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On a class of fuzzy sets defined by Orlicz functions

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Abstract. The idea of difference sequences of real (or complex) numbers was generalized by Et and Çolak [9]. In this paper, using the difference operator Δ^m and an Orlicz function, we introduce and examine a class of sequences of fuzzy numbers. We study some of their properties like completeness, solidity, symmetricity etc. We also give some inclusion relations related to this class.

1. Introduction

The concepts of fuzzy sets and fuzzy set operations were first introduced by Zadeh [30] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming. Matloka [20] introduced bounded and convergent sequences of fuzzy numbers, studied some of their properties and showed that every convergent sequence of fuzzy numbers is bounded. In addition, sequences of fuzzy numbers have been discussed by Altin *et al.* [1], Aytar and Pehlivan [3], Başarır and Mursaleen [4], Bilgin [5], Et *et al.* [8], Nuray and Savaş [22], Nuray [23], Savaş [27], Talo and Başar [28] and many others.

The study of Orlicz sequence spaces was initiated with a certain specific purpose in Banach space theory. Indeed, Lindberg [17] got interested in Orlicz spaces in connection with finding Banach spaces with symmetric Schauder bases having complementary subspaces isomorphic to c_0 or ℓ_p $(1 \leq p < \infty)$. Subsequently Lindenstrauss and Tzafriri [18] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to ℓ_p $(1 \leq p < \infty)$.

Recently, Parashar and Choudhary [26] have introduced and discussed some properties of the four sequence spaces defined by using an Orlicz function M, which generalized the well-known Orlicz sequence space ℓ_M and strongly summable sequence spaces $[C,1,p]$, $[C,1,p]_0$ and $[C,1,p]_\infty$. Later on, Mursaleen *et al.* [21], Nuray and Gülcü [24], Tripathy *et al.* [29] used the idea of an Orlicz function to construct some spaces of complex sequences. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in [14]. Orlicz spaces find a number of useful applications in the theory of nonlinear integral equations. Whereas the Orlicz sequence spaces are the generalizations of ℓ_p−spaces, the L_p−spaces find themselves enveloped in Orlicz spaces [11].

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The purpose of this paper is to introduce and study the sequence space $\ell(M, \Delta^m, p, F)$ which arises from the notation of generalized difference operator Δ^m and the concept of an Orlicz function. We examine some topological properties of this space and establish elementary connections about this space.

2. Definitions and Preliminaries

In this section, we give the following definitions which will be needed in the sequel.

Fuzzy sets are considered with respect to a nonempty base set X of elements of interest. The essential idea is that each element $x \in X$ is assigned a membership grade $u(x)$ taking values in $[0,1]$, with $u(x) = 0$ corresponding to nonmembership, $0 < u(x) < 1$ to partial membership, and $u(x) = 1$ to full membership. According to Zadeh a fuzzy subset of X is a nonempty subset $\{(x, u(x)) : x \in X\}$ of $X \times [0,1]$ for some function $u : X \longrightarrow [0,1]$. The function u itself is often used for the fuzzy set.

Let $C(\mathbb{R}^n)$ denote the family of all nonempty, compact, convex subsets of $\mathbb{R}^n$. If $\alpha, \beta \in \mathbb{R}$ and $A, B \in C(\mathbb{R}^n)$, then

$$\alpha(A+B) = \alpha A + \alpha B, \qquad (\alpha\beta)A = \alpha(\beta A), \qquad 1A = A$$

and if $\alpha, \beta \geq 0$, then $(\alpha+\beta)A = \alpha A + \beta A$. The distance between A and B is defined by the Haussdorff metric

$$\delta_\infty(A,B) = \max\{\sup_{a\in A} \inf_{b\in B} \| a-b \|, \sup_{b\in B} \inf_{a\in A} \| a-b \|\},$$

where $\| \,.\, \|$ denotes the usual Euclidean norm in $\mathbb{R}^n$. It is well known that $(C(\mathbb{R}^n), \delta_\infty)$ is a complete metric space.

