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I. Mihai* · R. Rosca · L. Verstraelen

On a class of Einstein hypersurfaces immersed in a Riemannian manifold

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Abstract. Let $x : M \to \tilde{M}$ be an isometric immersion of a hypersurface M into an $(n+1)$-dimensional Riemannian manifold $\tilde{M}$ and let ρ_i ($i \in \{1, ..., n\}$) be the principal curvatures of M. We denote by E and P the distinguished vector field and the curvature vector field of M, respectively, in the sense of [8].

If M is structured by a P-parallel connection [7], then it is Einsteinian. In this case, all the curvature 2-forms are exact and other properties induced by E and P are stated.

The principal curvatures ρ_i are isoparametric functions and the set $(\rho_1, ..., \rho_n)$ defines an isoparametric system [10].

In the last section, we assume that, in addition, M is endowed with an almost symplectic structure. Then, the dual 1-form $\pi = P^\flat$ of P is symplectic harmonic. If M is compact, then its 2nd Betti number $b_2 \geq 1$.

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1. Preliminaries

Let (M, g) be an n-dimensional Riemannian C^∞-manifold and ∇ the Levi–Civita connection with respect to g. We denote by $\Gamma(TM)$ the set of sections of the tangent bundle TM and by $\flat : TM \to T^*M$ the *musical isomorphism* defined by g and by $\sharp : T^*M \to TM$ its inverse.

A function $\rho : \mathbf{R}^n \to \mathbf{R}$ is called *isoparametric* [10] if $\|\nabla\rho\|^2$ and div $(\nabla\rho)$ are functions of ρ. Recall $\Delta\rho = -\mathrm{div}(\nabla\rho)$.

More generally, a system of functions $F = (\rho_1, ..., \rho_n)$ such that

$$<\nabla\rho_i, \nabla\rho_j> = A_{ij}(F), \tag{1.1}$$

$$\mathrm{div}\,(\nabla\rho_i) = B_i(F), \tag{1.2}$$

$$[\nabla\rho_i, \nabla\rho_j] = \sum_{k=1}^{n} C_{ij}^k \nabla\rho_k, \tag{1.3}$$

I. Mihai: Faculty of Mathematics, Str. Academiei 14, 70109 Bucharest, Romania and Departement Wiskunde, K.U. Leuven, Celestijnenlaan 200 B, B 3001 Leuven, Belgium

R. Rosca: 59 Avenue Emile Zola, 75015 Paris, France

L. Verstraelen: Departement Wiskunde, K.U. Leuven, Celestijnenlaan 200 B, B 3001 Leuven, Belgium

* Research Fellow of the Research Council of the K.U. Leuven

($i, j, k \in \{1, ..., n\}$), where A_{ij}, B_i and C_{ij}^k are smooth functions, is said to be an *isoparametric system* [10].

Let $\mathcal{O} = \{e_1, ..., e_n\}$ be a local field of adapted vectorial frames over M and let $\mathcal{O}^* = \{\omega^1, ..., \omega^n\}$ be its associated coframe.

Then, the *soldering form dp* (i.e. the canonical vector valued 1-form associated with $\mathcal{O}$) is expressed by

$$dp = \omega^i \otimes e_i, \tag{1.4}$$

and E. Cartan's structure equations, written in indexless form, are

$$\nabla e = \theta \otimes e, \tag{1.5}$$

$$d\omega = -\theta \wedge \omega, \tag{1.6}$$

$$d\theta = -\theta \wedge \theta + \Theta. \tag{1.7}$$

In the above equations, θ and Θ, respectively, are the local connection forms in the tangent bundle TM and the curvature 2-forms of M.

2. Parallel connection

Let $x : (M, g) \to (\tilde{M}, \tilde{g})$ be the immersion of a C^∞-hypersurface M into an $(n+1)$-dimensional Riemannian manifold $\tilde{M}$.

Let $\mathcal{B}(\tilde{M})$ be the bundle of orthonormal frames $\mathcal{O} = \{e_A;\ A = 1, ..., n+1\}$ over $\tilde{M}$ and $\tilde{\nabla}$ the Levi–Civita connection with respect to $\tilde{g}$.

