QUARTERLY OF APPLIED MATHEMATICS
VOLUME LV, NUMBER 1
MARCH 1997, PAGES 69-87

ON A CLASS OF LIMIT STATES OF FRICTIONAL JOINTS: FORMULATION AND EXISTENCE THEOREM

By

LARS-ERIK ANDERSSON (*Dept. of Mathematics, Linköping Institute of Technology, Linköping, Sweden*)

AND

ANDERS KLARBRING (*Dept. of Mechanical Engineering, Linköping Institute of Technology, Linköping, Sweden*)

Abstract. The model dealt with is a linear elastic body in frictional contact with a rigid support. Limit states of such an assemblage are characterized by deformations and forces such that a small perturbation may introduce a large change in configuration. The class of limit states considered here is specified by the possibility of superposing a time constant rigid body velocity field to a static deformation. The problem of finding such states (i.e., forces and static deformations) for a prescribed rigid body velocity is formulated, and for the case when the geometrically admissible rigid body displacements form a linear space an existence result is given. It is proved that under restrictions on the magnitude of the friction coefficient and in the case that an intuitively clear condition on the direction of the forces is satisfied, there exist a load multiplier and a corresponding static displacement.

1. Introduction. The present work is concerned with frictional joints treated from the point of view of linear elasticity. A frictional joint is an assemblage of bodies interacting through frictional forces. It may be regarded as an abstraction of various elements encountered in machine design, such as the shrink-fitted shaft and bushing assemblage or different types of brakes. As a model of such a joint we consider a linear elastic body in frictional contact with a rigid support. In fact, it is well known that in linear elasticity the multibody contact problem has the same mathematical structure as the present one-body problem. The objective is to analyze limit states of frictional joints, i.e., states where a small perturbation may introduce a large change of configuration. As a more precise definition of a limit state, the requirement that the forces and the static deformation are such that a time constant rigid body velocity field can be added to the deformation, is used. That is, the elastic body moves (or slides) in a steady state

Received March 15, 1994.
1991 *Mathematics Subject Classification.* Primary 35J85, 73C99.
The first author was supported by the Swedish Natural Science Research Council under contract F-FU-8448-366 and the second by The Center for Industrial Information Technology (CENIIT) at Linköping Institute of Technology.

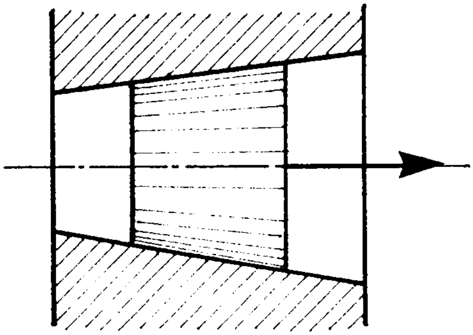

FIG. 1. A tapered joint

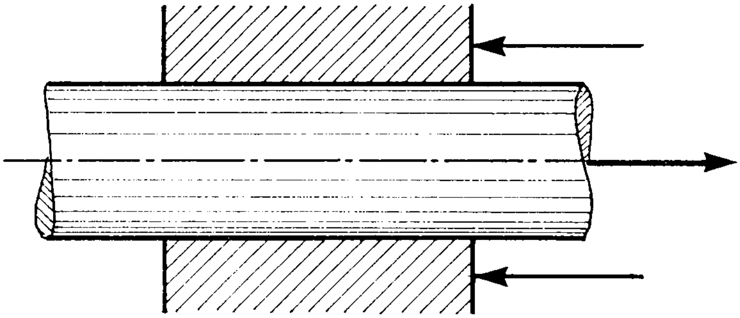

FIG. 2. A shrink-fitted shaft and bushing assemblage

fashion relative to the rigid support. To get insight into the nature of such a definition two examples are briefly discussed from a qualitative point of view:

First, consider the tapered joint shown in Fig. 1. For a friction coefficient that is large enough to make the joint self-locking, any force F from zero to infinity may be necessary to take the joint apart, depending on the previous load history. It is also clear that, unless the force is zero, the constant limit velocity field required by the definition will not be present. Rather, the joint will unassemble in a dynamic fashion.

As a second example, consider the shrink-fitted shaft and bushing assemblage in Fig. 2. Here it is clear that the definition makes sense for small friction coefficients. However, for large friction coefficients we may still have a "violent" limit state: consider the uncorking of a bottle.

Thus, the definition above covers a subclass of the situations that one would like to call limit states of frictional joints. Nevertheless, for this subclass some exact mathematical statements, which may form a starting point for a more complete theory, are given in this paper. The problem of finding the forces and the static deformation corresponding to a given limit velocity is formulated. The existence of a solution is shown under restrictions on the magnitude of the friction coefficient and the direction of the external force. This result is in agreement with our intuitive understanding of the problem.

