \leq CM_k(t-\tau)^{k-(Q+2)/\alpha}, \tag{4.19}$$

for some constant C. Then it suffices that $k_0 \geq (Q+2)/\alpha$. ■

Proof of Proposition 4.1. The convergence of the series (4.9) follows from the previous lemma (cf. (4.19)). Indeed the power series

$$\sum_{k\geq 1} M_{k_0+k} s^k \tag{4.20}$$

with M_k as in (4.12) has radius of convergence equal to infinity.

Then, proceeding as in Lemma 4.3, it is straightforward to prove that Φ verifies estimate (4.10) and solves (4.6). ■

Corollary 4.4. For every $\varepsilon > 0$ and $T > 0$, there exists a positive constant C, only dependent on ε, T, μ, and B, such that

$$\left|J(z,\zeta)\right| \leq C(t-\tau)^{\alpha/2}\Gamma^\varepsilon(z,\zeta), \tag{4.21}$$

and the fundamental solution Γ in (4.2) verifies estimate (1.28):

$$\Gamma(z,\zeta) \leq C\Gamma^\varepsilon(z,\zeta), \tag{4.22}$$

for any $z, \zeta \in \mathbb{R}^{N+1}$ with $0 < t-\tau \leq T$. □

Proof. We have

$$\begin{aligned}\left|J(z,\zeta)\right| &\leq \int_{S_{\tau,t}} Z(z,w)\left|\Phi(w,\zeta)\right|dw \\ &\leq C\int_\tau^t\int_{\mathbb{R}^N}\Gamma^\varepsilon(x,t,y,s)\frac{\Gamma^\varepsilon(y,s,\xi,\tau)}{(s-\tau)^{1-\alpha/2}}dy\,ds \quad (\text{by } (4.10)) \\ &= C\Gamma^\varepsilon(z,\zeta)\int_\tau^t\frac{ds}{(s-\tau)^{1-\alpha/2}} \quad (\text{by the reproduction property of } \Gamma^\varepsilon),\end{aligned} \tag{4.23}$$

and (4.21) follows. The estimate of Γ is a direct consequence of (4.21) and the estimate of Z in Proposition 3.1. ■

5 Potential estimates

We consider the potential

$$V_f(z) = \int_{S_{T_0,t}} Z(z,\zeta) f(\zeta) d\zeta, \quad S_{T_0,t} = \mathbb{R}^N \times \left]T_0, t\right[, \tag{5.1}$$

where $f \in C(S_{T_0,T_1})$ satisfies the growth estimate (1.20):

$$\left|f(x,t)\right| \le C_1 e^{C_1|x|^2}, \quad \forall (x,t) \in S_{T_0,T_1}, \tag{5.2}$$

and Z is the parametrix of (1.1). In this section, we aim to study the regularity properties of V_f by adapting the arguments used by Polidoro [31].

We first show that the integral in (5.1) is convergent in the strip $S_{T_0,T}$ for some $T \in \left]T_0, T_1\right]$. Indeed, by **Proposition 3.1**, we have

$$\begin{aligned} \left|V_f(x,t)\right| &\le C_2 \int_{T_0}^t \int_{\mathbb{R}^N} \Gamma^+(x,t,\xi,\tau) e^{C_1|\xi|^2} d\xi\, d\tau \\ &\le C_3 \int_{T_0}^t \int_{\mathbb{R}^N} \frac{1}{\sqrt{\det \mathcal{C}(s)}} \exp\left(-\frac{1}{4\mu}\left\langle \mathcal{C}^{-1}(s)\omega, \omega \right\rangle + C_1|\xi|^2 \right) d\xi\, d\tau \\ &\qquad\qquad \text{(denoting } s = t-\tau \text{ and } \omega = x - E(s)\xi) \\ &\le C_4 \int_{T_0}^t \int_{\mathbb{R}^N} \exp\left(-\frac{|\eta|^2}{4\mu} + C_1\left|E(-s)\left(x - \mathcal{C}^{1/2}(s)\eta\right)\right|^2 \right) d\eta\, d\tau \\ &\le C\left(t - T_0\right) e^{C|x|^2} \quad \text{(by the change of variables } \eta = \mathcal{C}^{-1/2}(s)\omega), \end{aligned} \tag{5.3}$$

for some positive constant C, assuming that $t \in \left]T_0, T\right]$ with $T - T_0$ suitably small and using the fact that $\|\mathcal{C}(s)\|$ tends to zero as $s \to 0$.

Proposition 5.1. There exist $\partial_{x_i} V_f \in C(S_{T_0,T})$ for $i = 1, \dots, p_0$ and it holds that

$$\partial_{x_i} V_f(x,t) = \int_{T_0}^t \int_{\mathbb{R}^N} \partial_{x_i} Z(x,t,\xi,\tau) f(\xi,\tau) d\xi\, d\tau. \tag{5.4}$$

□

Proof. By **Proposition 3.6** and the above argument, the integral in (5.4) is absolutely convergent and

$$\int_{T_0}^t \int_{\mathbb{R}^N} \left|\partial_{x_i} Z(x,t,\xi,\tau) f(\xi,\tau)\right| d\xi\, d\tau \le C\sqrt{t - T_0}\, e^{C|x|^2}. \tag{5.5}$$

Next we set

$$V_{f,\delta}(x,t) = \int_{T_0}^{t-\delta} \int_{\mathbb{R}^N} Z(x,t,\xi,\tau) f(\xi,\tau) d\xi\, d\tau, \quad 0 < \delta < t - T_0. \tag{5.6}$$

By Lebesgue's theorem, we have

$$\lim_{\delta\to 0^+} V_{f,\delta}(x,t) = V_f(x,t), \tag{5.7}$$

$$\partial_{x_i} V_{f,\delta}(x,t) = \int_{T_0}^{t-\delta}\int_{\mathbb{R}^N} \partial_{x_i} Z(x,t,\xi,\tau) f(\xi,\tau)\, d\xi\, d\tau. \tag{5.8}$$

In order to prove (5.4), it suffices to verify that

$$\lim_{\delta\to 0^+} \partial_{x_i} V_{f,\delta}(x,t) = \int_{T_0}^{t}\int_{\mathbb{R}^N} \partial_{x_i} Z(x,t,\xi,\tau) f(\xi,\tau)\, d\xi\, d\tau, \tag{5.9}$$

uniformly on $B_{R_1} \times]T_0, T]$. This is an easy consequence of (5.8) and (5.5), indeed we have

$$\begin{aligned} &\partial_{x_i} V_{f,\delta}(x,t) - \int_{T_0}^{t}\int_{\mathbb{R}^N} \partial_{x_i} Z(x,t,\xi,\tau) f(\xi,\tau)\, d\xi\, d\tau \\ &\quad = \int_{t-\delta}^{t}\int_{\mathbb{R}^N} \partial_{x_i} Z(x,t,\xi,\tau) f(\xi,\tau)\, d\xi\, d\tau \le C\sqrt{\delta} e^{C|x|^2}. \end{aligned} \tag{5.10}$$

■

Lemma 5.2. For every positive ε and T, there exists a constant $C > 0$ such that

$$\begin{aligned} &\left|Z_\zeta(z,\zeta) - Z_w(z,\zeta)\right| \le C\left\|\zeta^{-1}\circ w\right\|_B^\alpha \Gamma^\varepsilon(z,\zeta), \\ &\left|\partial_{x_i} Z_\zeta(z,\zeta) - \partial_{x_i} Z_w(z,\zeta)\right| \le C\frac{\left\|\zeta^{-1}\circ w\right\|_B^\alpha}{\sqrt{t-\tau}} \Gamma^\varepsilon(z,\zeta), \\ &\left|\partial_{x_i x_j} Z_\zeta(z,\zeta) - \partial_{x_i x_j} Z_w(z,\zeta)\right| \le C\frac{\left\|\zeta^{-1}\circ w\right\|_B^\alpha}{t-\tau} \Gamma^\varepsilon(z,\zeta), \end{aligned} \tag{5.11}$$

for any $i, j = 1, \dots, p_0$ and $z, \zeta, w \in \mathbb{R}^{N+1}$ with $0 < t - \tau \le T$. □

