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On a class of difference equations with interlacing indices

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Abstract

Some results on the long-term behavior of solutions to a class of difference equations, which includes numerous nonlinear difference equations of various orders that attracted some attention in the last 15 years, are presented. We also present a natural connection among these difference equations, compare some results on the equations with some other ones in the literature, and give a list of a considerable number of difference equations which can be treated in a similar way.

MSC: 39A10

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1 Introduction

1.1 Notation and some general facts

By $\mathbb{N}$ and $\mathbb{Z}$ we denote the sets of natural and whole numbers, respectively. If $k \in \mathbb{Z}$, then by $\mathbb{N}_k$ we denote the set of all $n \in \mathbb{Z}$ such that $k \le n$. If $l, m \in \mathbb{Z}$, then $j = \overline{l,m}$, denotes the set of all $j \in \mathbb{Z}$ such that $l \le j \le m$. If $l, m \in \mathbb{N}$, then $\gcd(l,m)$ denotes the greatest common divisor of numbers l and m.

During the last 30 years there has been a huge interest in investigating solutions to concrete difference equations and systems of difference equations (see, e.g., [1–36]).

1.2 On an influential result

The following difference equation is one of the concrete ones which attracted some attention:

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n}, \quad n \in \mathbb{N}_0. \tag{1}$$

A difficult problem on the existence of a positive solution to Eq. (1) converging to zero had been open for some time (the problem was posed for another difference equation which is equivalent to the equation).

Note that the linearization of Eq. (1) is

$$x_{n+1} - x_{n-1} = 0, \quad n \in \mathbb{N}_0,$$

and that the zeros of the associated characteristic polynomial

$$P(\lambda) = \lambda^2 - 1$$

to the equation are $\lambda_{1,2} = \pm 1$.

The difference equations and systems whose linearizations have some zeros with modulus equal to one are of some interest and have been studied a lot (see, e.g., [2, 7–9, 13, 15, 16, 26, 28]).

The above-mentioned problem was solved in [26] by considering a more general equation, for which it was shown, in a quite complex way, the existence of a positive solution converging to zero, from which the same result follows for Eq. (1) as a special case of the general equation. During the course of a long solution of the problem there have been proved several interesting formulas which are satisfied by solutions to the difference equation. Some other solutions to the problem were given in [2] and [13].

The next theorem, among other ones, was proved in [26].

Theorem 1 *For each positive solution to* (1) *the following claims are true.*

(a) $(x_{2n})_{n\in\mathbb{N}_0}$ *and* $(x_{2n+1})_{n\ge -1}$ *are decreasing and* $x_{2n} \to a$ *and* $x_{2n+1} \to b$, *as* $n \to +\infty$.

(b) *The sequence* $\widetilde{x}_{2n-1} = b$, $\widetilde{x}_{2n} = a$, $n \in \mathbb{N}_0$, *is a solution to* (1) *with period two.*

(c) $ab = 0$.

(d) *If* $x_n \ge x_{n+1}$ *for all* $n \ge n_1$ *and some* $n_1 \in \mathbb{N}$, *then* $\lim_{n\to\infty} x_n = 0$.

(e) *We have*

$$x_{2n} = x_0\left(1 - x_1 \sum_{j=1}^{n} \prod_{i=1}^{2j-1} \frac{1}{1+x_i}\right), \quad n \in \mathbb{N}_0, \tag{2}$$

$$x_{2n+1} = x_{-1}\left(1 - \frac{x_0}{1+x_0} \sum_{j=0}^{n} \prod_{i=1}^{2j} \frac{1}{1+x_i}\right), \quad n \ge -1. \tag{3}$$

(f) *If* $x_0 + x_0^2 \le x_{-1}$, *then* $\lim_{n\to+\infty} x_{2n} \ne 0$ *or* $\lim_{n\to+\infty} x_{2n+1} \ne 0$.

(g) *Each solution to* (1) *converging to zero decreases.*

Remark 1 The formulation of Theorem 1 *(f)* in [26] is, in fact, slightly different, but what was proved therein is stated above. An improvement of Theorem 1 *(f)* has been recently presented in [34]. Namely, the following result was proved therein.

Theorem 2 *Let* $(x_n)_{n\ge -1}$ *be a positive solution to* (1) *such that* $x_0 + x_0^2 \le x_{-1}$. *Then* $x_{2n} \to 0$ *and* $x_{2n+1} \to b \ne 0$, *as* $n \to \infty$.

1.3 A generalization of Eq. (1) and some important investigations

A natural generalization of Eq. (1) is

$$x_n = \frac{x_{n-k}}{1 + x_{n-1}\cdots x_{n-k+1}}, \quad n \in \mathbb{N}_0, \tag{4}$$

where $k \ge 2$.

In order to generalize some results obtained in [26], Stević in [28] and [31], and somewhat later Berg and Stević in [9] studied the existence of a positive solution to Eq. (4) which converges to zero. It is not easy to develop the method in [26] for the case of Eq. (4), since it is basically connected to second order difference equations, that is, to the case $k = 2$. In [2–6], Berg employed some asymptotic methods for solving similar problems. For more information on asymptotic methods see, for example, [37]. Stević developed Berg's ideas and applied them successfully in solving some related problems, for example, in [10, 27–30, 32] (see also [14]). Stević in [28] solved the problem by using asymptotic methods. The existence problem for the case $k = 3$ was further treated by Berg in [7], whereas the general case was further treated in Berg–Stević's paper [9].