Following this introduction a statement of the quasistatic contact problem with friction is given in Sec. 2. The cone of limit velocity fields is then defined in Sec. 3, which leads in Sec. 4 to a statement of a particular limit state problem where rigid body displacement is taking place in the direction of the x_1-axis. In Sec. 5 a variational formulation is given. In Sec. 6 a further geometrical specialization is introduced: it is assumed that the kinematically admissible rigid body displacements form a linear subspace of the configuration space. This leads to a decoupled formulation. In Sec. 7 we introduce the proper functional setting. Notably the functional representing the virtual work of the friction forces has to be carefully examined. In Sec. 8 we give the proof of the existence of a load multiplier and a displacement.

2. The quasi-static frictional contact problem. Since the interest is in analyzing constant velocity states, it is sufficient to start from the so-called quasi-static formulation, where dynamic forces are neglected.

Consider a linear elastic body that occupies a region Ω of $\mathbf{R}^3$. The boundary of Ω consists of disjoint parts S_t and S_c. The body is subjected to body forces $\alpha\mathbf{f} = \alpha(f_1, f_2, f_3)$ over Ω and surface tractions $\alpha\mathbf{t} = \alpha(t_1, t_2, t_3)$ over S_t, where α is a scalar load parameter and $\mathbf{f}$ and $\mathbf{t}$ are given vector fields. The following classical equations of linear elasticity are valid:

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \alpha f_i = 0 \qquad \text{in } \Omega, \tag{1}$$

$$\sigma_{ij} = E_{ijkl}\frac{\partial u_k}{\partial x_l} \qquad \text{in } \Omega, \tag{2}$$

$$\sigma_{ij}\nu_j = \alpha t_i \qquad \text{on } S_t. \tag{3}$$

Here $\mathbf{u} = (u_1, u_2, u_3)$ is the displacement vector, $\boldsymbol{\sigma} = \{\sigma_{ij}\}$ is the stress tensor, and ν_j are the components of the outward unit normal vector. E_{ijkl} are elasticity constants that satisfy the usual symmetry and ellipticity conditions. Furthermore, $i, j, k, l = 1, 2, 3$, the summation convention is used, and $(0, x_1, x_2, x_3)$ is the cartesian reference frame.

The boundary part S_c is the contact boundary where the body may come into contact with a rigid support. To state the laws of contact and friction we decompose the displacement and traction vectors on S_c into normal and tangential components:

$$\sigma_N = \sigma_{ij}n_i n_j, \quad \sigma_{Ti} = \sigma_{ij}n_j - \sigma_N n_i,$$
$$u_N = u_i n_i, \qquad u_{Ti} = u_i - u_N n_i.$$

Here n_i are the components of a unit vector that may be thought of as coinciding with ν_i and this is the interpretation preferred in this study. However, such an interpretation is not unique, as was seen in Klarbring et al. [1], where the kinematic constraint $(4)^2$ below was derived from an exact large deformation constraint by means of linearization. It was there shown that n_i may also be interpreted as pointing in the opposite direction of the outward unit normal vector of the rigid support. Nevertheless, ν_i and n_i must be almost coinciding for the theory to be physically admissible.

The contact law is taken to be the classical one of Signorini, i.e.,

$$\sigma_N \leq 0, \quad u_N - g \leq 0, \quad \sigma_N(u_N - g) = 0 \quad \text{on } S_c, \tag{4}$$

where g is the initial gap between the body and the rigid support. Note that there is no sign restriction for g.

The friction law is that of Coulomb, which can be written as

$$|\boldsymbol{\sigma}_T| \leq \mu|\sigma_N|, \tag{5}$$

$$\text{if } |\boldsymbol{\sigma}_T| < \mu|\sigma_N| \text{ then } \dot{\mathbf{u}}_T = \mathbf{0}, \text{ on } S_c, \tag{6}$$

$$\text{if } |\boldsymbol{\sigma}_T| = \mu|\sigma_N| > 0 \text{ then } \dot{\mathbf{u}}_T = -\lambda\boldsymbol{\sigma}_T, \quad \lambda \geq 0, \tag{7}$$

where μ is the friction coefficient, a superposed dot denotes time derivative, $\boldsymbol{\sigma}_T = (\sigma_{T1}, \sigma_{T2}, \sigma_{T3})$, and $\mathbf{u}_T = (u_{T1}, u_{T2}, u_{T3})$.

In conclusion, (1) through (7) constitute the quasi-static frictional contact problem.

3. The cone of limit velocity fields. A limit state of the mechanical system is a state in which a constant rigid velocity field can be superposed onto a static deformation. In this section we develop the kinematic conditions that must be placed on such a limit velocity field.