Proof. We only prove the third estimate. We use the usual notations $s = t - \tau$, $\omega = x - E(s)\xi$, $\eta = D_0(1/\sqrt{s})\omega$ and first note that

$$\partial_{x_i x_j} Z_w(z,\zeta) = \frac{Ce^{-s\,\mathrm{tr}\,B}}{\sqrt{\det \mathcal{C}_w(s)}} e^{-(1/4)\langle \mathcal{C}_w^{-1}(s)\omega,\omega\rangle} \Big(\big(\mathcal{C}_w^{-1}(s)\big)_{ij} + \big(\mathcal{C}_w^{-1}(s)\omega\big)_i \big(\mathcal{C}_w^{-1}(s)\omega\big)_j \Big). \tag{5.12}$$

Then the thesis follows from the following estimates:

$$\left|\frac{1}{\sqrt{\det \mathcal{C}_\zeta(s)}} - \frac{1}{\sqrt{\det \mathcal{C}_w(s)}}\right| \le C\frac{\|\zeta^{-1}\circ w\|_B^\alpha}{\sqrt{\det \mathcal{C}_\zeta(s)}}, \tag{5.13}$$

$$\left|e^{-(1/4)\langle \mathcal{C}_\zeta^{-1}(s)\omega,\omega\rangle} - e^{-(1/4)\langle \mathcal{C}_w^{-1}(s)\omega,\omega\rangle}\right| \le C\|\zeta^{-1}\circ w\|_B^\alpha e^{-(1/4(\mu+\varepsilon))\langle \mathcal{C}^{-1}(s)\omega,\omega\rangle}, \tag{5.14}$$

$$\left|\left(\mathcal{C}_\zeta^{-1}(s)\right)_{ij} - \left(\mathcal{C}_w^{-1}(s)\right)_{ij}\right| \le \frac{C}{s}\|\zeta^{-1}\circ w\|_B^\alpha, \tag{5.15}$$

$$\left|\left(\mathcal{C}_\zeta^{-1}(s)\omega\right)_i\left(\mathcal{C}_\zeta^{-1}(s)\omega\right)_j - \left(\mathcal{C}_w^{-1}(s)\omega\right)_i\left(\mathcal{C}_w^{-1}(s)\omega\right)_j\right| \le \frac{C}{s}\|\zeta^{-1}\circ w\|_B^\alpha|\eta|^2, \tag{5.16}$$

where $\mathcal{C}$ denotes the matrix in (3.2) with $A \equiv \left(\begin{smallmatrix} I_{p_0} & 0 \\ 0 & 0\end{smallmatrix}\right)$.

By Remark 3.3, (5.13) is equivalent to

$$\begin{aligned}
&\frac{\left|\det \mathcal{C}_\zeta(s) - \det \mathcal{C}_w(s)\right|}{s^Q} \\
&\quad \le C\left|\det\left(D_0\left(\frac{1}{\sqrt{s}}\right)\mathcal{C}_\zeta(s)D_0\left(\frac{1}{\sqrt{s}}\right)\right) - \det\left(D_0\left(\frac{1}{\sqrt{s}}\right)\mathcal{C}_w(s)D_0\left(\frac{1}{\sqrt{s}}\right)\right)\right| \\
&\quad \le C\|\zeta^{-1}\circ w\|_B^\alpha.
\end{aligned} \tag{5.17}$$

A general result from linear algebra states that

$$\left|\det M_1 - \det M_2\right| \le C\|M_1 - M_2\|, \tag{5.18}$$

where the constant C only depends on the dimension of the matrices M_1, M_2 and on $\|M_1\|$, $\|M_2\|$. Then (5.17) follows from the estimate

$$\sup_{|\xi|=1}\left|\left\langle\left(\mathcal{C}_\zeta(s) - \mathcal{C}_w(s)\right)D_0\left(\frac{1}{\sqrt{s}}\right)\xi, D_0\left(\frac{1}{\sqrt{s}}\right)\xi\right\rangle\right| \le C\|\zeta^{-1}\circ w\|_B^\alpha\|\mathcal{C}(1)\|. \tag{5.19}$$

This concludes the proof of (5.13). Next we consider (5.14). An elementary inequality yields

$$\begin{aligned}
&\left|e^{-(1/4)\langle \mathcal{C}_\zeta^{-1}(s)\omega,\omega\rangle} - e^{-(1/4)\langle \mathcal{C}_w^{-1}(s)\omega,\omega\rangle}\right| \\
&\quad \le \left|\left\langle\left(\mathcal{C}_\zeta^{-1}(s) - \mathcal{C}_w^{-1}(s)\right)\omega,\omega\right\rangle\right|e^{-(1/4\mu)\langle \mathcal{C}^{-1}(s)\omega,\omega\rangle} \\
&\quad \le \left\|D_0(\sqrt{s})\left(\mathcal{C}_\zeta^{-1}(s) - \mathcal{C}_w^{-1}(s)\right)D_0(\sqrt{s})\right\||\eta|^2 e^{-(1/4\mu)\langle \mathcal{C}^{-1}(s)\omega,\omega\rangle} \\
&\quad \le C\left\|D_0(\sqrt{s})\left(\mathcal{C}_\zeta^{-1}(s) - \mathcal{C}_w^{-1}(s)\right)D_0(\sqrt{s})\right\|e^{-(1/4(\mu+\varepsilon))\langle \mathcal{C}^{-1}(s)\omega,\omega\rangle}.
\end{aligned} \tag{5.20}$$

On the other hand,

$$\begin{aligned}
&\left\|D_0(\sqrt{s})\left(\mathcal{C}_\zeta^{-1}(s)-\mathcal{C}_w^{-1}(s)\right)D_0(\sqrt{s})\right\| \\
&\qquad\le \left\|D_0(\sqrt{s})\mathcal{C}_\zeta^{-1}(s)D_0(\sqrt{s})\right\|\left\|D_0\left(\frac{1}{\sqrt{s}}\right)\left(\mathcal{C}_\omega(s)-\mathcal{C}_\zeta(s)\right)D_0\left(\frac{1}{\sqrt{s}}\right)\right\| \\
&\qquad\quad\cdot\left\|D_0(\sqrt{s})\mathcal{C}_w^{-1}(s)D_0(\sqrt{s})\right\| \\
&\qquad\le C\left\|\zeta^{-1}\circ w\right\|_B^{\alpha},
\end{aligned} \tag{5.21}$$

and this proves (5.14). We omit the proofs of (5.15) and (5.16) which are analogous. ■

Proposition 5.3. Under the hypotheses of Theorem 1.4, there exist $\partial_{x_i x_j} V_f \in C(S_{T_0,T})$ for $i,j = 1,\dots,p_0$, and it holds that

$$\partial_{x_i x_j} V_f(x,t) = \int_{T_0}^{t}\int_{\mathbb{R}^N} \partial_{x_i x_j} Z(x,t,\xi,\tau) f(\xi,\tau)\, d\xi\, d\tau. \tag{5.22}$$