Although cited in [9] as reference [14] therein and circulated among some experts on difference equations, preprint [31] has never been published. One of the reasons for this was to avoid publishing two papers on the same problem.

1.4 Some equations influenced by [26]

In [25] has been recently studied the following difference equation:

$$x_{n+1} = \frac{x_{n-17}}{1 + x_{n-5}x_{n-11}}, \quad n \in \mathbb{N}_0. \tag{5}$$

It is said therein that some of these authors and their collaborators studied also the following difference equations:

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-1}x_{n-3}}, \quad n \in \mathbb{N}_0, \tag{6}$$

$$x_{n+1} = \frac{x_{n-3}}{1 + x_n x_{n-1}x_{n-2}}, \quad n \in \mathbb{N}_0, \tag{7}$$

$$x_{n+1} = \frac{x_{n-15}}{1 + x_{n-3}x_{n-7}x_{n-11}}, \quad n \in \mathbb{N}_0, \tag{8}$$

$$x_{n+1} = \frac{x_{n-11}}{1 + x_{n-2}x_{n-5}x_{n-8}}, \quad n \in \mathbb{N}_0, \tag{9}$$

$$x_{n+1} = \frac{x_{n-11}}{1 + x_{n-1}x_{n-3}x_{n-5}x_{n-7}x_{n-9}}, \quad n \in \mathbb{N}_0, \tag{10}$$

$$x_{n+1} = \frac{x_{n-(5k+9)}}{1 + x_{n-4}x_{n-9}\cdots x_{n-(5k+4)}}, \quad n \in \mathbb{N}_0, \tag{11}$$

$$x_{n+1} = \frac{x_{n-(4k+3)}}{1 + \prod_{t=0}^{2} x_{n-(k+1)t-k}}, \quad n \in \mathbb{N}_0, \tag{12}$$

$$x_{n+1} = \frac{x_{n-(k+1)}}{1 + x_n x_{n-1}\cdots x_{n-k}}, \quad n \in \mathbb{N}_0. \tag{13}$$

They also studied some other equations not mentioned in [25] such as

$$x_{n+1} = \frac{x_{n-23}}{1 + x_{n-5}x_{n-11}x_{n-17}}, \quad n \in \mathbb{N}_0, \tag{14}$$

and the following quite recently studied equations:

$$x_{n+1} = \frac{x_{n-14}}{1 + x_{n-2}x_{n-5}x_{n-8}x_{n-11}}, \quad n \in \mathbb{N}_0, \tag{15}$$

$$x_{n+1} = \frac{x_{n-20}}{1 + x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}x_{n-17}}, \quad n \in \mathbb{N}_0, \tag{16}$$

$$x_{n+1} = \frac{x_{n-13}}{1 + x_{n-1}x_{n-3}x_{n-5}x_{n-7}x_{n-9}x_{n-11}}, \quad n \in \mathbb{N}_0, \tag{17}$$

$$x_{n+1} = \frac{x_{n-pl+1}}{1 + \prod_{j=1}^{p-1} x_{n-jl+1}}, \quad n \in \mathbb{N}_0. \tag{18}$$

At first sight the difference equations in (5)–(18) form a large collection of various equations of different orders, and one can expect that these investigations of the equations produced many different results. It should be also said that a detailed inspection of the results and their proofs shows that the main motivation for all these studies has been the proof of Theorem 1 in Stević's paper [26].

1.5 The aim of the paper

We show that the results on all the difference equations in (5)–(18) in the above quoted papers follow from the results on only one of the equations, namely on the results on Eq. (13) only, and that they are essentially obtained by some slight modifications of the results in Theorem 1 from [26]. This fact has been noticed long time ago in [31] for the case of few of these equations. The problem was essentially noticed also in [7]. Few of the results in [31] were mentioned in [9]. We present one of the main results in [31] in its original form (some of the statements are more general than the corresponding ones in many published papers on the difference equations of this type).

2 Equations with interlacing indices and some examples

In this section we explain the notion of difference equations with interlacing indices, present some examples of such equations which can be found in the literature, and give some comments on the equations.

2.1 Equations with interlacing indices

From a difference equation or system it is always possible to construct a sequence of difference equations or systems by using a simple method.

The general equation

$$x_{n+1} = g\bigl(x_{n+1-(k+1)}, x_{n+1-2(k+1)}, \dots, x_{n+1-l(k+1)}\bigr), \quad n \in \mathbb{N}_0, \tag{19}$$

where $l \in \mathbb{N}$ and $k \in \mathbb{N}_0$, is a *difference equation with interlacing indices* (for some details and examples see [35, 36]).

Let $\mathcal{A}_j^l := \{(m-l)(k+1) + 1 + j : m \in \mathbb{N}_0\}$, $j = \overline{0,k}$. Then $\mathcal{A}_i^l \cap \mathcal{A}_j^l = \emptyset$, $i \neq j$, and $\bigcup_{j=0}^k \mathcal{A}_j^l = \mathbb{N}_{-l(k+1)+1}$. If we use the notation

$$x_m^{(j)} := x_{m(k+1)+1+j}, \quad m \geq -l,$$

where $j = \overline{0,k}$, then Eq. (19) becomes

$$x_m^{(j)} = g\bigl(x_{m-1}^{(j)}, x_{m-2}^{(j)}, \dots, x_{m-l}^{(j)}\bigr), \quad m \in \mathbb{N}_0,$$

for $j = \overline{0,k}$.