Denote

$$L(\mathbb{R}^n) = \{u : \mathbb{R}^n \longrightarrow [0,1] \mid u \; satisfies \; (i)-(iv) \; below\},$$

where

i) u is normal, that is, there exists an $x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$;

ii) u is fuzzy convex, that is, for $x, y \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1$, $u(\lambda x + (1-\lambda)y) \geq \min[u(x), u(y)]$;

iii) u is upper semicontinuous;

iv) the closure of $\{x \in \mathbb{R}^n : u(x) > 0\}$, denoted by $[u]^0$, is compact.

If $u \in L(\mathbb{R}^n)$, then u is called a fuzzy number, and $L(\mathbb{R}^n)$ is said to be a fuzzy number space.

For $0 < \alpha \leq 1$, the α-level set $[u]^\alpha$ is defined by

$$[u]^\alpha = \{x \in \mathbb{R}^n : u(x) \geq \alpha\}.$$

Then from $(i)-(iv)$, it follows that the α-level sets $[u]^\alpha \in C(\mathbb{R}^n)$.
For the addition and scalar multiplication in $L(\mathbb{R}^n)$, we have

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha, \qquad [ku]^\alpha = k[u]^\alpha$$

where $u, v \in L(\mathbb{R}^n)$, $k \in \mathbb{R}$.

Define, for each $1 \leq q < \infty$,

$$d_q(u,v) = \left(\int_0^1 (\delta_\infty([u]^\alpha, [v]^\alpha))^q \, d\alpha \right)^{1/q}$$

and $d_\infty(u,v) = \sup_{0\leq\alpha\leq 1} \delta_\infty([u]^\alpha, [v]^\alpha)$, where δ_∞ is the Haussdorff metric. Clearly $d_\infty(u,v) = \lim_{q\longrightarrow\infty} d_q(u,v)$ with $d_q \leq d_s$ if $q \leq s$ ([6], [16]).

A sequence $X = (X_k)$ of fuzzy numbers is a function X from the set $\mathbb{N}$ of all positive integers into $L(\mathbb{R}^n)$. Thus, a sequence of fuzzy numbers X is a correspondence from the set of positive integers to a set of fuzzy numbers, i.e., to each positive integer k there corresponds a fuzzy number $X(k)$. It is more common to write X_k rather than $X(k)$ and to denote the sequence by $X = (X_k)$ rather than X. The fuzzy number X_k is called the k-th term of the sequence.

A sequence $X = (X_k)$ of fuzzy numbers is said to be bounded if the set $\{X_k : k \in \mathbb{N}\}$ of fuzzy numbers is bounded, and X is said to be convergent to the fuzzy number X_0, written as $\lim_k X_k = X_0$, if for every $\varepsilon > 0$ there exists a positive integer k_0 such that $d(X_k, X_0) < \varepsilon$ for $k > k_0$. Let $\ell_\infty(F)$ and $c(F)$ denote the set of all bounded sequences and all convergent sequences of fuzzy numbers, respectively.

The difference sequnce spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$, consisting of all real valued sequences $x = (x_k)$ such that $\Delta x = (x_k - x_{k+1})$ in the sequence spaces ℓ_∞, c and c_0, were defined by Kızmaz [15]. The idea of difference sequences was generalized by Et and Çolak [9] and studied by Altay and Başar [2], Et *et al.* ([7],[10],[12]), Isik [13], Malkowsky *et al.* [19] and many others.

A fuzzy sequence space $E(F)$ is said to be solid (or normal) if $(Y_k) \in E(F)$, for some $(X_k) \in E(F)$, whenever $d(Y_k, \bar{0}) \leq d(X_k, \bar{0})$ for all $k \in \mathbb{N}$.

Remark. If a fuzzy sequence space $E(F)$ is solid, then $E(F)$ is monotone.

A sequence space $E(F)$ is said to be symmetric if $\left(X_{\pi(n)}\right) \in E(F)$, whenever $(X_k) \in E(F)$, where π is a permutation of $\mathbb{N}$.

A sequence space $E(F)$ is said to be sequence algebra if $(X_k \otimes Y_k) \in E(F)$, whenever $(X_k), (Y_k) \in E(F)$.