Let

$$Q = \{\mathbf{w} \mid w_i = \alpha_i + \beta_{ij}x_j, \ \beta_{ij} = -\beta_{ji}\}$$

be the set of rigid body velocities. Here α_i and β_{ij} are constants. The kinematic contact condition $u_N - g \leq 0$ must be satisfied both at the instance when the limit velocity is added to the static deformation and when such a motion has continued for some time. For a given static deformation we therefore define the cone of limit velocity fields as

$$\begin{aligned} K^\infty &= \{\mathbf{w} \in Q \mid u_N + \lambda w_N \leq g, \ u_N \leq g \text{ on } S_c, \ \forall\lambda > 0, \ \mathbf{w} \neq \mathbf{0}\} \\ &= \{\mathbf{w} \in Q \mid w_N \leq 0 \text{ on } S_c, \ \mathbf{w} \neq \mathbf{0}\}, \end{aligned}$$

where w_N is defined similarly to u_N. Similar constructions to this, related to different problems, can be found in Baiocchi et al. [2], and Ciarlet and Nečas [3].

A velocity $\mathbf{w} \in K^\infty$ divides S_c into two complementary parts:

$$\begin{aligned} S_c^+(\mathbf{w}) &= \{\mathbf{x} \in S_c \mid w_N < 0\}, \\ S_c^0(\mathbf{w}) &= \{\mathbf{x} \in S_c \mid w_N = 0\}. \end{aligned}$$

Obviously, a set $S_c^0(\mathbf{w})$ cannot be completely general. To fix ideas, let it be open and simply connected, and let $n_i = \nu_i$. Then the work of Hlaváček and Nečas [4] shows that $S_c^0(\mathbf{w})$ must be a part of one of the following: a helicoidal surface, a surface of revolution, a cylinder, a plane, or a sphere. If it is multiply connected it can consist of, for instance, two cylinders with a common orientation. If n_i is taken as pointing in the opposite direction of the outward unit normal vector of the rigid support, then the corresponding surface of the rigid support is one of the above-stated geometrical objects.

4. A limit state problem. Consider a velocity $\mathbf{w} \in K^\infty$ that is added to a static deformation at time τ equal to zero. Then, for $\tau > 0$ and $\mathbf{x} \in S_c^+(\mathbf{w})$ we see from (4) that $\sigma_N(\mathbf{x}) = 0$. We also have from (5) that $\boldsymbol{\sigma}_T(\mathbf{x}) = \mathbf{0}$. Thus, when stating a problem defining the static deformation of the body at the limit state we only have to consider contact and friction conditions on $S_c^0(\mathbf{w})$.

As a particular case of a limit-state problem we will consider a situation when $\mathbf{w} = k\mathbf{e}_1$, where $\mathbf{e}_1 = (1, 0, 0)$ is a natural base vector of $\mathbf{R}^3$, $k \in \mathbf{R}$, $k \neq 0$ and $\mathbf{w} \in K^\infty$. This means that if $S_c^0(\mathbf{w})$ is an open set and $n_i = \nu_i$, then each of its connected parts will be part of cylindrical or plane surfaces parallel with the x_1-axis; see Hlaváček and Nečas [4]. Since the constant k enters only as sign(k) in the following, we set for simplicity $k = 1$.

Substituting the displacement $\mathbf{u} = \mathbf{u}^0 + \tau\mathbf{e}_1$, where $\mathbf{u}^0$ is independent of time, into (2) and (4) we obtain

$$\sigma_{ij} = E_{ijkl}\frac{\partial u_k^0}{\partial x_l} \quad \text{in } \Omega, \tag{8}$$

$$\sigma_N \leq 0, \quad u_N^0 - g \leq 0, \quad \sigma_N(u_N^0 - g) = 0 \quad \text{on } S_c^0(\mathbf{w}). \tag{9}$$

Furthermore, the friction law (5) through (7) implies that

$$\boldsymbol{\sigma}_T = -\mu \mid \sigma_N \mid \mathbf{e}_1 \quad \text{on } S_c^0(\mathbf{w}). \tag{10}$$

The problem of finding the fields $\boldsymbol{\sigma}$ and $\mathbf{u}^0$ and the scalar α such that (1), (3), and (8) through (10) are satisfied constitutes a limit-state problem that will be considered in subsequent sections. The given data are $\mathbf{f}, \mathbf{t}$, E_{ijkl}, ν_i, n_i, and μ. Note that for the chosen $\mathbf{w}$, the problem can, at least for a flat $S_c^0(\mathbf{w})$, be interpreted as a steady sliding problem. Generally, this is the case when no rotations are involved, i.e., if $\beta_{ij} = 0$. If $\beta_{ij} \neq 0$, this interpretation is not possible due to the incapability of linear elasticity to model large rotations.