So $(x_m^{(j)})_{m\ge -l}$, $j=\overline{0,k}$, are the $k+1$ solutions to the equation

$$x_n = g(x_{n-1}, x_{n-2}, \dots, x_{n-l}), \quad n \in \mathbb{N}_0, \tag{20}$$

with the initial values $x_{-t}^{(j)}$, $t=\overline{1,l}$, for $j=\overline{0,k}$, respectively.

If we go back from Eq. (20), in this way we obtain Eq. (19), which consists of $k+1$ non-related samples of Eq. (20). Thus, if we know some results about solutions to Eq. (20), then the results are almost trivially transferred to the solutions to Eq. (19). So, there is no essential need to publish the corresponding results on Eq. (19).

2.2 Some examples of difference equations with interlacing indices

Here we show that some of Eqs. (5)–(18) are also examples of difference equations with interlacing indices, and to each of such equations we determine the corresponding difference equation with non-interlacing indices, and give a list of several equations with interlacing indices belonging to the same class.

Example 1 Equation (6) is a difference equation with interlacing indices. Indeed, from (6) we see that x_{n+1} is presented in terms of x_{n-1}, x_{n-3} and x_{n-5}, and that the indices of the subsequent members form an arithmetic progression with the difference equal to 2, since $n+1-(n-1)=n-1-(n-3)=n-3-(n-5)=2$. This means the set of indices $\mathbb{N}_{-5} := \{n \in \mathbb{Z} : n \ge -5\}$ is partitioned into two disjoint subsets. The set of even and odd indices belonging to $\mathbb{N}_{-5}$. Hence, the subsequences $(x_{2m})_{m\ge -2}$ and $(x_{2m-1})_{m\ge -2}$ are two solutions to the following difference equation with non-interlacing indices:

$$y_{m+1} = \frac{y_{m-2}}{1 + y_m y_{m-1}}. \tag{21}$$

By using the method described above, from (21) is obtained the following "general" equation with interlacing indices:

$$y_{n+1} = \frac{y_{n-(3k+2)}}{1 + y_{n-k} y_{n-(2k+1)}}, \quad n \in \mathbb{N}_0, \tag{22}$$

where $k \in \mathbb{N}_0$.

Equation (22), beside Eq. (6), which is obtained for $k=1$, contains, for example, the following special cases, which are obtained for $k=2,3,4,5,6$, respectively:

$$y_{n+1} = \frac{y_{n-8}}{1 + y_{n-2} y_{n-5}}, \quad n \in \mathbb{N}_0, \tag{23}$$

$$y_{n+1} = \frac{y_{n-11}}{1 + y_{n-3} y_{n-7}}, \quad n \in \mathbb{N}_0, \tag{24}$$

$$y_{n+1} = \frac{y_{n-14}}{1 + y_{n-4} y_{n-9}}, \quad n \in \mathbb{N}_0, \tag{25}$$

$$y_{n+1} = \frac{y_{n-17}}{1 + y_{n-5} y_{n-11}}, \quad n \in \mathbb{N}_0 \tag{26}$$

$$y_{n+1} = \frac{y_{n-20}}{1 + y_{n-6} y_{n-13}}, \quad n \in \mathbb{N}_0. \tag{27}$$

Example 2 Equation (8) is also another example of a difference equation with interlacing indices. Indeed, from Eq. (8) we see that x_{n+1} is presented in terms of the following four members: x_{n-3}, x_{n-7}, x_{n-11} and x_{n-15}, and also note that their indices form an arithmetic progression with the difference equal to four (simply note that the following sequence of equalities holds: $n+1-(n-3) = n-3-(n-7) = n-7-(n-11) = n-11-(n-15) = 4$). This means that the set of indices $\mathbb{N}_{-15}$ is partitioned into four disjoint subsets. Hence, the subsequences $(x_{4m+j})_{m\geq -4}$, $j = \overline{1,4}$, are four solutions to the following difference equation with non-interlacing indices:

$$y_{m+1} = \frac{y_{m-3}}{1 + y_m y_{m-1} y_{m-2}}. \tag{28}$$

By using the method described above, from (28) is obtained the following sequence of equations with interlacing indices:

$$y_{n+1} = \frac{y_{n-(4k+3)}}{1 + y_{n-k} y_{n-(2k+1)} y_{n-(3k+2)}}, \quad n \in \mathbb{N}_0, \tag{29}$$

where $k \in \mathbb{N}_0$.

Equation (29), beside Eq. (8), which is obtained for $k = 3$, contains, for example, the following special cases, which are obtained for $k = 2, 4, 5, 6, 7$, respectively:

$$y_{n+1} = \frac{y_{n-11}}{1 + y_{n-2} y_{n-5} y_{n-8}}, \quad n \in \mathbb{N}_0, \tag{30}$$

$$y_{n+1} = \frac{y_{n-19}}{1 + y_{n-4} y_{n-9} y_{n-14}}, \quad n \in \mathbb{N}_0, \tag{31}$$

$$y_{n+1} = \frac{y_{n-23}}{1 + y_{n-5} y_{n-11} y_{n-17}}, \quad n \in \mathbb{N}_0, \tag{32}$$

$$y_{n+1} = \frac{y_{n-27}}{1 + y_{n-6} y_{n-13} y_{n-20}}, \quad n \in \mathbb{N}_0, \tag{33}$$

$$y_{n+1} = \frac{y_{n-31}}{1 + y_{n-7} y_{n-15} y_{n-23}}, \quad n \in \mathbb{N}_0. \tag{34}$$

Remark 2 Note that such obtained Eq. (30), is in fact Eq. (9), whereas Eq. (32) is, in fact, Eq. (14). Thus the above analysis given in Example 2 shows that Eqs. (8), (9) and (14) are obtained from the same difference equation, that is, from Eq. (29), by using the method described above. Note also that Eq. (12) is nothing but Eq. (29), which is also obtained from Eq. (28) by the method described above. Finally, note that Eq. (7) is, in fact, Eq. (28).