Let $w(F)$ be the set of all sequences of fuzzy numbers. The operator $\Delta^m : w(F) \to w(F)$ is defined by

$$\left(\Delta^0 X\right)_k = X_k, \left(\Delta^1 X\right)_k = \Delta^1 X_k = X_k - X_{k+1}, (\Delta^m X)_k = \Delta^1 \left(\Delta^{m-1} X\right)_k, (m \geq 2), \text{ for all } k \in \mathbb{N}.$$

Definition 2.1 [8] Let $X = (X_k)$ be a sequence of fuzzy numbers. Then the sequence $X = (X_k)$ is said to be Δ^m−bounded if the set $\{\Delta^m X_k : k \in \mathbb{N}\}$ of fuzzy numbers is bounded, and X is said to be Δ^m−convergent to the fuzzy number X_0, written as $\lim_k \Delta^m X_k = X_0$, if for every $\varepsilon > 0$ there exists a positive integer k_0 such that $d(\Delta^m X_k, X_0) < \varepsilon$ for all $k > k_0$. By $\ell_\infty(\Delta^m, F)$ and $c(\Delta^m, F)$ we denote the sets of all Δ^m− bounded sequences and all Δ^m−convergent sequences of fuzzy numbers, respectively

Recall ([11],[14],[25]) that an Orlicz function is a function $M : [0, \infty) \to [0, \infty)$, which is continuous, non decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

Lindenstrauss and Tzafriri [18] used the idea of Orlicz function to construct the sequence space

$$\ell_M = \left\{x \in w : \textstyle\sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \quad \text{for some } \rho > 0\right\}.$$

The space ℓ_M is a Banach space with the norm

$$\|x\| = \inf\left\{\rho > 0 : \textstyle\sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\}$$

and this space is called an Orlicz sequence space. For $M(t) = t^p$, $1 \leq p < \infty$, the space ℓ_M coincides with the classical sequence space ℓ_p.

Definition 2.2 Two Orlicz functions M_1 and M_2 are said to be equivalent if there are positive constants α, β and x_0 such that $M_1(\alpha x) \leq M_2(x) \leq M_1(\beta x)$ for all x with $0 \leq x \leq x_0$.

The existing literature on Orlicz spaces appears to have been restricted to real or complex sequences. Now we will extend the idea to apply to sequences of fuzzy numbers.

Definition 2.3 Let M be an Orlicz function and $p = (p_k)$ be any sequence of strictly positive real numbers. We define the following set

$$\ell(M, \Delta^m, p, F) = \left\{X = (X_k) : \sum_{k=1}^{\infty} \left[M\left(\frac{d(\Delta^m X_k, \bar{0})}{\rho}\right)\right]^{p_k} < \infty, \text{ for some } \rho > 0\right\},$$

where

$$\bar{0}(t) = \begin{cases} 1, & t = (0, 0, 0, ..., 0) \\ 0, & \text{otherwise} \end{cases}.$$

We get the following sequence spaces from $\ell(M,\Delta^m,p,F)$ by giving particular values to p and M. Taking $p_k = 1$ for all $k \in \mathbb{N}$ we have

$$\ell(M,\Delta^m,F) = \left\{X = (X_k) : \sum_{k=1}^{\infty}\left[M\left(\frac{d(\Delta^m X_k,\bar{0})}{\rho}\right)\right] < \infty, \text{ for some } \rho > 0\right\},$$

if we take $M(x) = x$, then we have

$$\ell(\Delta^m,p,F) = \left\{X = (X_k) : \sum_{k=1}^{\infty}\left[d(\Delta^m X_k,\bar{0})\right]^{p_k} < \infty\right\},$$

if we take $p_k = 1$ for all $k \in \mathbb{N}$ and $M(x) = x$, then we have

$$\ell(\Delta^m,F) = \left\{X = (X_k) : \sum_{k=1}^{\infty} d(\Delta^m X_k,\bar{0}) < \infty\right\},$$

The sequence space $\ell(M,\Delta^m,p,F)$ contains some unbounded sequences of fuzzy numbers. To show this let $M(x) = x$, $p_k = 1$ for all $k \in \mathbb{N}$. Then the sequence $X = (X_k) = \left(\bar{k}^{m-1}\right)$ belongs to $\ell(M,\Delta^m,p,F)$. Actually, if $X = \left(\bar{k}^{m-1}\right)$, then $\Delta^m\left(\bar{k}^{m-1}\right) = \bar{0}$ and $d(\Delta^m X_k,\bar{0}) = 0$, and thus $\sum_{k=1}^{\infty} d(\Delta^m X_k,\bar{0}) < \infty$, but the sequence X is divergent and is not bounded.