□

Proof. We first show that the integral in (5.22) exists. Fixed $R > 0$, we consider $x \in \mathbb{R}^N$ such that $|x| < R$ and denote by B_R the Euclidean ball in $\mathbb{R}^N$ centered at the origin. For a suitable $R_1 > R$ to be determined later, we split the integral in (5.22) as follows:

$$\begin{aligned}
&\int_{T_0}^{t}\int_{\mathbb{R}^N} \partial_{x_i x_j} Z(x,t,\xi,\tau) f(\xi,\tau)\, d\xi\, d\tau \\
&\quad= \int_{T_0}^{t}\int_{B_{R_1}} \partial_{x_i x_j} Z(x,t,\xi,\tau) f(\xi,\tau)\, d\xi\, d\tau + \int_{T_0}^{t}\int_{\mathbb{R}^N\setminus B_{R_1}} \partial_{x_i x_j} Z(x,t,\xi,\tau) f(\xi,\tau)\, d\xi\, d\tau \\
&\quad\equiv K_1 + K_2.
\end{aligned} \tag{5.23}$$

We consider K_1. For every $\tau \in]T_0, t[$ and $y \in \mathbb{R}^N$, denoting $w = (y,\tau)$, we have

$$\begin{aligned}
&\int_{B_{R_1}} \partial_{x_i x_j} Z(x,t,\xi,\tau) f(\xi,\tau)\, d\xi \\
&\quad= \int_{B_{R_1}} \partial_{x_i x_j} Z(x,t,\xi,\tau)\left(f(\xi,\tau)-f(y,\tau)\right) d\xi \\
&\qquad+ f(y,\tau)\int_{B_{R_1}} \partial_{x_i x_j}\left(Z(x,t,\xi,\tau)-Z_w(x,t,\xi,\tau)\right) d\xi \\
&\qquad+ f(y,\tau)\int_{B_{R_1}} \partial_{x_i x_j} Z_w(x,t,\xi,\tau)\, d\xi \\
&\quad= I_1 + I_2 + I_3.
\end{aligned} \tag{5.24}$$

We put $y = E(\tau - t)x$ and by Proposition 3.6 and the regularity properties of f, we get

$$\left|I_1\right| \le C \int_{\mathbb{R}^N} \frac{\Gamma^\varepsilon(x,t,\xi,\tau)}{t-\tau} \left|\xi - E(\tau - t)x\right|_B^\beta d\xi \le C \int_{\mathbb{R}^N} \frac{\Gamma^\varepsilon(x,t,\xi,\tau)}{(t-\tau)^{1-\beta/2}} |\eta|_B^\beta d\xi, \tag{5.25}$$

since

$$\left|\xi - E(\tau - t)x\right|_B \le C\sqrt{t-\tau}|\eta|_B, \tag{5.26}$$

for some constant C, where $\eta = D_0(1/\sqrt{t-\tau})(x - E(t-\tau)\xi)$. Now, by Proposition 3.5, we have

$$|\eta|^\beta \Gamma^\varepsilon(x,t,\xi,\tau) \le C\Gamma^{2\varepsilon}(x,t,\xi,\tau), \tag{5.27}$$

and since

$$\int_{\mathbb{R}^N} \Gamma^{2\varepsilon}(x,t,\xi,\tau) d\xi = 1, \quad t > \tau, \tag{5.28}$$

we finally deduce

$$\left|I_1\right| \le \frac{C}{(t-\tau)^{1-\beta/2}}. \tag{5.29}$$

Next we consider I_2. By Lemma 5.2 and the growth estimate (1.20), we have

$$\begin{aligned}
\left|I_2\right| &\le C_1 \left|f(y,\tau)\right| \int_{B_{R_1}} \frac{\Gamma^\varepsilon(x,t,\xi,\tau)}{t-\tau} |\xi - y|_B^\alpha d\xi \\
&\le C_2 e^{C_2|x|^2} \int_{\mathbb{R}^N} \frac{\Gamma^\varepsilon(x,t,\xi,\tau)}{t-\tau} \left|\xi - E(\tau - t)x\right|_B^\alpha d\xi \\
&\le \frac{C}{(t-\tau)^{1-\alpha/2}} \quad \text{(by the previous argument).}
\end{aligned} \tag{5.30}$$

We now consider I_3. We first remark that we have

$$\begin{aligned}
\partial_{x_i} Z_w(x,t,\xi,\tau) &= -\frac{1}{2} Z_w(x,t,\xi,\tau) \big(\mathcal{C}_w^{-1}(t-\tau)\big(x - E(t-\tau)\xi\big)\big)_i, \\
\partial_{\xi_i} Z_w(x,t,\xi,\tau) &= \frac{1}{2} Z_w(x,t,\xi,\tau) \sum_{j=1}^N \big(\mathcal{C}_w^{-1}(t-\tau)\big(x - E(t-\tau)\xi\big)\big)_j E_{ji}(t-\tau) \\
&= -\sum_{j=1}^N \partial_{x_j} Z_w(x,t,\xi,\tau) E_{ji}(t-\tau).
\end{aligned} \tag{5.31}$$

Thus it holds that

$$\nabla_\xi Z_w(x,t,\xi,\tau) = -\nabla_x Z_w(x,t,\xi,\tau)E(t-\tau) \tag{5.32}$$

and inverting the matrix E, we finally get

$$\nabla_x Z_w(x,t,\xi,\tau) = -\nabla_\xi Z_w(x,t,\xi,\tau)E(\tau-t). \tag{5.33}$$

Therefore, we have

$$\begin{aligned}
&\int_{B_{R_1}} \partial_{x_i x_j} Z_w(x,t,\xi,\tau)d\xi \\
&\quad = -\sum_{k=1}^{N}\int_{B_{R_1}} \partial_{x_i \xi_k} Z_w(x,t,\xi,\tau)E_{kj}(\tau-t)d\xi \\
&\quad = -\sum_{k=1}^{N}\int_{\partial B_{R_1}} \partial_{x_i} Z_w(x,t,\xi,\tau)\nu_k E_{kj}(\tau-t)d\sigma(\xi) \\
&\qquad \text{(by the divergence theorem and denoting by } \nu \text{ the outer normal to } B_{R_1}\text{);}
\end{aligned} \tag{5.34}$$

thus, by Proposition 3.6, we conclude that

$$|I_3| \le \frac{C}{\sqrt{t-\tau}}. \tag{5.35}$$

We consider K_2. We first note that

$$E(s) = I_N + O(s), \quad \text{as } s \longrightarrow 0. \tag{5.36}$$

Then for some positive constant C, we have

$$\left|x - E(t-\tau)\xi\right| \ge C|\xi| - |x| \ge CR_1 - R \equiv R_2 > 0, \tag{5.37}$$

since $|x| < R$ and assuming $|\xi| \ge R_1$ with R_1 suitably large. Then we have

$$\begin{aligned}
|K_2| &\le C\int_{T_0}^{t}\int_{\mathbb{R}^N\setminus B_{R_1}} \frac{\Gamma^\varepsilon(x,t,\xi,\tau)}{t-\tau} e^{C_1|\xi|^2} d\xi\, d\tau \\
&\le Ce^{|x|^2}\int_{T_0}^{t}\int_{|\omega|\ge R_2} \frac{1}{(t-\tau)^{(Q+2)/2}} \exp\left(-\frac{\left|\mathcal{C}^{-1/2}(t-\tau)\omega\right|^2}{4\mu} + C_2|\omega|^2\right) d\omega\, d\tau \\
&\qquad \text{(by the change of variable } \omega = x - E(t-\tau)\xi\text{).}
\end{aligned} \tag{5.38}$$

Keeping in mind the asymptotic estimate of Lemma 3.2, clearly the last integral converges (provided that $T - T_0$ is suitably small).