Example 3 Equation (15) is also another difference equation with interlacing indices. Indeed, from the relation in (15) we see that the member x_{n+1} is presented in terms of the following ones: x_{n-2}, x_{n-5}, x_{n-8}, x_{n-11} and x_{n-14}, and that the indices of the subsequent ones form an arithmetic progression with the difference equal to three, since $n+1-(n-2) = n-2-(n-5) = n-5-(n-8) = n-8-(n-11) = n-11-(n-14) = 3$. This means that the set of indices $\mathbb{N}_{-14}$ is partitioned into three disjoint subsets. Hence, the subsequences $(x_{3m+j})_{m\geq -5}$, $j = \overline{1,3}$, are three solutions to the following difference equation with non-interlacing indices:

$$y_{m+1} = \frac{y_{m-4}}{1 + y_m y_{m-1} y_{m-2} y_{m-3}}. \tag{35}$$

By using the method described above, from (35) is obtained the following equation with interlacing indices:

$$y_{n+1} = \frac{y_{n-(5k+4)}}{1 + y_{n-k}y_{n-(2k+1)}y_{n-(3k+2)}y_{n-(4k+3)}}, \quad n \in \mathbb{N}_0, \tag{36}$$

where $k \in \mathbb{N}_0$.

Equation (36), beside Eq. (15), which is obtained for $k = 2$, contains, for example, the following special cases, which are obtained for $k = 3, 4, 5, 6, 7$, respectively:

$$y_{n+1} = \frac{y_{n-19}}{1 + y_{n-3}y_{n-7}y_{n-11}y_{n-15}}, \quad n \in \mathbb{N}_0, \tag{37}$$

$$y_{n+1} = \frac{y_{n-24}}{1 + y_{n-4}y_{n-9}y_{n-14}y_{n-19}}, \quad n \in \mathbb{N}_0, \tag{38}$$

$$y_{n+1} = \frac{y_{n-29}}{1 + y_{n-5}y_{n-11}y_{n-17}y_{n-23}}, \quad n \in \mathbb{N}_0, \tag{39}$$

$$y_{n+1} = \frac{y_{n-34}}{1 + y_{n-6}y_{n-13}y_{n-20}y_{n-27}}, \quad n \in \mathbb{N}_0, \tag{40}$$

$$y_{n+1} = \frac{y_{n-39}}{1 + y_{n-7}y_{n-15}y_{n-23}y_{n-31}}, \quad n \in \mathbb{N}_0. \tag{41}$$

Example 4 Equation (10) is also a difference equation with interlacing indices. Indeed, from (10) we see that the member x_{n+1} is presented in terms of the following ones: x_{n-1}, x_{n-3}, x_{n-5}, x_{n-7}, x_{n-9} and x_{n-11}, and that the indices of the subsequent members form an arithmetic progression with the difference equal to two. This means that the set of indices $\mathbb{N}_{-11}$ is partitioned into two disjoint subsets. Hence, the subsequences $(x_{2m+j})_{m\geq -6}$, $j = \overline{1,2}$, are two solutions to the following difference equation with non-interlacing indices:

$$y_{m+1} = \frac{y_{m-5}}{1 + y_m y_{m-1}y_{m-2}y_{m-3}y_{m-4}}. \tag{42}$$

By using the method described above, from (42) is obtained the following equation with interlacing indices:

$$y_{n+1} = \frac{y_{n-(6k+5)}}{1 + y_{n-k}y_{n-(2k+1)}y_{n-(3k+2)}y_{n-(4k+3)}y_{n-(5k+4)}}, \quad n \in \mathbb{N}_0, \tag{43}$$

where $k \in \mathbb{N}_0$.

Equation (43), beside Eq. (10), which is obtained for $k = 1$, contains, for example, the following special cases, which are obtained for $k = 2, 3, 4, 5, 6$, respectively:

$$y_{n+1} = \frac{y_{n-17}}{1 + y_{n-2}y_{n-5}y_{n-8}y_{n-11}y_{n-14}}, \quad n \in \mathbb{N}_0, \tag{44}$$

$$y_{n+1} = \frac{y_{n-23}}{1 + y_{n-3}y_{n-7}y_{n-11}y_{n-15}y_{n-19}}, \quad n \in \mathbb{N}_0, \tag{45}$$

$$y_{n+1} = \frac{y_{n-29}}{1 + y_{n-4}y_{n-9}y_{n-14}y_{n-19}y_{n-24}}, \quad n \in \mathbb{N}_0, \tag{46}$$

$$y_{n+1} = \frac{y_{n-35}}{1 + y_{n-5}y_{n-11}y_{n-17}y_{n-23}y_{n-29}}, \quad n \in \mathbb{N}_0, \tag{47}$$

$$y_{n+1} = \frac{y_{n-41}}{1 + y_{n-6}y_{n-13}y_{n-20}y_{n-27}y_{n-34}}, \quad n \in \mathbb{N}_0. \tag{48}$$