We will denote the induced elements on M by suppressing ˜ and put $N = e_{n+1}$ for the unit normal vector field of M. Then, the second fundamental form l associated with x is expressed by

$$l = - < dp, \nabla N > = \sum_{i=1}^{n} \theta_i^{n+1} \otimes \omega^i. \tag{2.1}$$

We assume (this is always possible) that l is diagonal, i.e. each vector e_i ($i = 1, ..., n$) is an eigenvector of l. Then, we may write

$$\theta_i^{n+1} = \rho_i \omega^i, \tag{2.2}$$

(no summation over i), where $\rho_i \in C^\infty(M)$ are the *principal curvatures* of M.

In [8], $P_0 = E = \sum_{i=1}^{n} e_i$ and $P_1 = P = \sum_{i=1}^{n} \rho_i e_i$ have been called, respectively, the *distinguished vector* field and the *curvature vector* field associated with x. More generally, $P_r = \sum_{i=1}^{n} \rho_i^r e_i$ is called the *r-th curvature vector* field associated with x, and following [9],

$$h_r = \frac{1}{n} \sum_{i=1}^{n} \rho_i^r = \frac{1}{n} g(E, P_r)$$

is the *r-th associated mean curvature* of M.

By (2.2) and the structure equation (1.5), one gets

$$\nabla N = -\sum_{i=1}^{n} \rho_i \omega^i \otimes e_i, \tag{2.3}$$

and it is seen that if all the principal curvatures ρ_i are equal, N is an umbilical section.

Denote by $L : T_pM \to T_pM$ the *Weingarten map*, i.e. $LZ = \nabla_Z N$ (L is linear and self adjoint).

One has

$$\nabla_{P_{r-1}} N = -P_r,$$

which shows that all the curvature vector fields P_r are (up to the sign) related by the Weingarten map.

We assume in this paper that M is structured by a *P-parallel connection*, in the sense of [6] and [7]. Consequently, the connection forms θ_i^j satisfy

$$\theta_i^j =< P, e_i \wedge e_j >= \rho_i \omega^j - \rho_j \omega^i, \tag{2.4}$$

where $\wedge$ denotes the wedge product of vector fields.

On behalf of the structure equations (1.6), we easily get, by (2.4), that

$$d\omega^i = \pi \wedge \omega^i, \tag{2.5}$$

where $\pi = P^\flat$ denotes the dual form of P.

Since all the 1-forms ω^i are exterior recurrent [3], having the same recurrence form π, it follows that π is closed, i.e. $d\pi = 0$, or equivalently,

$$\sum_{i=1}^{n} d\rho_i \wedge \omega^i = 0. \tag{2.6}$$

On the other hand, by the structure equations (1.5) and (2.4), one gets

$$\nabla \rho_i = \rho_i dp - \omega^i \otimes P + \rho_i \omega^i \otimes N, \tag{2.7}$$

(no summation over i), and since $\nabla_P e_i = \rho_i^2 N$, it follows that all the vector fields e_i, ($i = 1, ..., n$) are *P-parallel* on *TM*.

Theorem 1. *Let $x : M \to \tilde{M}$ be an isometric immersion of a hypersurface M into an $(n+1)$-dimensional Riemannian manifold $\tilde{M}$ and let ρ_i ($i = 1, ..., n$) be the principal curvatures associated with x.*

Let E and P, respectively, be the distinguished vector field and the curvature vector field of M.

If M is structured by a P-parallel connection, then M is an Einstein hypersurface of scalar curvature $2(n-1)\rho^2$ ($\rho^2 = ||P||^2 = constant$) and P is an invariant section for the canonical vector-valued 1-form ω_2 of the set of second-order frames on M.

In addition, the following properties hold:

i) *E and P commute and ρ_i represent, up to $(n-1)$, the divergences of the vector basis $\mathcal{O} = \{e_i\}$ on M (the indices correspond) and are first integrals of P;*
ii) *all the curvature forms on M are exact 2-forms and P is a recurrent dynamical system (in the sense of H. Poincaré) on M;*
iii) *the retracting space corresponding to $\Theta_i^j, \omega^i, \omega^j$ is of dimension 3;*
iv) *the brackets $[e_i, e_j]$ define infinitesimal conformal transformations of the curvature forms Θ_i^j (the indices correspond).* □