For the classical number sequences, (x_k) converges to ℓ implies $(\Delta^m x_k)$ converges to 0, but this case does not hold for the sequences of fuzzy numbers. For this see the following example.

Example 1. Consider the sequence $X = (X_k)$ as follows:

$$X_k(x) = \begin{cases} \frac{k}{k+1}x + \frac{1-k}{1+k}, & \text{if } x \in \left[\frac{k-1}{k}, 2\right] \\ -\frac{k}{k+1}x + \frac{3k+1}{1+k}, & \text{if } x \in (2, \frac{3k+1}{k}] \\ 0, & \text{otherwise} \end{cases}$$

Then the sequence $X = (X_k)$ is convergent to fuzzy number ℓ_1, where

$$\ell_1 = \begin{cases} x-1, & \text{if } x \in [1,2] \\ -x+3, & \text{if } x \in (2,3] \\ 0, & \text{otherwise} \end{cases}.$$

We find the $\alpha-$level set of X_k and ΔX_k as follows respectively:

$$[X_k]^\alpha = \left[\frac{k-1}{k} + \frac{k+1}{k}\alpha, \frac{3k+1}{k} - \frac{k+1}{k}\alpha\right]$$

and

$$[\Delta X_k]^\alpha = \left[\frac{-2k^2-4k-1}{k^2+k} + \left(\frac{k+1}{k} + \frac{k+2}{k+1}\right)\alpha, \frac{2k^2+4k+1}{k^2+k} - \left(\frac{k+1}{k} + \frac{k+2}{k+1}\right)\alpha\right].$$

Then we have $\Delta X_k \to L$, where $[L]^\alpha = [-2+2\alpha, 2-2\alpha] \neq \bar{0}$.

3. Main Results

In this section we prove the main results of this paper related to $\ell(M, \Delta^m, p, F)$. We study their different properties and obtain some inclusion relations involving these sets.

Theorem 3.1 Let (p_k) be bounded. Then $\ell(M, \Delta^m, p, F)$ is closed under the operations of addition and scalar multiplication.

Proof. Omitted.

Theorem 3.2 The space $\ell(\Delta^m, p, F)$ is complete metric space with the metric

$$\delta_\Delta(X, Y) = \sum_{i=1}^{m} d(X_i, Y_i) + \left(\sum_{k=1}^{\infty} [d(\Delta^m X_k, \Delta^m Y_k)]^{p_k}\right)^{\frac{1}{K}}$$

where $K = \max\left(1, H = \sup_k p_k\right)$.

Proof. Let (X^s) be a Cauchy sequence in $\ell(\Delta^m, p, F)$, where $X^s = \left(X_i^s\right)_i = \left(X_1^s, X_2^s, ...\right) \in \ell(\Delta^m, p, F)$ for each $s \in \mathbb{N}$. Then

$$\delta_\Delta\left(X^s, X^t\right) = \sum_{i=1}^{m} d\left(X_i^s, X_i^t\right) + \left(\sum_{k=1}^{\infty} [d(\Delta^m X_k, \Delta^m Y_k)]^{p_k}\right)^{\frac{1}{K}} \to 0, \text{ as } s, t \to \infty.$$

Therefore $\sum_{i=1}^{m} d\left(X_i^s, X_i^t\right) \to 0$ and $\sum\limits_{k=1}^{\infty} [d(\Delta^m X_k, \Delta^m Y_k)]^{p_k} \to 0$ as $s, t \to \infty$.

Hence $\sum_{i=1}^{m} d\left(X_i^s, X_i^t\right) \to 0$ and $d\left(\Delta X_k^s, \Delta X_k^t\right) \to 0$ as $s, t \to \infty$, for each fixed $k \in \mathbb{N}$.