So far we have proved the existence of the integral in (5.22), next we prove (5.22). We set

$$V_f(z) = V_f^{(1)}(z) + V_f^{(2)}(z), \tag{5.39}$$

where

$$\begin{aligned} V_f^{(1)}(x,t) &= \int_{T_0}^{t} \int_{B_{R_1}} Z(x,t,\xi,\tau) f(\xi,\tau)\, d\xi\, d\tau, \\ V_f^{(2)}(x,t) &= \int_{T_0}^{t} \int_{\mathbb{R}^N \setminus B_{R_1}} Z(x,t,\xi,\tau) f(\xi,\tau)\, d\xi\, d\tau. \end{aligned} \tag{5.40}$$

By Lebesgue's theorem, we have

$$\partial_{x_i x_j} V_f^{(2)}(x,t) = \int_{T_0}^{t} \int_{\mathbb{R}^N \setminus B_{R_1}} \partial_{x_i x_j} Z(x,t,\xi,\tau) f(\xi,\tau)\, d\xi\, d\tau. \tag{5.41}$$

In order to prove that

$$\partial_{x_i x_j} V_f^{(1)}(x,t) = \int_{T_0}^{t} \int_{B_{R_1}} \partial_{x_i x_j} Z(x,t,\xi,\tau) f(\xi,\tau)\, d\xi\, d\tau, \tag{5.42}$$

we set

$$V_{f,\delta}^{(1)}(x,t) = \int_{T_0}^{t-\delta} \int_{B_{R_1}} Z(x,t,\xi,\tau) f(\xi,\tau)\, d\xi\, d\tau, \quad 0 < \delta < t - T_0. \tag{5.43}$$

By the dominated convergence theorem and Proposition 5.1, we have

$$\begin{aligned} \lim_{\delta \to 0^+} \partial_{x_i} V_{f,\delta}^{(1)}(x,t) &= \lim_{\delta \to 0^+} \int_{T_0}^{t-\delta} \int_{B_{R_1}} \partial_{x_i} Z(x,t,\xi,\tau) f(\xi,\tau)\, d\xi\, d\tau \\ &= \int_{T_0}^{t} \int_{B_{R_1}} \partial_{x_i} Z(x,t,\xi,\tau) f(\xi,\tau)\, d\xi\, d\tau \\ &= \partial_{x_i} V_f^{(1)}(x,t). \end{aligned} \tag{5.44}$$

Hence, in order to show (5.42), it suffices to prove that

$$\lim_{\delta \to 0^+} \partial_{x_i x_j} V_{f,\delta}^{(1)}(x,t) = \int_{T_0}^{t} \int_{B_{R_1}} \partial_{x_i x_j} Z(x,t,\xi,\tau) f(\xi,\tau)\, d\xi\, d\tau, \tag{5.45}$$

uniformly on $B_{R_1} \times]T_0, T]$. Denoting $w = (y, \tau)$ for $y \in \mathbb{R}^N$, we have

$$\begin{aligned} \partial_{x_i x_j} V^{(1)}_{f,\delta}(x,t) - \int_{T_0}^{t} \int_{B_{R_1}} \partial_{x_i x_j} Z(x,t,\xi,\tau) f(\xi,\tau)\, d\xi\, d\tau \\ = \int_{t-\delta}^{t} \big(J_1(\tau) + J_2(\tau) + J_3(\tau)\big)\, d\tau, \end{aligned} \tag{5.46}$$

where

$$\begin{aligned} J_1(\tau) &= \int_{B_{R_1}} \partial_{x_i x_j} Z(x,t,\xi,\tau)\big(f(\xi,\tau) - f(y,\tau)\big)\, d\xi, \\ J_2(\tau) &= f(y,\tau) \int_{B_{R_1}} \partial_{x_i x_j} \big(Z(x,t,\xi,\tau) - Z_w(x,t,\xi,\tau)\big)\, d\xi, \\ J_3(\tau) &= f(y,\tau) \int_{B_{R_1}} \partial_{x_i x_j} Z_w(x,t,\xi,\tau)\, d\xi. \end{aligned} \tag{5.47}$$

Proceeding as in the estimate of I_1 in (5.25) by choosing $y = E(\tau - t)x$, we obtain

$$\int_{t-\delta}^{t} \big|J_1(\tau)\big|\, d\tau \le C \int_{t-\delta}^{t} \frac{1}{(t-\tau)^{1-\beta/2}}\, d\tau. \tag{5.48}$$

Analogously the terms J_2 and J_3 can be treated as I_2 and I_3 in (5.25), thus (5.45) follows straightforwardly. ∎

Proposition 5.4. Under the hypotheses of Theorem 1.4, there exists $YV_f \in C(S_{T_0,T})$ and it holds that

$$YV_f(z) = \int_{S_{T_0,t}} YZ(z,\zeta) f(\zeta)\, d\zeta - f(z). \tag{5.49}$$

□

Proof. The proof is analogous to that of [31, Proposition 3.3]. As in the proof of Proposition 5.3, we split the domain of the integral in (5.49) in $]T_0, t[\times(\mathbb{R}^N \setminus B_{R_1})$ and $]T_0, t[\times B_{R_1}$ and we only consider the second integral since the other one is straightforward.

We set

$$V_{f,\delta}(x,t) = \int_{T_0}^{t-\delta} \int_{B_{R_1}} Z(x,t,\xi,\tau) f(\xi,\tau)\, d\xi\, d\tau \tag{5.50}$$

and consider the integral path of $-Y$ starting from z:

$$\gamma : \mathbb{R} \longrightarrow \mathbb{R}^{N+1}, \quad \gamma(s) = \big(x(s), t(s)\big) = \big(E(s)x, t+s\big). \tag{5.51}$$

Clearly, $\gamma(0) = z$ and $\dot{\gamma}(s) = (-B^T x(s), 1) = -Y(\gamma(s))$. We show that

$$YV_{f,\delta}(x,t) = \int_{T_0}^{t-\delta} \int_{B_{R_1}} YZ(x,t,\xi,\tau) f(\xi,\tau) d\xi\, d\tau - \int_{B_{R_1}} Z(x,t,\xi,t-\delta) f(\xi,t-\delta) d\xi. \tag{5.52}$$

Indeed, for $|s| < \delta/2$, we have

$$\begin{aligned} \frac{V_{f,\delta}\big(\gamma(s)\big) - V_{f,\delta}\big(\gamma(0)\big)}{s} &= \int_{T_0}^{t-\delta} \int_{B_{R_1}} \frac{Z\big(\gamma(s),\xi,\tau\big) - Z\big(\gamma(0),\xi,\tau\big)}{s} f(\xi,\tau) d\xi\, d\tau \\ &\quad + \frac{1}{s} \int_{t-\delta}^{t+s-\delta} \int_{B_{R_1}} Z\big(\gamma(s),\xi,\tau\big) f(\xi,\tau) d\xi\, d\tau. \end{aligned} \tag{5.53}$$

Since $Z(z,\zeta)$ is the fundamental solution of L_ζ, there exists s^* such that

$$\begin{aligned} \frac{Z\big(\gamma(s),\zeta\big) - Z\big(\gamma(0),\zeta\big)}{s} &= \frac{d}{ds} Z\big(\gamma(s),\zeta\big)\Big|_{s=s^*} = -YZ\big(\gamma(s^*),\zeta\big) \\ &= \sum_{i,j=1}^{p_0} a_{ij}(\zeta) \partial_{x_i x_j} Z\big(\gamma(s^*),\zeta\big). \end{aligned} \tag{5.54}$$