Example 5 Equation (16) is also a difference equation with interlacing indices. Indeed, from (16) we see that x_{n+1} is presented in terms of x_{n-2}, x_{n-5}, x_{n-8}, x_{n-11}, x_{n-14}, x_{n-17} and x_{n-20}, and that their indices form an arithmetic progression with the difference equal to three. This means that the set of indices $\mathbb{N}_{-20}$ is partitioned into three disjoint subsets. Hence, the subsequences $(x_{3m+j})_{m\geq -7}$, $j=\overline{1,3}$, are three solutions to the following difference equation with non-interlacing indices:

$$y_{m+1}=\frac{y_{m-6}}{1+y_m y_{m-1} y_{m-2} y_{m-3} y_{m-4} y_{m-5}}. \tag{49}$$

By using the method described above, from (49) is obtained the following equation with interlacing indices:

$$y_{n+1}=\frac{y_{n-(7k+6)}}{1+y_{n-k} y_{n-(2k+1)} y_{n-(3k+2)} y_{n-(4k+3)} y_{n-(5k+4)} y_{n-(6k+5)}}, \quad n\in\mathbb{N}_0, \tag{50}$$

where $k\in\mathbb{N}_0$.

Equation (50), beside Eq. (16), which is obtained for $k=2$, contains, for example, the following special cases, which are obtained for $k=1,3,4,5,6$, respectively:

$$y_{n+1}=\frac{y_{n-13}}{1+y_{n-1} y_{n-3} y_{n-5} y_{n-7} y_{n-9} y_{n-11}}, \quad n\in\mathbb{N}_0, \tag{51}$$

$$y_{n+1}=\frac{y_{n-27}}{1+y_{n-3} y_{n-7} y_{n-11} y_{n-15} y_{n-19} y_{n-23}}, \quad n\in\mathbb{N}_0, \tag{52}$$

$$y_{n+1}=\frac{y_{n-34}}{1+y_{n-4} y_{n-9} y_{n-14} y_{n-19} y_{n-24} y_{n-29}}, \quad n\in\mathbb{N}_0, \tag{53}$$

$$y_{n+1}=\frac{y_{n-41}}{1+y_{n-5} y_{n-11} y_{n-17} y_{n-23} y_{n-29} y_{n-35}}, \quad n\in\mathbb{N}_0, \tag{54}$$

$$y_{n+1}=\frac{y_{n-48}}{1+y_{n-6} y_{n-13} y_{n-20} y_{n-27} y_{n-34} y_{n-41}}, \quad n\in\mathbb{N}_0. \tag{55}$$

Remark 3 Note that such obtained Eq. (51) is, in fact, Eq. (17). Thus the analysis in Example 5 shows that both, Eq. (16) and Eq. (17) are obtained from the same equation, that is, from Eq. (50), by using the method described above.

Example 6 Equation (11) is also a difference equation with interlacing indices. Indeed, from (11) we see that x_{n+1} is presented in terms of the $k+2$ members $x_{n-(5j+4)}$, $j=\overline{0,k+1}$, and that the indices of the subsequent members form an arithmetic progression with the difference equal to five, since $n-(5(j-1)+4)-(n-(5j+4))=5$, $j=\overline{1,k+1}$. This means that the set of indices $\mathbb{N}_{-(5k+9)}$ is partitioned into five disjoint subsets. Hence, the subsequences $(x_{5m+j})_{m\geq -(k+2)}$, $j=\overline{1,5}$, are five solutions to the following difference equation with non-interlacing indices:

$$y_{m+1}=\frac{y_{m-(k+1)}}{1+y_m y_{m-1}\cdots y_{m-k}}. \tag{56}$$

By using the method described above, from (56) is obtained the following equation with interlacing indices:

$$y_{n+1}=\frac{y_{n-(k+2)(s+1)+1}}{1+y_{n-s} y_{n-(2s+1)}\cdots y_{n-(k+1)(s+1)+1}}, \quad n\in\mathbb{N}_0, \tag{57}$$

where $k \in \mathbb{N}_0$. Note that if we choose $k = p - 2$ and $s = l - 1$ Eq. (57) becomes Eq. (18).

Equation (57), beside Eq. (11), which is obtained for $s = 4$, contains, for example, the following special cases, which are obtained for $s = 2, 3, 5, 6, 7$, respectively:

$$y_{n+1} = \frac{y_{n-3(k+2)+1}}{1 + y_{n-2}y_{n-5}\cdots y_{n-3(k+1)+1}}, \quad n \in \mathbb{N}_0, \tag{58}$$

$$y_{n+1} = \frac{y_{n-4(k+2)+1}}{1 + y_{n-3}y_{n-7}\cdots y_{n-4(k+1)+1}}, \quad n \in \mathbb{N}_0, \tag{59}$$

$$y_{n+1} = \frac{y_{n-6(k+2)+1}}{1 + y_{n-5}y_{n-11}\cdots y_{n-6(k+1)+1}}, \quad n \in \mathbb{N}_0, \tag{60}$$

$$y_{n+1} = \frac{y_{n-7(k+2)+1}}{1 + y_{n-6}y_{n-13}\cdots y_{n-7(k+1)+1}}, \quad n \in \mathbb{N}_0, \tag{61}$$

$$y_{n+1} = \frac{y_{n-8(k+2)+1}}{1 + y_{n-7}y_{n-15}\cdots y_{n-8(k+1)+1}}, \quad n \in \mathbb{N}_0. \tag{62}$$