Now from

$$d\left(X_{k+m}^s, X_{k+m}^t\right) \leq d\left(\Delta^m X_k^s, \Delta^m X_k^t\right) + \binom{m}{0} d\left(X_k^s, X_k^t\right) + ... + \binom{m}{m-1} d\left(X_{k+m-1}^s, X_{k+m-1}^t\right)$$

we have $d\left(X_k^s, X_k^t\right) \to 0$, as $s, t \to \infty$, for each $k \in \mathbb{N}$. Therefore $\left(X_k^s\right)_s = \left(X_k^1, X_k^2, ...\right)$ is a Cauchy sequence in $L(\mathbb{R}^n)$. Since $L(\mathbb{R}^n)$ is complete, it is convergent

$$\lim_s X_k^s = X_k$$

say, for each $k \in \mathbb{N}$. Since (X^s) is a Cauchy sequence, for each $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon)$ such that

$$\delta_\Delta\left(X^s, X^t\right) < \varepsilon \text{ for all } s, t \geq n_0.$$

Hence we get

$$\sum_{i=1}^{m} d\left(X_i^s, X_i^t\right) < \varepsilon \text{ and } \sum_{k=1}^{\infty} [d(\Delta^m X_k, \Delta^m Y_k)]^{p_k} < \varepsilon^K, \text{ for all } s, t \geq n_0.$$

So we have

$$\lim_t \sum_{i=1}^{m} d\left(X_i^s, X_i^t\right) = \sum_{i=1}^{m} d\left(X_i^s, X_i\right) < \varepsilon$$

and

$$\lim_t \sum_{k=1}^{\infty} \left[d\left(\Delta^m X_k^s, \Delta^m X_k^t\right)\right]^{p_k} = \sum_{k=1}^{\infty} \left[d\left(\Delta^m X_k^s, \Delta^m X_k\right)\right]^{p_k} < \varepsilon^K$$

for all $n \in \mathbb{N}$ and $s \geq n_0$. This implies that $\delta_\Delta(X^s, X) < 2\varepsilon$, for all $s \geq n_0$, that is $X^s \to X$ as $s \to \infty$, where $X = (X_k)$. Since

$$\begin{aligned}\sum_{k=1}^{\infty}\left[d\left(\Delta^m X_k, \bar{0}\right)\right]^{p_k} &\leq \left\{\sum_{k=1}^{\infty}\left[d\left(\Delta^m X_k^{n_0}, \bar{0}\right) + d\left(\Delta^m X_k^{n_0}, \Delta^m X_k\right)\right]^{p_k}\right\} \\ &\leq D\sum_{k=1}^{\infty}\left[d\left(\Delta^m X_k^{n_0}, \bar{0}\right)\right]^{p_k} \\ &\quad + D\sum_{k=1}^{\infty}\left[d\left(\Delta^m X_k^{n_0}, \Delta^m X_k\right)\right]^{p_k}\end{aligned}$$

where $D = \max\left(1, 2^{H-1}\right)$, we obtain $X \in \ell(\Delta^m, p, F)$. Therefore $\ell(\Delta^m, p, F)$ is a complete metric space.

Theorem 3.3 Let $0 < p_k \leq q_k < \infty$ for each $k \in \mathbb{N}$. Then $\ell(M, \Delta^m, p, F) \subset \ell(M, \Delta^m, q, F)$.

Proof. Let $x \in \ell(M, \Delta^m, p, F)$. Then there exists some $\rho > 0$

$$\sum_{k=1}^{\infty}\left[M\left(\frac{d\left(\Delta^m X_k, \bar{0}\right)}{\rho}\right)\right]^{p_k} < \infty.$$

This implies that $M\left(\frac{d(\Delta^m X_k, \bar{0})}{\rho}\right) \leq 1$ for sufficiently large values of k, say $k \geq k_0$ for some fixed $k_0 \in \mathbb{N}$. Since $p_k \leq q_k$ for each $k \in \mathbb{N}$ we get

$$\left[M\left(\frac{d\left(\Delta^m X_k, \bar{0}\right)}{\rho}\right)\right]^{q_k} \leq \left[M\left(\frac{d\left(\Delta^m X_k, \bar{0}\right)}{\rho}\right)\right]^{p_k}$$

for all $k \geq k_0$, and therefore

$$\sum_{k \geq k_0}^{\infty}\left[M\left(\frac{d\left(\Delta^m X_k, \bar{0}\right)}{\rho}\right)\right]^{q_k} \leq \sum_{k \geq k_0}^{\infty}\left[M\left(\frac{d\left(\Delta^m X_k, \bar{0}\right)}{\rho}\right)\right]^{p_k}.$$

Hence we have

$$\sum_{k=1}^{\infty}\left[M\left(\frac{d\left(\Delta^m X_k, \bar{0}\right)}{\rho}\right)\right]^{q_k} < \infty,$$

and so that $X \in \ell(M, \Delta^m, q, F)$.