By Proposition 3.6 and since $|s^*| < \delta/2$, the last term in (5.54) is a bounded function of $\zeta \in \mathbb{R}^N \times]T_0, t-\delta[$. Thus we have

$$\begin{aligned} &\lim_{s \to 0} \int_{T_0}^{t-\delta} \int_{B_{R_1}} \frac{Z\big(\gamma(s),\xi,\tau\big) - Z\big(\gamma(0),\xi,\tau\big)}{s} f(\xi,\tau) d\xi\, d\tau \\ &\quad = -\int_{T_0}^{t-\delta} \int_{B_{R_1}} YZ(x,t,\xi,\tau) f(\xi,\tau) d\xi\, d\tau. \end{aligned} \tag{5.55}$$

On the other hand,

$$\begin{aligned} &\int_{B_{R_1}} Z(x,t,y,t-\delta) f(y,t-\delta) d\xi - \frac{1}{s} \int_{t-\delta}^{t+s-\delta} \int_{B_{R_1}} Z\big(\gamma(s),\xi,\tau\big) f(\xi,\tau) d\xi\, d\tau \\ &\quad = \int_0^1 \int_{B_{R_1}} \big(Z(x,t,\xi,t-\delta) - Z(\gamma(s),\xi,t-\delta+\rho s)\big) f(\xi,t-\delta) d\xi\, dr \\ &\qquad + \int_0^1 \int_{B_{R_1}} Z\big(\gamma(s),\xi,t-\delta+\rho s\big) \big(f(\xi,t-\delta) - f(\xi,t-\delta+\rho s)\big) d\xi\, dr \\ &\qquad\qquad \left(\text{by setting } \rho = \frac{\tau - t + \delta}{s}\right) \\ &\quad = I(z,s) + J(z,s). \end{aligned} \tag{5.56}$$

Since $|s| < \delta/2$, then the integrand of I is a bounded function of $(\xi, \rho) \in B_{R_1} \times [0, 1]$, therefore

$$\lim_{s \to 0} I(z, s) = 0. \tag{5.57}$$

Analogously we have

$$\lim_{s \to 0} J(z, s) = 0. \tag{5.58}$$

This concludes the proof of (5.52).

Next we prove that

$$\lim_{\delta \to 0^+} YV_{f,\delta}(x, t) = \int_{T_0}^{t} \int_{B_{R_1}} YZ(x, t, \xi, \tau) f(\xi, \tau) d\xi\, d\tau - f(x, t), \tag{5.59}$$

uniformly on $B_R \times]T_0, T[$. To this end, it suffices to note that, since $Z(z, \zeta)$ is the fundamental solution of L_ζ, we have

$$\begin{aligned} &\left| \int_{t-\delta}^{t} \int_{B_{R_1}} YZ(x, t, \xi, \tau) f(\xi, \tau) d\xi\, d\tau \right| \\ &\quad \le \sum_{i,j=1}^{p_0} \int_{t-\delta}^{t} \int_{B_{R_1}} \left| a_{ij}(\xi, \tau) \partial_{x_i x_j} Z(x, t, \xi, \tau) f(\xi, \tau) \right| d\xi\, d\tau \\ &\quad \le C \int_{t-\delta}^{t} \frac{1}{(t-\tau)^{1-\beta/2}} d\tau \\ &\qquad \text{(proceeding as in the proof of Proposition 5.3, cf. (5.29)).} \end{aligned} \tag{5.60}$$

Finally, since f is a continuous and bounded function on $B_R \times]T_0, T[$, we have

$$\lim_{\delta \to 0^+} \int_{B_{R_1}} Z(x, t, \xi, t - \delta) f(\xi, t - \delta) d\xi = f(x, t), \tag{5.61}$$

uniformly on $B_R \times]T_0, T[$ and this concludes the proof. ■

6 Proof of Theorems 1.4 and 1.5

In this section, we prove Theorems 1.4 and 1.5. We begin by a preliminary result.

Lemma 6.1. For every $\varepsilon > 0$ and $T > 0$, there exists a positive constant C such that

$$\left| \Phi(x, t, \xi, \tau) - \Phi(y, t, \xi, \tau) \right| \le C \frac{|x - y|_B^{\alpha/2}}{(t - \tau)^{1-\alpha/4}} \left(\Gamma^\varepsilon(x, t, \xi, \tau) + \Gamma^\varepsilon(y, t, \xi, \tau) \right), \tag{6.1}$$

for any $(\xi, \tau) \in \mathbb{R}^{N+1}$, $t \in]\tau, \tau + T]$, and $x, y \in \mathbb{R}^N$. □

Proof. We set $w = (y, t)$ and note that if $|x-y|_B \geq \sqrt{t-\tau}$, then we have the trivial estimate

$$\big|LZ(z,\zeta) - LZ(w,\zeta)\big| \leq \frac{C}{(t-\tau)^{1-\alpha/2}}\big(\Gamma^{\varepsilon}(z,\zeta) + \Gamma^{\varepsilon}(w,\zeta)\big). \tag{6.2}$$

In the case $|x-y|_B < \sqrt{t-\tau}$, we first prove the following estimates:

$$\begin{aligned}
&\big|Z(z,\zeta) - Z(w,\zeta)\big| \leq \frac{C}{\sqrt{t-\tau}}\Gamma^{\varepsilon/2}(z,\zeta),\\
&\big|\partial_{x_k} Z(z,\zeta) - \partial_{x_i} Z(w,\zeta)\big| \leq C\frac{|x-y|_B}{t-\tau}\Gamma^{\varepsilon/2}(z,\zeta),\\
&\big|\partial_{x_i x_j} Z(z,\zeta) - \partial_{x_i x_j} Z(w,\zeta)\big| \leq C\frac{|x-y|_B}{(t-\tau)^{3/2}}\Gamma^{\varepsilon/2}(z,\zeta).
\end{aligned} \tag{6.3}$$

Since the proof is similar, we only consider the third estimate in (6.3). By using the mean-value theorem, we have

$$\big|\partial_{x_i x_j} Z(z,\zeta) - \partial_{x_i x_j} Z(w,\zeta)\big| \leq \max_{\rho\in[0,1]} \sum_{h=1}^{N} \big|\partial_{x_h x_i x_j} Z\big(x+\rho(x-y), t, \xi, \tau\big)(x-y)_h\big|. \tag{6.4}$$

Denoting $s = t-\tau$, $\omega = x - E(s)\xi$, and $\mathcal{C} = \mathcal{C}_\zeta(s)$, a short computation shows

$$\begin{aligned}
\partial_{x_h x_i x_j} Z(z,\zeta) = Z(z,\zeta)\Big(&\mathcal{C}^{-1}_{ih}\big(\mathcal{C}^{-1}\omega\big)_j + \big(\mathcal{C}^{-1}\omega\big)_i \mathcal{C}^{-1}_{jh} + \big(\mathcal{C}^{-1}\omega\big)_h \mathcal{C}^{-1}_{ij}\\
&+ \big(\mathcal{C}^{-1}\omega\big)_h\big(\mathcal{C}^{-1}\omega\big)_i\big(\mathcal{C}^{-1}\omega\big)_j\Big)\\
\equiv Z(z,\zeta)\big(&a_h(\omega) + b_h(\omega) + c_h(\omega) + d_h(\omega)\big).
\end{aligned} \tag{6.5}$$

Then we put $v = x - y$, $\widetilde{\omega} = \omega + \rho v$ and, by Lemma 3.4, we get

$$\begin{aligned}
\left|\sum_{h=1}^{N} v_h a_h(\widetilde{\omega})\right| &\leq \sum_{h=1}^{N}\left|\mathcal{C}^{-1}_{ih} v_h \big(\mathcal{C}^{-1}\widetilde{\omega}\big)_j\right| = \left|\big(\mathcal{C}^{-1}v\big)_i\right|\left|\big(\mathcal{C}^{-1}\widetilde{\omega}\big)_j\right|\\
&\leq \frac{C}{s}\left|D_0\left(\frac{1}{\sqrt{s}}\right)\right|\left|D_0\left(\frac{1}{\sqrt{s}}\right)\widetilde{\omega}\right|.
\end{aligned} \tag{6.6}$$