Proof. i) Taking the covariant differential of P, one finds, by (2.7),

$$\nabla P = \sum_{i=1}^{n} d\rho_i \otimes e_i + \rho^2 dp - \pi \otimes P + \upsilon \otimes N, \tag{2.8}$$

where $\rho^2 = \|P\|^2$ and

$$\upsilon = \sum_{i=1}^{n} \rho_i^2 \omega^i = P_2^\flat. \tag{2.9}$$

Hence, in order that P be also parallel in TM, it is necessary and sufficient that the differential of the principal curvatures ρ_i satisfy

$$d\rho_i = \rho_i \pi - \rho^2 \omega^i. \tag{2.10}$$

One derives that $\rho^2 =$ constant and

$$\nabla P = \upsilon \otimes N. \tag{2.11}$$

Since, in general, $\operatorname{tr} \nabla Z = \operatorname{div} Z$, one derives, by (2.7),

$$\operatorname{div} e_i = \sum_{j=1}^{n} g(\nabla_{e_j} e_i, e_j) = (n-1)\rho_i,$$

i.e. the components of P represent, up to $(n-1)$, the divergences of the vector basis $\mathcal{O} = \{e_i\}$.

Taking the covariant differential of the distinguished vector field E, one has, by (2.7),

$$\nabla E = n h_1 dp - \omega \otimes P + \pi \otimes N, \tag{2.12}$$

and remarking that $\upsilon(E) = \pi(P) = \rho^2$, we get, by (2.11),

$$[E, P] = 0, \tag{2.13}$$

that is, E and P commute.

Since, by (2.5), we may write

$$\omega^i \wedge \omega^j \wedge d\omega^i = 0,$$

it follows that all the 2-planes $\{e_i, e_j\}$ are integrable.

In addition, one easily finds $\mathcal{L}_P \rho_i = 0$, i.e. all the principal curvatures ρ_i are *first integrals* of P.

ii) Further, taking the exterior differential of the connection forms θ_i^j, one obtains, by (2.5) and (2.10),

$$d\theta_i^j = 2\pi \wedge \theta_i^j - 2\rho^2 \omega^i \wedge \omega^j. \tag{2.14}$$

Now, making use of the structure equations (1.7), after a standard calculation, we get

$$\Theta_i^j = \frac{1}{2} d\theta_i^j. \tag{2.15}$$

Thus, all the curvature forms are exact.

With respect to the local Riemannian volume element η of M and the local star operator, $*$, determined by a local orientation of M, one infers, by (2.10),

$$*\text{div } P = (\text{div } P)\eta = \mathcal{L}_P \eta = 0,$$

where $\mathcal{L}$ denotes the Lie derivative.

Hence, we may say that P defines a *recurrent dynamical system* (Poincaré's theorem of non-return).

We recall that the Ricci tensor field $\mathcal{R}$ and the scalar curvature S are defined by

$$\mathcal{R}(Z, Z') = \sum_{i=1}^{n} g\big(R(e_i, Z)Z', e_i\big),$$

(R is the curvature tensor of ∇),

$$S = \sum_{i=1}^{n} \mathcal{R}(e_i, e_i) = \sum_{i=1}^{n} R_{ii},$$

respectively.

By (2.14) and (2.15), one derives $R_{ii} = 2(\rho^2 - \rho_i^2)$. Consequently, the scalar curvature of the hypersurface M is given by

$$S = 2(n-1)\rho^2 = \text{ constant.}$$

This proves that M is an *Einstein* hypersurface.

By (2.4), (2.5), (2.10) and (2.14), one derives, after some calculations,

$$\mathcal{L}_P \omega^i = 0, \quad \mathcal{L}_P \theta_i^j = 0. \tag{2.16}$$

Then, if $\{e_i^k;\ i, k = 1, ..., n\}$ is a basis of the Lie algebra $gl(n, \mathbf{R})$ acting on $\mathbf{R}^n$, the last equations reveal that P is an *invariant section* for the canonical vector-valued 1-form

$$\omega_2 = \omega^i \otimes e_i + \theta_k^i \otimes e_i^k$$

of the set of 2-frames $\mathcal{O}^2 M$, i.e. the frames of second order (see also [4]) on M.

iii) We recall the following definition: let $\mathcal{I}$ be an ideal spanned by the linearly independent elements $\omega^i \in \mathcal{O}^*$ and a 2-form $\psi \in \wedge^2 M$. Then, following [2], if q is the smallest integer such that

$$\psi^{q+1} \wedge \omega^1 \wedge ... \wedge \omega^s = 0,$$

the retracting space $H(\mathcal{I})$ corresponding to $\mathcal{I}$ is of dimension $2q + s$ and has the Grassmannian coordinate vector $\phi^q \wedge \omega^1 \wedge ... \wedge \omega^s$.