The following result is a consequence of the above theorem.

Corollary 3.4 i) If $0 < p_k \leq 1$ for all $k \in \mathbb{N}$. Then $\ell(M, \Delta^m, p, F) \subset \ell(M, \Delta^m, F)$,
ii) If $1 \leq p_k < \infty$ for all $k \in \mathbb{N}$. Then $\ell(M, \Delta^m, F) \subset \ell(M, \Delta^m, p, F)$.

The proof of the following result follows from Definition 2.2.

Theorem 3.5 Let M_1 and M_2 be two Orlicz functions. If M_1 and M_2 are equivalent then $\ell(M_1, \Delta^m, p, F) = \ell(M_2, \Delta^m, p, F)$.

Theorem 3.6 Let M_1 and M_2 be two Orlicz functions. Then $\ell(\Delta^m, F, M_1, p) \cap \ell(\Delta^m, F, M_2, p) \subset \ell(\Delta^m, F, M_1 + M_2, p)$.

Proof. Let $x \in \ell(\Delta^m, F, M_1, p) \cap \ell(\Delta^m, F, M_2, p)$, then we have

$$\begin{aligned}\left[(M_1 + M_2)\left(\frac{d\left(\Delta^m X_k, \bar{0}\right)}{\rho}\right)\right]^{p_k} &= \left[M_1\left(\frac{d\left(\Delta^m X_k, \bar{0}\right)}{\rho}\right) + M_2\left(\frac{d\left(\Delta^m X_k, \bar{0}\right)}{\rho}\right)\right]^{p_k} \\ &\leq D\left[M_1\left(\frac{d\left(\Delta^m X_k, \bar{0}\right)}{\rho}\right)\right]^{p_k} + D\left[M_2\left(\frac{d\left(\Delta^m X_k, \bar{0}\right)}{\rho}\right)\right]^{p_k}\end{aligned}$$

where $D = \max\left(1, 2^{H-1}\right)$, taking summation from $k = 1$ to ∞ in the above inequality, we get $X \in \ell\left(\Delta^m, F, M_1 + M_2, p\right)$.

Theorem 3.7 The inclusion $\ell\left(\Delta^{m-1}, F, M\right) \subseteq \ell\left(\Delta^m, F, M\right)$ is strict, for $m \geq 1$. In general $\ell\left(\Delta^i, F, M\right) \subseteq \ell\left(\Delta^m, F, M\right)$ for $i = 1, 2, ..., m-1$ and the inclusion is strict.

Proof. Choose $\rho = 2\rho_1$. Then we observe that $(X_k) \in \ell\left(\Delta^{m-1}, F, M\right)$ implies $(X_k) \in \ell\left(\Delta^m, F, M\right)$ from the following inequality

$$\left[M\left(\frac{d\left(\Delta^m X_k, \bar{0}\right)}{\rho}\right)\right] \leq \frac{1}{2}\left\{\left[M\left(\frac{d\left(\Delta^{m-1} X_k, \bar{0}\right)}{\rho_1}\right)\right] + \left[M\left(\frac{d\left(\Delta^{m-1} X_{k+1}, \bar{0}\right)}{\rho_1}\right)\right]\right\}.$$

We get

$$\ell\left(\Delta^i, F, M\right) \subseteq \ell\left(\Delta^m, F, M\right)$$

for $i = 0, 1, ..., m-1$ by applying induction. The sequence $\left(\bar{k}\right)$ belongs to $\ell\left(\Delta^2, F\right)$, but does not belong to $\ell\left(\Delta, F\right)$, for $M(x) = x$ and $p_k = 1$ for all $k \in \mathbb{N}$. Therefore the inclusion is strict for $= 0, 1, ..., m-1$. Actually, if $X = \left(\bar{k}\right)$, then $\Delta^2\left(\bar{k}\right) = \bar{0}$ and $\Delta\left(\bar{k}\right) = -\bar{1}$ and thus $d\left(\Delta^2 X_k, \bar{0}\right) = 0$ and $d\left(\Delta X_k, \bar{0}\right) = 1$. Therefore $\sum_{k=1}^{\infty} d\left(\Delta^2 X_k, \bar{0}\right) < \infty$, but $\sum_{k=1}^{\infty} d\left(\Delta X_k, \bar{0}\right) = \infty$.