Since $|v|_B < \sqrt{s}$, we have $|D_0(1/\sqrt{s})v| \leq C|D_0(1/\sqrt{s})v|_B = C(|v|_B/\sqrt{s})$, therefore

$$\left|\sum_{h=1}^{N} v_h a_h(\widetilde{\omega})\right| \leq C\frac{|v|_B|\widetilde{\eta}|_B}{s^{3/2}}, \tag{6.7}$$

where $\widetilde{\eta} = D_0(1/\sqrt{s})\widetilde{\omega}$. The same estimate holds substituting a_h with b_h or c_h. Moreover,

$$\left|\sum_{h=1}^{N} v_h d_h(\widetilde{\omega})\right| \leq \sum_{h=1}^{N}\left|\big(\mathcal{C}^{-1}\widetilde{\omega}\big)_h v_h \big(\mathcal{C}^{-1}\widetilde{\omega}\big)_i\big(\mathcal{C}^{-1}\widetilde{\omega}\big)_j\right| \leq \frac{|v|_B|\widetilde{\eta}|_B^3}{s^{3/2}}. \tag{6.8}$$

Collecting all the terms and using Proposition 3.5, we obtain

$$\begin{aligned}\left|\partial_{x_i x_j} Z(z,\zeta) - \partial_{x_i x_j} Z(w,\zeta)\right| &\leq \frac{|v|_B\left(|\widetilde{\eta}|_B + |\widetilde{\eta}|_B^3\right)}{s^{3/2}} Z(x+\rho v, t, \xi, \tau) \\ &\leq \frac{|x-y|_B}{s^{3/2}} \Gamma^{\varepsilon/3}(x+\rho v, t, \xi, \tau).\end{aligned} \tag{6.9}$$

By a standard argument, we have that, if $|x-y|_B < \sqrt{t-\tau}$, then

$$\Gamma^{\varepsilon/3}(x+v, t, \xi, \tau) \leq \Gamma^{\varepsilon/2}(x, t, \xi, \tau). \tag{6.10}$$

This concludes the proof of the third inequality in (6.3) at least for $|x-y|_B < \sqrt{t-\tau}$. Next we show how to deduce from (6.3) an estimate similar to (6.2). We recall that $(w)^{-1} \circ z = (x-y, 0)$ and we have

$$\begin{aligned}&\left|LZ(z,\zeta) - LZ(w,\zeta)\right| \\ &\quad = \Bigg| \sum_{i,j=1}^{p_0} a_{ij}(z) \partial_{x_i x_j} Z(z,\zeta) + \sum_{i=1}^{p_0} a_i(z) \partial_{x_i} Z(z,\zeta) \\ &\qquad - \sum_{i,j=1}^{p_0} a_{ij}(w) \partial_{x_i x_j} Z(w,\zeta) - \sum_{i=1}^{p_0} a_i(w) \partial_{x_i} Z(w,\zeta) \\ &\qquad + YZ(z,\zeta) - YZ(w,\zeta) + c(z)Z(z,\zeta) - c(w)Z(w,\zeta) \\ &\qquad - L_\zeta Z(z,\zeta) + L_\zeta Z(w,\zeta) \Bigg| \\ &\quad \leq \sum_{i,j=1}^{p_0} \left|a_{ij}(z) - a_{ij}(w)\right| \left|\partial_{x_i x_j} Z(w,\zeta)\right| \\ &\qquad + \sum_{i,j=1}^{p_0} \left|a_{ij}(z) - a_{ij}(\zeta)\right| \left|\partial_{x_i x_j} Z(z,\zeta) - \partial_{x_i x_j} Z(w,\zeta)\right| \\ &\qquad + \sum_{i=1}^{p_0} \left|a_i(z) - a_i(w)\right| \left|\partial_{x_i} Z(w,\zeta)\right| \\ &\qquad + \sum_{i=1}^{p_0} \left|a_i(w)\right| \left|\partial_{x_i} Z(z,\zeta) - \partial_{x_i} Z(w,\zeta)\right| \\ &\qquad + \left|c(z) - c(w)\right| \left|Z(w,\zeta)\right| + \left|c(z)\right| \left|Z(z,\zeta) - Z(w,\zeta)\right|\end{aligned}$$

$$\leq C\left(\frac{|x-y|_B^\alpha}{t-\tau}\Gamma^{\varepsilon/2}(w,\zeta)+\left\|\zeta^{-1}\circ z\right\|_B^\alpha\frac{|x-y|_B}{(t-\tau)^{3/2}}\Gamma^{\varepsilon/2}(z,\zeta)\right.$$
$$+\frac{|x-y|_B^\alpha}{\sqrt{t-\tau}}\Gamma^{\varepsilon/2}(w,\zeta)+\frac{|x-y|_B}{t-\tau}\Gamma^{\varepsilon/2}(z,\zeta)$$
$$\left.+|x-y|_B^\alpha\Gamma^{\varepsilon/2}(w,\zeta)+\frac{|x-y|_B}{\sqrt{t-\tau}}\Gamma^{\varepsilon/2}(z,\zeta)\right)$$

(by the regularity properties of the coefficients,

by Proposition 3.6, and by (6.3)).

(6.11)

Since

$$\left\|\zeta^{-1}\circ z\right\|_B^\alpha=(t-\tau)^{\alpha/2}\left(1+\left|D_0\left((t-\tau)^{-1/2}\right)\left(x-E(t-\tau)\xi\right)\right|_B^\alpha\right),\tag{6.12}$$

we may use Proposition 3.5 to deduce

$$\left|LZ(z,\zeta)-LZ(w,\zeta)\right|\leq C\left(\frac{|x-y|_B}{(t-\tau)^{(3-\alpha)/2}}+\frac{|x-y|_B^\alpha}{t-\tau}\right)\left(\Gamma^\varepsilon(z,\zeta)+\Gamma^\varepsilon(w,\zeta)\right).\tag{6.13}$$

On the other hand, if $|x-y|_B<\sqrt{t-\tau}$, it holds that

$$\frac{|x-y|_B}{(t-\tau)^{(3-\alpha)/2}}+\frac{|x-y|_B^\alpha}{t-\tau}\leq\frac{|x-y|_B}{(t-\tau)^{(3-\alpha)/2}}\left(\frac{|x-y|_B}{\sqrt{t-\tau}}\right)^{-1+\alpha/2}+\frac{|x-y|_B^\alpha}{t-\tau}\left(\frac{|x-y|_B}{\sqrt{t-\tau}}\right)^{-\alpha}$$
$$=\frac{|x-y|_B^{\alpha/2}}{(t-\tau)^{1-\alpha/4}}.\tag{6.14}$$

Combining (6.2), (6.13), and (6.14), finally we get

$$\left|LZ(z,\zeta)-LZ(w,\zeta)\right|\leq C\frac{|x-y|_B^{\alpha/2}}{(t-\tau)^{1-\alpha/4}}\left(\Gamma^\varepsilon(z,\zeta)+\Gamma^\varepsilon(w,\zeta)\right).\tag{6.15}$$