Remark 4 To the best of our knowledge the following difference equations have not been studied separately yet:

$$y_{n+1} = \frac{y_{n-8}}{1 + y_{n-1}y_{n-2}y_{n-3}y_{n-4}y_{n-5}y_{n-6}y_{n-7}}, \quad n \in \mathbb{N}_0, \tag{63}$$

$$y_{n+1} = \frac{y_{n-15}}{1 + y_{n-1}y_{n-3}y_{n-5}y_{n-7}y_{n-9}y_{n-11}y_{n-13}}, \quad n \in \mathbb{N}_0, \tag{64}$$

$$y_{n+1} = \frac{y_{n-23}}{1 + y_{n-2}y_{n-5}y_{n-8}y_{n-11}y_{n-14}y_{n-17}y_{n-20}}, \quad n \in \mathbb{N}_0, \tag{65}$$

$$y_{n+1} = \frac{y_{n-31}}{1 + y_{n-3}y_{n-7}y_{n-11}y_{n-15}y_{n-19}y_{n-23}y_{n-27}}, \quad n \in \mathbb{N}_0, \tag{66}$$

$$y_{n+1} = \frac{y_{n-39}}{1 + y_{n-4}y_{n-9}y_{n-14}y_{n-19}y_{n-24}y_{n-29}y_{n-34}}, \quad n \in \mathbb{N}_0, \tag{67}$$

$$y_{n+1} = \frac{y_{n-47}}{1 + y_{n-5}y_{n-11}y_{n-17}y_{n-23}y_{n-29}y_{n-35}y_{n-41}}, \quad n \in \mathbb{N}_0, \tag{68}$$

$$y_{n+1} = \frac{y_{n-55}}{1 + y_{n-6}y_{n-13}y_{n-20}y_{n-27}y_{n-34}y_{n-41}y_{n-48}}, \quad n \in \mathbb{N}_0. \tag{69}$$

But the above quite detailed explanation shows that they really do not deserve separate investigations if they only consider the corresponding statements listed in Theorem 1.

Remark 5 Now note that the main equations in Examples 1–5, that is, Eqs. (21), (28), (35), (42) and (49) are special cases of the main equation in Example 6, that is, of Eq. (56), and that all other equations including (23)–(27), (30)–(34), (37)–(41), (44)–(48), (51)–(55), (58)–(62), are obtained by the method described above for forming difference equations with interlacing indices. The same observation holds for Eqs. (63)–(69).

2.3 Some comments and a theorem

The observations given in Remark 5 shows that the only difference equation which deserved to be studied is Eq. (56), since the corresponding results for the other difference equations with interlacing indices easily follow from obtained results on the equation. Note also that Eq. (56) with $k = 0$ reduces to Eq. (1), which, as we have already mentioned, was studied in [26].

Far the most difficult problem concerning solutions to Eq. (56), out of the studied ones, was the problem of the existence of a positive solution to Eq. (56) converging to zero. Recall that the problem was solved in [28] by using asymptotic methods, more precisely, by finding a finite asymptotic expansion of a suitably chosen solution to the equation.

Later in [9] were found some other solutions, and it was also shown that some of the corresponding statements in Theorem 1 also hold (see [9, p.583] for the corresponding statements *(a)–(c)*). The properties corresponding to the statements *(d)* and *(e)* of the theorem were proved in [31], but are not explicitly mentioned in [9] (recall that [31] was cited as reference [14] in [9] and as reference [7] in [7]). One of the reasons for not including these two statements in [9] was the fact that they are not used in the proofs of the main results in [9].

As we have already mentioned, preprint [31] has never been published, and although it circulated among a number of experts its content has not become widely known. Beside this, our impression was that the statements are proved by some arguments similar to the corresponding ones in the proof of Theorem 1 in [26], so that the results might not be much interesting, especially since in [28] the main problem had already been solved. However, the number of papers devoted to the investigation of the class of difference equations shows that the impression of ours was wrong. These facts, among other things, motivate us to present here the statements and their proofs.

Now we formulate and prove the generalization of Theorem 1 presented in [31]. Shorter versions of the proofs of the first two statements in the generalization of Theorem 1 are given in [9]. We present here also detailed proofs of these statements for the completeness and the benefit of the reader.