Theorem 3.8 The sequence space $\ell\left(M, F\right)$ is solid and hence monotone, but the sequence space $\ell\left(M, \Delta^m, p, F\right)$ is not solid.

Proof. Let $(X_k) \in \ell\left(M, F\right)$ and (Y_k) be such that $d\left(Y_k, \bar{0}\right) \leq d\left(X_k, \bar{0}\right)$ for each $k \in \mathbb{N}$. Since M is non-decreasing, we get

$$\sum_{k=1}^{\infty} M\left(\frac{d\left(Y_k, \bar{0}\right)}{\rho}\right) \leq \sum_{k=1}^{\infty} M\left(\frac{d\left(X_k, \bar{0}\right)}{\rho}\right).$$

Hence $\ell\left(M, F\right)$ is solid and hence monotone. It follows from the following example that the space $\ell\left(\Delta^m, F, M, p\right)$ is not solid.

Example 2. Let $M(x) = x$, and $p_k = 1$ for all $k \in \mathbb{N}$ and consider the sequences $X = (X_k) = (\bar{1})$ and $Y = (Y_k) = \left((-\bar{1})^k\right)$. Then $d\left(Y_k, \bar{0}\right) = d\left(X_k, \bar{0}\right) = 1$, $\Delta X = (\bar{0})$ and so that $X \in \ell\left(\Delta, F\right)$, but $d\left(Y_k, \bar{0}\right) = 2$ and so that $X \notin \ell\left(\Delta, F\right)$. Hence $\ell\left(\Delta, F\right)$ is not solid.

Theorem 3.9 The sequence space $\ell\left(\Delta^m, F, M, p\right)$ is not symmetric.

Proof. It follows from the following example that the space $\ell\left(\Delta^m, F, M, p\right)$ is not symmetric.

Example 3. Let $M(x) = x$, and $p_k = 1$ for all $k \in \mathbb{N}$, then the sequence $X = \left(\bar{k}\right) \in \ell\left(M, \Delta^m, p, F\right)$. Let (Y_k) be a rearrangement of (X_k), which is defined as follows:

$$(Y_k) = \{X_1, X_2, X_4, X_3, X_9, X_5, X_{16}, X_6, X_{25}, X_7, X_{36}, X_8, X_{49}, X_{10,} ...\}.$$

Then $(Y_k) \notin \ell\left(M, \Delta^m, p, F\right)$.

Theorem 3.10 The sequence space $\ell\left(\Delta^m, F, M, p\right)$ is not sequence algebra.

Proof. This follows from the following example.

Example 4. Let $M(x) = x$, and $p_k = 1$ for all $k \in \mathbb{N}$, then the sequence $X = \left(\bar{k}\right)$ and $Y = \left(\bar{k}\right)$ belong to $\ell\left(M, \Delta^2, p, F\right)$, but $(X_k \otimes Y_k) \notin \ell\left(M, \Delta^2, p, F\right)$.

4. Conclusion

In this paper, using an Orlicz function we have introduced some of fairly wide classes of sequences of fuzzy numbers. Giving particular values to the sequence $p = (p_k)$, M and m we obtain some sequence spaces which are the special cases of the sequence space that we have defined. The most of the results proved in the previous sections will be true for these spaces.

References

[1] Y. Altin, M. Et and R. Çolak, Lacunary statistical and lacunary strongly convergence of generalized difference sequences of fuzzy numbers, Comput. Math. Appl. 52 (6-7) (2006), 1011–1020.

[2] B. Altay and F. Başar, On the fine spectrum of the difference operator Δ on c_0 and c, *Inform. Sci.* **168**(1-4) (2004), 217–224.

[3] S. Aytar and S. Pehlivan, Statistically monotonic and statistically bounded sequences of fuzzy numbers, *Inform. Sci.* **176**(6) (2006), 734–744

[4] M. Başarır and M. Mursaleen, Some sequence spaces of fuzzy numbers generated by infinite matrices, *J. Fuzzy Math.* **11**(3) (2003), 757–764.