Coming back to the case under discussion, it follows, on behalf of (2.14) and (2.15), that the retracting space $H(\mathcal{I})$ corresponding to $\Theta_i^j, \omega^i, \omega^j$ is of dimension 3.

iv) On the other hand, by (2.7), one finds

$$[e_i, e_j] = \rho_j e_i - \rho_i e_j, \tag{2.17}$$

and taking account of (2.10) and of the fact that Θ_i^j are closed, one derives

$$\mathcal{L}_{[e_i, e_j]} \Theta_i^j = (\rho_j - \rho_i) \Theta_i^j. \tag{2.18}$$

Hence, the brackets $[e_i, e_j]$ define infinitesimal conformal transformations of the curvature forms Θ_i^j $(i, j \in \{1, ..., n\})$. □

Theorem 2. *The vector field $P = \pi^\sharp$ is harmonic and all its components ρ_i (the principal curvatures of M) are eigenfunctions of Δ and have, as spectrum, $(n-1)\rho^2$.*

Moreover, ρ_i are isoparametric functions and the set $(\rho_1, ..., \rho_n)$ defines an isoparametric system on M. □

Proof. Taking the star operator, $*$, of $\pi = P^\flat = \rho_i \omega^i$, one has by a standard formula

$$*\pi = \sum_{i=1}^{n} (-1)^{i-1} \rho_i \omega^1 \wedge ... \wedge \hat{\omega}^i \wedge ... \wedge \omega^n. \tag{2.19}$$

Then, using (2.4), one immediatly gets

$$d(*\pi) = 0 \Rightarrow \delta\pi = 0, \tag{2.20}$$

(δ is the adjoint of d), which shows that π is a harmonic 1-form, or that P is a harmonic vector field.

Further, since, by (2.10), one has

$$\nabla \rho_i = \rho_i P - \rho^2 e_i, \tag{2.21}$$

and recalling that div $e_i = (n-1)\rho_i$, one derives

$$\Delta \rho_i = (n-1)\rho^2 \rho_i, \tag{2.22}$$

which shows that all the principal curvatures ρ_i of M are *eigenfunctions* of Δ corresponding to the same *eigenvalue* $(n-1)\rho^2$.

Hence, following a known definition, $(n-1)\rho^2$ is in the *spectrum* of Δ on $\wedge^1 M$ and $\{\rho_i\}$ form the *eigenspace* of the eigenvalue $(n-1)\rho^2$.

Further, one has

$$||\nabla\rho_i||^2 = \rho^2\big(\rho^2 - \rho_i^2\big).$$

Therefore, according to [10], it follows that ρ_i are isoparametric functions.

Moreover, since $\pi = P^\flat$, one gets by (2.21) that $\pi(\nabla\rho_i) = 0$, from which we derive

$$< \nabla\rho_i, \nabla\rho_j >= -\rho^2\rho_i\rho_j$$

and

$$[\nabla\rho_i, \nabla\rho_j] = \rho^2(\rho_i\nabla\rho_j - \rho_j\nabla\rho_i) + \rho_i\rho_j\big(\rho_i^3 - \rho_j^3\big)N.$$

Accordingly, by references to (1.1)–(1.3), one may say that the principal curvatures define an isoparametric system on M. □

3. Almost symplectic structure

We assume in this section that M is of even dimension, say $n = 2m$, and consider the almost symplectic form

$$\Omega = \sum_{a=1}^{m}\omega^a \wedge \omega^{a^*},\ \ a^* = a + m.$$

Taking the exterior differential of Ω, one finds, by (2.5),

$$d\Omega = 2\pi \wedge \Omega. \tag{3.1}$$

This proves that M is endowed with a *conformal symplectic structure* $CSp(m;\mathbf{R})$ having π as a covector of Lee (see also [6], [7]).

Denoting now, as usual, by $\Omega^\flat : TM \to T^*M$, $Z \mapsto -i_Z\Omega = {}^\flat Z$, $Z \in \Gamma(TM)$, the *symplectic isomorphism* and, in order to simplify, setting ${}^\flat P = \gamma$, one derives

$$d\gamma = 2\pi \wedge \gamma + 2\rho^2\Omega. \tag{3.2}$$

Next, after a short calculation, we find

$$d(\mathcal{L}_P\Omega) = 2\rho^2\pi \wedge \Omega. \tag{3.3}$$

Hence, by reference to [6], [7], P defines a relative conformal transformation of Ω.

It should also be noted that, by (3.1) and (3.3) and taking account that $\rho^2 =$ constant, one may state that $\mathcal{L}_P\Omega$ and $\rho^2\Omega$ are homologous.