Theorem 3 *Consider Eq.* (4). *Let*

$$\min\{x_{-k}, x_{-k+1}, \dots, x_{-1}\} > 0. \tag{70}$$

Then for every solution $(x_n)_{n\ge -k}$ *to Eq.* (4) *satisfying condition* (70), *the following statements hold.*

(a) *The subsequences* $(x_{km+r})_{m\ge -1}$, $r = \overline{0, k-1}$, *are decreasing and bounded.*

(b) *Let*

$$p_r = \lim_{m\to\infty} x_{km+r}, \quad r = \overline{0, k-1}. \tag{71}$$

Then the sequence defined by

$$\widehat{x}_{km+r} := p_r, \quad m \ge -1, r = \overline{0, k-1}, \tag{72}$$

is a nonnegative periodic solution to Eq. (4), *such that*

$$\prod_{i=0}^{k-1} p_i = 0. \tag{73}$$

(c) *Assume that* $l \in \{1, 2, \dots, k-1\}$, $\gcd(k, l) = 1$ *and there is* $n_0 \in \mathbb{N}$ *such that the following condition holds*:

$$x_n \le x_{n-l}, \quad n \ge n_0. \tag{74}$$

Then

$$\lim_{n\to+\infty} x_n = 0. \tag{75}$$

(d) *The following formula holds*:

$$x_{mk+r} = x_{r-k}\left(1 - \frac{x_{r-1}\cdots x_{r-(k-1)}}{1+x_{r-1}\cdots x_{r-(k-1)}}\sum_{l=0}^{m}\prod_{i=0}^{lk-1}\frac{1}{1+\prod_{j=0}^{k-2}x_{lk+r-1-i-j}}\right)$$

for each $r \in \{0,1,\dots,k-1\}$.

Proof *(a)* Let $(x_n)_{n\ge -k}$ be such a solution to Eq. (4). Then from (4) and (70), we have

$$0 < x_n = \frac{x_{n-k}}{1+x_{n-1}\cdots x_{n-(k-1)}} < x_{n-k}, \quad n \in \mathbb{N}_0,$$

from which it follows that the subsequences $(x_{km+r})_{m\ge -1}$, $r = \overline{0,k-1}$, are decreasing and bounded, as claimed.

(b) First note that in view of *(a)* and a well-known theorem, it follows that the limits

$$\lim_{m\to+\infty} x_{km+r},$$

$r = \overline{0,k-1}$, are finite numbers. That (72) is a nonnegative sequence is clear. To prove that it is a periodic solution to Eq. (4) easily follows by taking the limits in the following relations obtained from (4):

$$x_{mk+j} = \frac{x_{(m-1)k+j}}{1+x_{mk+j-1}\cdots x_{mk+j-(k-1)}},$$

where j can be any nonnegative integer.

If we assume that some of the numbers $p_0, p_1, \dots, p_{k-1}$ are equal to zero, then clearly Eq. (73) follows. Now assume that there is $i_0 \in \{0,1,\dots,k-1\}$ such that $p_{i_0} \ne 0$. Then by letting $m \to +\infty$ in the following consequence of Eq. (4):

$$x_{mk+i_0} = \frac{x_{(m-1)k+i_0}}{1+x_{mk+i_0-1}\cdots x_{mk+i_0-(k-1)}},$$

it follows that

$$p_{i_0} = \frac{p_{i_0}}{1+p_{i_0-1}\cdots p_{i_0-(k-1)}}$$

and consequently

$$\prod_{j=0}^{k-1} p_{i_0-j} = 0,$$

from which together with the periodicity of the sequence defined in (72), or simply by using the convention $p_l = p_s$ if $l = s \pmod{k}$, Eq. (73) follows.

(c) From (74) it follows that the subsequences (x_{ml+s}), $s = \overline{0, l-1}$, are convergent. Let

$$q_s = \lim_{m\to+\infty} x_{ml+s}, \quad s = \overline{0, l-1}.$$

Then clearly the sequence defined by

$$\widetilde{x}_{lm+s} := q_s,$$

for $m \in \mathbb{Z}$ and $s = \overline{0, l-1}$ such that $ml + s \geq -k$, is an l-periodic nonnegative solution to Eq. (4).

Let $(L_{-i})_{i\in\mathbb{Z}}$ be a full limiting sequence [11, 12] of the solution $(x_n)_{n\geq-k}$. Employing the statement in *(b)* we see that $(L_{-i})_{i\in\mathbb{Z}}$ is a k-periodic solution to Eq. (4) and from the above-mentioned fact it follows that $(L_{-i})_{i\in\mathbb{Z}}$ is an l-periodic solution to Eq. (4). From this we see that $(L_{-i})_{i\in\mathbb{Z}}$ is periodic by periods k and l, and consequently it is periodic by period $\gcd(k, l) = 1$ (see, for example, [38]). Hence, the solution $(x_n)_{n\geq-k}$ is convergent, from which it follows that $\lim_{n\to\infty} x_n = 0$.

(d) From (4) we have

$$x_n - x_{n-k} = -x_n x_{n-1} \cdots x_{n-(k-1)}. \tag{76}$$

Further, from (4) and by using the Eq. (76) where n is replaced by $n-1$, it follows that

$$\begin{aligned} x_n - x_{n-k} &= -\frac{x_{n-1}\cdots x_{n-(k-1)}x_{n-k}}{1 + x_{n-1}\cdots x_{n-(k-1)}} \\ &= \frac{x_{n-1} - x_{n-(k+1)}}{1 + x_{n-1}\cdots x_{n-(k-1)}}, \end{aligned}$$

for $n \in \mathbb{N}$, which implies that

$$x_n - x_{n-k} = (x_r - x_{r-k}) \prod_{i=0}^{n-1-r} \frac{1}{1 + \prod_{j=0}^{k-2} x_{n-1-i-j}}, \tag{77}$$

for each $r \in \{0, 1 \ldots, k-1\}$.