By (6.15) and an inductive argument, it is possible to show that if M_1 is the constant in (4.11) such that $|LZ(z,\zeta)|\leq M_1(\Gamma^\varepsilon(z,\zeta)/(t-\tau)^{1-\alpha/2})$, then we have

$$\left|(LZ)_k(z,\zeta)-(LZ)_k(w,\zeta)\right|\leq\widetilde{M}_k\frac{|x-y|_B^{\alpha/2}}{(t-\tau)^{1-\alpha/4}}\left(\Gamma^\varepsilon(z,\zeta)+\Gamma^\varepsilon(w,\zeta)\right)M_1^k(t-\tau)^k,\tag{6.16}$$

where

$$\widetilde{M}_k=C_0\Gamma_E^k\left(\frac{\alpha}{2}\right)\frac{\Gamma_E\left(\dfrac{\alpha}{4}\right)}{\Gamma_E\left(\dfrac{\alpha}{2}\left(k+\dfrac{1}{2}\right)\right)},\tag{6.17}$$

for some positive constant C_0. The thesis follows since the power series with coefficients $\widetilde{M}_k$ has radius of convergence equal to infinity. ■

Proof of Theorem 1.4. Let Γ be the function defined in (4.2), (4.3), and (4.7) by means of Proposition 4.1:

$$\Gamma(z,\zeta) = Z(z,\zeta) + \int_{S_{\tau,t}} Z(z,w)\Phi(w,\zeta)dw, \quad z \neq \zeta. \tag{6.18}$$

(1) By Corollary 4.4 and Proposition 4.1, it is clear that $\Gamma(\cdot,\zeta) \in L^1_{loc}(\mathbb{R}^{N+1}) \cap C(\mathbb{R}^{N+1} \setminus \{\zeta\})$ for every $\zeta \in \mathbb{R}^{N+1}$.

(2) Thanks to estimate (4.10) and Lemma 6.1, we may apply Propositions 5.1, 5.3, and 5.4 to conclude that the following derivatives exist and are continuous functions for $z \neq \zeta$:

$$\begin{aligned} \partial_{x_i}\Gamma(z,\zeta) &= \partial_{x_i}Z(z,\zeta) + \int_{S_{\tau,t}} \partial_{x_i}Z(z,w)\Phi(w,\zeta)dw, \\ \partial_{x_ix_j}\Gamma(z,\zeta) &= \partial_{x_ix_j}Z(z,\zeta) + \int_{S_{\tau,t}} \partial_{x_ix_j}Z(z,w)\Phi(w,\zeta)dw, \\ Y\Gamma(z,\zeta) &= YZ(z,\zeta) + \int_{S_{\tau,t}} \partial_{x_i}YZ(z,w)\Phi(w,\zeta)dw - \Phi(z,\zeta), \end{aligned} \tag{6.19}$$

for every $i, j = 1, \dots, p_0$. By using the above formulas, we directly obtain

$$L\Gamma(z,\zeta) = LZ(z,\zeta) + \int_{S_{\tau,t}} LZ(z,w)\Phi(w,\zeta)dw - \Phi(z,\zeta) = 0 \tag{6.20}$$

for $z \neq \zeta$, since Φ satisfies the integral equation (4.6).

(3) Formula (1.19) can be proved following [31, Proposition 2.5].

(4) By the results in Section 4, the function u in (1.22) is well defined in $S_{T_0,T}$ for $T - T_0 > 0$ suitably small. We set

$$V(z) = \int_{S_{T_0,t}} \Gamma(z,\zeta)f(\zeta)d\zeta, \tag{6.21}$$

and we prove that

$$LV = -f, \quad \text{in } S_{T_0,T}. \tag{6.22}$$

Using expression (6.18) of Γ, we rewrite $V = V_f + V_{\hat{f}}$ where V_f is the potential in (5.1) and

$$\hat{f}(z) = \int_{S_{T_0,t}} \Phi(z,\zeta)f(\zeta)d\zeta. \tag{6.23}$$

In order to apply Propositions 5.1, 5.3, and 5.4 to the potential $V_{\widehat{f}}$, we show that $\widehat{f}$ verifies estimates (1.20) and (1.21). By (4.10) we have

$$\begin{aligned}\left|\widehat{f}(z)\right| &\le C\int_{S_{T_0,t}} \frac{\Gamma^{\varepsilon}(z,\zeta)}{(t-\tau)^{1-\alpha/2}}\left|f(\zeta)\right|d\zeta \\ &\le C\left(t-T_0\right)^{\alpha/2}e^{C|x|^2} \quad \text{(proceeding as in the proof of (5.3)).}\end{aligned} \tag{6.24}$$

On the other hand, by Lemma 6.1, we infer

$$\begin{aligned}&\left|\widehat{f}(x,t)-\widehat{f}(y,t)\right| \\ &\quad\le \int_{T_0}^{t}\int_{\mathbb{R}^N}\left|\Phi(x,t,\xi,\tau)-\Phi(y,t,\xi,\tau)\right|\left|f(\xi,\tau)\right|d\xi\, d\tau \\ &\quad\le C|x-y|_B^{\alpha/2}\int_{T_0}^{t}\frac{1}{(t-\tau)^{1-\alpha/4}}\int_{\mathbb{R}^N}\left(\Gamma^{\varepsilon}(x,t,\xi,\tau)+\Gamma^{\varepsilon}(y,t,\xi,\tau)\right)\left|f(\xi,\tau)\right|d\xi\, d\tau \\ &\quad\le C\left(t-T_0\right)^{\alpha/4}|x-y|_B^{\alpha/2}e^{C(|x|^2+|y|^2)}.\end{aligned} \tag{6.25}$$

Therefore, we can apply Propositions 5.1, 5.3, and 5.4 and we get, for $z\in S_{T_0,T}$,

$$\begin{aligned}LV(z) &= LV_f(z)+LV_{\widehat{f}}(z) \\ &= -f(z)-\widehat{f}(z)+\int_{S_{T_0,t}} LZ(z,\zeta)\left(f(\zeta)+\widehat{f}(\zeta)\right)d\zeta \\ &= -f(z)+\int_{S_{T_0,t}} f(\zeta)\left(-\Phi(z,\zeta)+LZ(z,\zeta)+\int_{S_{\tau,t}} LZ(z,w)\Phi(w,\zeta)dw\right)d\zeta \\ &= -f(z),\end{aligned} \tag{6.26}$$

by (4.6). Since, for $t>T_0$, it holds that

$$L\int_{\mathbb{R}^N}\Gamma\left(x,t,\xi,T_0\right)g(\xi)d\xi = \int_{\mathbb{R}^N} L\Gamma\left(x,t,\xi,T_0\right)=0, \tag{6.27}$$

by Step (2), we conclude that $Lu=f$ in $S_{T_0,T}$. Moreover, by Corollary 4.4,

$$\begin{aligned}\left|V(z)\right| &\le C\int_{S_{T_0,t}}\Gamma^{\varepsilon}(z,\zeta)\left|f(\zeta)\right|d\zeta \\ &\le C\left(t-T_0\right)e^{C|x|^2} \quad \text{(proceeding as in the proof of (5.3)),}\end{aligned} \tag{6.28}$$

therefore, by Step (3), we have that $u\in C(\mathbb{R}^N\times[T_0,T[)$ and $u(\cdot,T_0)=g$.

(5)–(7) The uniqueness result can be proved proceeding exactly as in the classical parabolic case (see, e.g., [18]). Then the reproduction property (1.24) and formula (1.25) follow immediately.