Therefore, if M is compact, then its Betti number of order 2, $b_2 \geq 1$.

Since the eigenvalue corresponding to the eigenfunctions ρ_i is positive, we may assume that M is compact.

More generally, let $X = \sum_{A=1}^{2m} X^A e_A \in \Gamma(TM)$ be any tangent vector field and denote $\alpha = X^\flat$ and $\beta = -{}^\flat X$. By (2.7), we have

$$\nabla X = dX^A \otimes e_A + g(X, P)dp - \alpha \otimes P + X^A \rho_A \omega^A \otimes N.$$

If we assume that the differential of the components of X satisfy

$$\begin{cases} dX^a = x_a \omega^{a^*} + t\omega^a, \\ dX^{a^*} = x_{a^*}\omega^a + t\omega^{a^*}, \end{cases}$$

$x_a, x_{a^*}, t \in C^\infty(M)$, then, since

$$\beta = i_X \Omega = \sum_{a=1}^{m} (X^a \omega^{a^*} - X^{a^*} \omega^a),$$

one derives, by (2.5),

$$d\beta = \pi \wedge \beta + 2t\Omega,$$

which infers $dt + t\pi = 0$, i.e. the existence of such X implies that π is an exact form.

By a straightforward calculation, we find

$$\mathcal{L}_X \Omega = 2(g(P, X) + t)\Omega - \pi \wedge \beta.$$

By exterior differentiation, one gets

$$d(\mathcal{L}_X \Omega) = (df + 2(t + f)\pi) \wedge \Omega, \;\; f = g(P, X);$$

the above equation shows that the vector field X defines a relative conformal transformation of Ω.

Moreover, since π is an exact form, then, by reference to [1], X also defines a *weak automorphism* of Ω.

Further, recall that the Poisson bracket $(u, v)_{\mathcal{P}}$ of two Pfaffians u and v is defined by

$$(u, v)_{\mathcal{P}} = i_{[u^\sharp, v^\sharp]}\Omega,$$

$[u^\sharp, v^\sharp]$ is called the associated vector field of $(u, v)_{\mathcal{P}}$.

Coming back to the case under discussion, one finds, by (2.7),

$$(\omega^a, \omega^{a^*})_{\mathcal{P}} = i_{[e_a, e_{a^*}]}\Omega = \rho_a \omega^a + \rho_{a^*}\omega^{a^*},$$

(no summation), and taking the Lie derivative, one derives, by (2.5) and (2.10),

$$\mathcal{L}_{[e_a, e_{a^*}]}\Omega = 0.$$

Hence, the structure 2-form Ω is invariant by the vector fields associated with the Poisson brackets $(\omega^a, \omega^{a^*})_{\mathcal{P}}$.

Finally, we recall that if $u \in \wedge^1 M$ is any 1-form and $\tilde{*}$ and $\tilde{\delta}$, respectively, denote the *symplectic adjoint operator* and the *symplectic differentiation operator*, then one has

$$\tilde{*}u = \frac{1}{(2m-1)!} u \wedge \Omega^{2m-1}, \;\; \tilde{\delta} = \tilde{*}d\tilde{*}u.$$

In the case under discussion, we quickly find $\tilde{\delta}\pi = 0$.

This shows that π is also a *symplectic harmonic* form.

Summing up, we state the following:

Theorem 3. *If the Einstein hypersurface M defined in Sect. 2 is endowed with an almost symplectic structure defined by the 2-form Ω, then this structure is necessarily conformal symplectic, having $\pi = P^\flat$ as the associated Lee covector.*

This structure induces the following properties:

i) *the curvature vector field P defines a relative conformal transformation of Ω and $\mathcal{L}_P\Omega$ and $\rho^2\Omega$ are homologous;*

ii) *if X is any vector field such that its components X^a, X^{a^*} satisfy*

$$dX^a = x_a\omega^{a^*} + t\omega^a,\ dX^{a^*} = x_{a^*}\omega^a + t\omega^{a^*},$$

$(x_a, x_{a^*}, t \in C^\infty M,\ a \in \{1, ..., m\},\ a^* = a + m)$, then π is an exact form and X defines a weak automorphism of Ω;

iii) *Ω is invariant by the associated vector fields of the Poisson brackets $(\omega^a, \omega^{a^*})_{\mathcal{P}}$;*

iv) *π is symplectic harmonic.* □

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