[5] T. Bilgin, Lacunary strongly Δ–convergent sequences of fuzzy numbers, *Inform. Sci.* **160**(1-4) (2004), 201-206.

[6] P. Diamond and P. Kloeden, Metric spaces of fuzzy sets, *Fuzzy Sets and Systems* **35** (1990), 241-249.

[7] M. Et, H. Altinok and Y. Altin, On some generalized sequence spaces, *Appl. Math. Comput.* **154**(1) (2004), 167–173.

[8] M. Et, Y. Altin and H. Altinok, On almost statistical convergence of generalized difference sequences of fuzzy numbers, *Math. Model. Anal.* **10**(4) (2005), 345–352.

[9] M. Et and R. Çolak, On some generalized difference sequence spaces, *Soochow J. Math.* **21**(4) (1995), 377-386.

[10] M. Et, H. Altinok and R. Çolak, On λ-statistical convergence of difference sequences of fuzzy numbers, *Inform. Sci.* **176**(15) (2006), 2268–2278.

[11] P. K. Kamthan and M. Gupta, Sequence Spaces and Series, Marcel Dekker Inc., New York, 1981.

[12] M. Güngör and M. Et, Δ^m-strongly almost summable sequences defined by Orlicz functions, *Indian J. Pure Appl. Math.* **34**(8) 2003), 1141–1151.

[13] M. Işık, Generalized vector-valued sequence spaces defined by modulus functions. J. Inequal. Appl. 2010, Art. ID 457892, 7 pp.

[14] M. A. Krasnoselskii and Y. B. Rutickii, Convex Functions and Orlicz Spaces, Groningen, Netherlands, 1961.

[15] H. Kızmaz, On certain sequence spaces, *Canad. Math. Bull.* **24**(2) (1981), 169–176.

[16] V. Lakshmikantham and R. N. Mohapatra, Theory of Fuzzy Differential Equations and Inclusions, Taylor and Francis, New York, 2003.

[17] K. Lindberg, On subspaces of Orlicz sequence spaces, *Studia Math.* **45** (1973), 379-390.

[18] J. Lindenstrauss and L. Tzafriri, On Orlicz sequence spaces, *Israel J. Math.* 10 (1971), 379-390.

[19] E. Malkowsky, M. Mursaleen, S. Suantai, The dual spaces of sets of difference sequences of order m and matrix transformations. *Acta Math. Sin.* (Engl. Ser.) 23(3) (2007), 521–532.

[20] M. Matloka, Sequences of fuzzy numbers, *BUSEFAL* **28** (1986), 28-37.

[21] M. Mursaleen, Q. A. Khan and T. A. Chishti, Some new convergent sequences spaces defined by Orlicz functions and statistical convergence, *Ital. J. Pure Appl. Math.* **9** (2001), 25-32.

[22] F. Nuray and E. Savaş, Statistical convergence of fuzzy numbers, *Math. Slovaca* **45**(3) (1995), 269-273.

[23] F. Nuray, Lacunary statistical convergence of sequences of fuzzy numbers, *Fuzzy Sets and Systems* **99**(3) (1998), 353–355

[24] F. Nuray and A. Gülcü, Some new sequence spaces defined by Orlicz functions, *Indian J. Pure Appl. Math.* **26**(12) (1995), 1169–1176.

[25] W. Orlicz, Über Räume (L^M), *Bull. Int.Acad. Polon. Sci.* **Ser A** (1936), 93-107.

[26] S. D. Parashar and B. Choudhary, Sequence spaces defined by Orlicz functions, *Indian J. Pure and Appl. Math.* **25**(4) (1994), 419-428.

[27] E. Savaş, A note on sequence of fuzzy numbers. *Inform. Sci.* **124**(1-4) (2000), 297-300.

[28] O. Talo and F. Başar, Quasilinearity of the classical sets of sequences of fuzzy numbers and some related results, *Taiwanese J. Math.* **14**(5) (2010), 1799–1819.

[29] B. C. Tripathy, Y. Altin and M. Et, Generalized difference sequence spaces on seminormed space defined by Orlicz functions, *Math. Slovaca* **58**(3) (2008), 315–324.

[30] L. A. Zadeh, Fuzzy sets, *Inform. and Control* **8** (1965), 338-353.