(8) Estimate (1.27) is included in Corollary 4.4. Analogously, by Proposition 3.6 and (4.10), we have

$$\left|\partial_{x_i}\Gamma(z,\zeta)\right| \le \frac{C\Gamma^{\varepsilon}(z,\zeta)}{\sqrt{t-\tau}} + C\Gamma^{\varepsilon}(z,\zeta)\int_{\tau}^{t}\frac{1}{(t-s)^{1/2}}\frac{1}{(s-\tau)^{1-\alpha/2}}ds \le C\frac{\Gamma^{\varepsilon}(z,\zeta)}{\sqrt{t-\tau}}, \quad (6.29)$$

for any $i = 1,\dots,p_0$ and $z,\zeta \in \mathbb{R}^{N+1}$ with $0 < t-\tau < T$. The proof of (1.29) is less trivial:

$$\begin{aligned}
&\left|\partial_{x_i x_j}\Gamma(z,\zeta)\right| \\
&\quad \le \left|\partial_{x_i x_j}Z(z,\zeta)\right| + \left|\partial_{x_i x_j}J(z,\zeta)\right| \\
&\quad \le C\frac{\Gamma^{\varepsilon}(z,\zeta)}{t-\tau} + \left|\int_{S_{\tau,t}}\partial_{x_i x_j}Z(z,w)\Phi(w,\zeta)dw\right| \\
&\qquad\quad \text{(by Propositions 3.6 and 5.3)} \\
&\quad \le C\frac{\Gamma^{\varepsilon}(z,\zeta)}{t-\tau} + C\int_{\tau}^{t}\frac{1}{(t-s)^{\alpha/4}}\frac{1}{(s-\tau)^{\alpha/4}}ds \\
&\qquad\quad \text{(managing the singularity of the integral as in the proof of Proposition 5.3)} \\
&\quad \le C\frac{\Gamma^{\varepsilon}(z,\zeta)}{t-\tau}.
\end{aligned} \quad (6.30)$$

■

Proof of Theorem 1.5. The proof of the existence and the properties of Γ^* is analogous to that of Theorem 1.4. In order to prove (1.32), we first note that the Green's identity holds:

$$\begin{aligned}
vLu - uL^*v = &\sum_{i,j=1}^{p_0}\partial_{x_i}\left(a_{ij}\left(v\partial_{x_j}u - u\partial_{x_j}v\right) + uv\left(a_i - \partial_{x_j}a_{ij}\right)\right) \\
&+ \sum_{i,j=1}^{N}\partial_{x_j}\left(b_{ij}x_i uv\right) - \partial_t(uv),
\end{aligned} \quad (6.31)$$

for any $u, v \in C_0^{\infty}(\mathbb{R}^{N+1})$. Then we consider the functions

$$u(w) = \Gamma(w,\zeta), \qquad v(w) = \Gamma^*(w,z) \quad (6.32)$$

for $w = (y,s)$ with $\tau < s < t$. Given $R, \delta > 0$, we integrate the identity (6.31) over the domain $\{(y,s) \mid |y| < R,\ \tau+\delta < s < t-\delta\}$ and we obtain

$$\int_{|y|<R}u(y,t-\delta)v(y,t-\delta)dy - \int_{|y|<R}u(y,\tau+\delta)v(y,\tau+\delta)dy = I_{R,\delta}, \quad (6.33)$$

where

$$I_{R,\delta}=\sum_{i,j=1}^{p_0}\int_{\tau+\delta}^{t-\delta}\int_{|y|=R}\left(a_{ij}\left(v\partial_{y_j}u-u\partial_{y_j}v\right)-uv\partial_{y_j}a_{ij}\right)\nu_i\,d\sigma(w)$$
$$+\sum_{i,j=1}^{N}\int_{\tau+\delta}^{t-\delta}\int_{|y|=R}b_{ij}y_i\nu_j uv\,d\sigma(w). \tag{6.34}$$

By (1.28) and (1.29) (and the analogous estimates for Γ^*), we get

$$\lim_{R\to+\infty}I_{R,\delta}=0, \tag{6.35}$$

so that

$$\int_{\mathbb{R}^N}u(y,t-\delta)v(y,t-\delta)dy=\int_{\mathbb{R}^N}u(y,\tau+\delta)v(y,\tau+\delta)dy \tag{6.36}$$

and the thesis follows by letting $\delta\to 0^+$. ■

Proof of Theorem 1.6. We only sketch the proof since it suffices to proceed as in [18, Theorem 16, page 29], by using Theorem 1.5 and the estimates (1.28) and (1.29) in Theorem 1.4.

It is not restrictive to assume $T_0=0$. We first prove that $u=0$ in a suitable thin strip $S_{0,\varepsilon}$. Fixed $(y,s)\in S_{0,\varepsilon}$, for any $R>|y|$, we consider $h_R\in C_0^\infty(B_{R+1})$, $0\le h_R\le 1$, such that $h_R\equiv 1$ on B_R and with the first- and second-order derivatives bounded uniformly with respect to R. We integrate the Green's identity (6.31) with $u=u(\zeta)$ and $v(\xi,\tau)=h_R(\xi)\Gamma(y,s,\xi,\tau)$ over the domain $\{\zeta\in\mathbb{R}^{N+1}:\xi\in B_{R+1},\ 0<\tau<s-\delta\}$, for some $\delta>0$. Since $Lu=0$, we have

$$-\int_0^{s-\delta}\int_{B_{R+1}}u(\xi,\tau)L^*v(\xi,\tau)d\xi\,d\tau$$
$$=\int_0^{s-\delta}\int_{B_{R+1}}\left(vLu-uL^*v\right)(\xi,\tau)d\xi\,d\tau$$
$$=-\int_{B_{R+1}}u(\xi,s-\delta)h(\xi)\Gamma(y,s,\xi,s-\delta)d\xi+\int_{B_{R+1}}u(\xi,0)h(\xi)\Gamma(y,s,\xi,0)d\xi$$
$$+\sum_{i,j=1}^{p_0}\int_0^{s-\delta}\int_{\partial B_{R+1}}\left(a_{ij}\left(v\partial_{\xi_j}u-u\partial_{\xi_j}v\right)-uv\partial_{\xi_j}a_{ij}\right)d\sigma(\zeta)$$
$$+\sum_{i,j=1}^{N}\int_0^{s-\delta}\int_{B_{R+1}}b_{ij}\xi_i uv\nu_j\,d\sigma(\zeta)\quad\text{(by the divergence theorem).} \tag{6.37}$$

The last three terms in (6.37) are null by hypothesis, then letting $\delta \to 0^+$, we get

$$\begin{aligned} u(y,s) &= \lim_{\delta\to 0^+} \int_{B_{R+1}} u(\xi, s-\delta) h(\xi) \Gamma(y,s,\xi,s-\delta) d\xi \\ &= \int_0^s \int_{B_{R+1}} u(\xi,\tau) L^* v(\xi,\tau) d\xi\, d\tau. \end{aligned} \tag{6.38}$$

Since $L^*\Gamma(y,s,\xi,\tau) = 0$, we deduce

$$\begin{aligned} u(y,s) = \int_0^s \int_{B_{R+1}\setminus B_R} u(\xi,\tau) \Bigg(\sum_{i,j=1}^{p_0} a_{ij}(\xi,\tau) \big(2\partial_{\xi_i} h_R(\xi) \partial_{\xi_j} \Gamma(y,s,\xi,\tau) \\ + \Gamma(y,s,\xi,\tau) \partial_{\xi_i \xi_j} h_R(\xi) \big) \\ - \sum_{i=1}^{p_0} a_i(\xi,\tau) \Gamma(y,s,\xi,\tau) \partial_{\xi_i} h_R(\xi) \\ - \sum_{i,j=1}^{N} b_{ij} \xi_i \partial_{\xi_j} h_R(\xi) \Gamma(y,s,\xi,\tau) \Bigg) d\xi\, d\tau. \end{aligned} \tag{6.39}$$

By means of Theorem 1.5 and (1.28) and (1.29), it is straightforward to conclude that if ε is suitably small, then the integral at the right-hand side of (6.39) tends to zero as $R \to +\infty$, so that $u(y,s) = 0$. The thesis follows by repeating the previous argument finitely many times. ∎

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