Setting $n = jk + r$ for $j = \overline{0, m}$, in (77) and summing such obtained equalities we obtain

$$\begin{aligned} x_{mk+r} &= x_{r-k} + (x_r - x_{r-k}) \sum_{l=0}^{m} \prod_{i=0}^{lk-1} \frac{1}{1 + \prod_{j=0}^{k-2} x_{lk+r-1-i-j}} \\ &= x_{r-k} - \frac{x_{r-1}\cdots x_{r-k}}{1 + x_{r-1}\cdots x_{r-(k-1)}} \sum_{l=0}^{m} \prod_{i=0}^{lk-1} \frac{1}{1 + \prod_{j=0}^{k-2} x_{lk+r-1-i-j}} \\ &= x_{r-k}\left(1 - \frac{x_{r-1}\cdots x_{r-(k-1)}}{1 + x_{r-1}\cdots x_{r-(k-1)}} \sum_{l=0}^{m} \prod_{i=0}^{lk-1} \frac{1}{1 + \prod_{j=0}^{k-2} x_{lk+r-1-i-j}}\right), \end{aligned}$$

finishing the proof of the theorem. □

Remark 6 Equation (56), which is equivalent to Eq. (4) (the delay k in Eq. (4) corresponds to the delay $k+2$ in Eq. (56)), was also studied in [24], where it was noticed that some slight modifications of the proofs of the statements in Theorem 1 in [26], prove some of the corresponding results for the statements.

Remark 7 Our statement *(c)* in Theorem 3 is more general than the corresponding ones in [24, 25], as well as statements in other papers dealing with Eqs. (6)–(18). Here we give two examples from [24] and [25].

Case of Eq. (13). The essential delay of Eq. (5) studied in [25], that is, the delay of the corresponding non-interlacing difference equation, is equal to three, so $k = 3$, whereas from the assumption $x_{n+1} \leq x_{n-11}$, $n \geq n_0$, posed in Theorem 2.1 *(d)* therein, it follows that the corresponding non-interlacing assumption is $x_n \leq x_{n-2}$, which means that $l = 2$. Since $\gcd(3, 2) = 1$, their statement follows from our Theorem 3 *(d)*.

Case of Eq. (5). The delay of Eq. (13) studied in [24] is equal to $k + 2$, whereas from the assumption

$$x_{n+1} \leq x_{n-k}, \quad n \geq n_0,$$

posed in Theorem 1 *(d)* therein, we see that $l = k + 1$. Since

$$\gcd(k + 2, k + 1) = 1,$$

their statement follows from our Theorem 3 *(d)*.

In the case of the difference equations in (6)–(12), (14)–(18), the situation with the corresponding statements is similar. Namely, in all these cases to the corresponding delay, which we denote by k, the delay l in the corresponding assumption is equal to $k - 1$, and since

$$\gcd(k, k - 1) = 1,$$

all the results follow from our Theorem 3 *(d)*.

Remark 8 Regarding the statements corresponding to Theorem 1 *(f)* in [26], which appear in [24, 25], as well as in other papers dealing with the Eqs. (6)–(12), (14)–(18), it should be said that they are true, but they trivially follow from the statements corresponding to Theorem 1 *(c)* appearing in the papers, so the proofs given there are not necessary. Moreover, the proofs do not prove what the statements claim. Now we explain it for the case of the equations treated in [24] and [25].

Case of Eq. (13). Theorem 1 *(f)* in [24] states that, if

$$\lim_{n\to+\infty} x_{(k+2)n+j} := a_j \neq 0, \tag{78}$$

for $j = \overline{1, k+1}$, then

$$\lim_{n\to+\infty} x_{(k+2)n+k+2} := a_{k+2} = 0. \tag{79}$$

However, in Theorem 1 *(c)* therein, it was proved that

$$\prod_{j=1}^{k+2} a_j = 0. \tag{80}$$

But, if the assumptions in (78) hold, then from this and (80) the reation (79) is directly obtained.

Case of Eq. (5). Theorem 2.1 *(f)* in [25] states that, if

$$\lim_{n\to+\infty} x_{18n+j} := a_j \neq 0 \quad \text{and} \quad \lim_{n\to+\infty} x_{18n+j+6} := a_{j+6} \neq 0, \tag{81}$$

for $j = \overline{1,6}$, then

$$\lim_{n\to+\infty} x_{18n+j+12} := a_{j+12} = 0, \tag{82}$$

for $j = \overline{1,6}$.

On the other hand, in Theorem 2.1 *(c)* therein, it was proved that

$$\prod_{i=0}^{2} a_{j+6i} = 0, \quad j = \overline{1,6}. \tag{83}$$

However, if the relations in (81) hold, then from this and (83) Eq. (82) is directly obtained.

On the other hand, the proofs of these two statements given in [24] and [25], respectively, seem wrongly use the argument in the proof of Theorem 1 *(f)* in [26]. Namely, to prove Theorem 1 *(f)* in [24] it is started with the assumption: "Suppose that $a_1 = a_2 = \cdots = a_{k+2} = 0$" and there is obtained a contradiction. However, since $k \geq 3$ the assumption is not the negation of the assumptions in (78). Similarly, to prove Theorem 2.1 *(f)* in [25] they started with the assumption: "Suppose that $a_{k+1} = a_{k+7} = a_{k+13} = 0$ for $k = \overline{0,5}$" and there is obtained a contradiction. However, the assumption is not the negation of the assumptions in (81). The same situation holds for the proofs of the corresponding statements in the papers dealing with the Eqs. (6)–(12), (14)–(18).

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