5. Variational formulations. The problem of the previous section will be formulated as a variational inequality. To that end, let $\widehat{V}$ be a space of sufficiently smooth functions defined on the closure of Ω. The following Green's formula holds for all $\mathbf{v} \in \widehat{V}$ and is obtained from (8):

$$a(\mathbf{u}^0, \mathbf{v}) = -\int_\Omega \frac{\partial \sigma_{ij}}{\partial x_j} v_i\, dx + \int_{\partial\Omega} \sigma_{ij}\nu_j v_i\, ds, \tag{11}$$

where the bilinear form is defined as

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega E_{ijkl}\frac{\partial u_i}{\partial x_j}\frac{\partial v_k}{\partial x_l}\, dx.$$

The convex set of admissible deformations is defined as

$$\widehat{K} = \{\mathbf{u} \in \widehat{V} \mid u_N - g_N \leq 0 \text{ on } \Gamma_c\}.$$

For technical reasons the initial gap is here defined by means of a function $\mathbf{g} = (g_1, g_2, g_3) \in \widehat{V}$ such that $g = g_N|_{\Gamma_c} = g_i n_i|_{\Gamma_c}$. Then we may write

$$\widehat{K} = \mathbf{g} + \widehat{K}_0$$

where

$$\widehat{K}_0 = \{\mathbf{u} \in \widehat{V} \mid u_N \leq 0 \text{ on } \Gamma_c\}$$

is a convex cone with vertex in the origin. Here and in the sequel we use for simplicity the notation $\Gamma_c := S_c^0(\mathbf{w})$.

Assuming sufficient regularity, the complementarity conditions (9) can be equivalently expressed as the variational inequality

$$\mathbf{u}^0 \in \widehat{K}, \qquad \int_{\Gamma_c} \sigma_N (v_N - u_N^0)\, ds \geq 0 \quad \forall \mathbf{v} \in \widehat{K}. \tag{12}$$

From (1), (3), (8), (10), (11), and (12) we obtain the following problem:

$$\begin{cases} \text{Find } \mathbf{u}^0 \in \widehat{K} \text{ and } \alpha \in \mathbf{R} \text{ such that for all } \mathbf{v} \in \widehat{K} \\ a(\mathbf{u}^0, \mathbf{v} - \mathbf{u}^0) - \int_{\Gamma_c} \mu\sigma_N(\mathbf{u}^0)(v_1 - u_1^0)\, ds \geq \alpha\langle \mathbf{F}, \mathbf{v} - \mathbf{u}^0\rangle, \end{cases}$$

or equivalently, in terms of K_0,

$$\begin{cases} \text{Find } \hat{\mathbf{u}} \in \widehat{K}_0 \text{ and } \alpha \in \mathbf{R} \text{ such that for all } \mathbf{v} \in \widehat{K}_0 \\ a(\hat{\mathbf{u}}, \mathbf{v} - \hat{\mathbf{u}}) - \int_{\Gamma_c} \mu\sigma_N(\hat{\mathbf{u}})(v_1 - \hat{u}_1)\, ds \\ \geq \alpha\langle \mathbf{F}, \mathbf{v} - \hat{\mathbf{u}}\rangle + \langle l_g, \mathbf{v} - \hat{\mathbf{u}}\rangle - a(\mathbf{g}, \mathbf{v} - \hat{\mathbf{u}}), \end{cases} \tag{$\widehat{\mathrm{P}}$}$$

where

$$\langle \mathbf{F}, \mathbf{v}\rangle = \int_\Omega f_i v_i\, dx + \int_{S_t} t_i v_i\, ds, \qquad \langle l_g, \mathbf{v}\rangle = \int_{\Gamma_c} \mu\sigma_N(\mathbf{g}) v_1\, ds$$

and $\sigma_N(\mathbf{u}^0), \sigma_N(\hat{\mathbf{u}})$, and $\sigma_N(\mathbf{g})$ are defined by (8). The solutions $\mathbf{u}^0$ and $\hat{\mathbf{u}}$ are related as $\mathbf{u}^0 = \hat{\mathbf{u}} + \mathbf{g}$. The force l_g may be interpreted as being due to the "shrink-fitting" (if $g < 0$).

6. Decoupled variational formulations. We will here make a further assumption on the geometry of the problem which will result in a replacement of the inequality of problem ($\widehat{\mathrm{P}}$) by a variational inequality related to the displacement and an equation related to the load multiplier.

It will be assumed that $\mathcal{L} := Q \cap \widehat{K}_0$ is a *linear subspace* of $\widehat{V}$. It is then clear that any $\mathbf{v} \in \mathcal{L}$ has the property that $v_N = 0$ on Γ_c. Thus, we can conclude that $K^\infty \subset \mathcal{L}\backslash\{\mathbf{0}\}$, implying that it is compatible with our previous particular choice of limit velocity to make the further geometrical *assumption* that

$$\mathcal{L} = \{\mathbf{v} \in \widehat{V} \mid \mathbf{v} = k\mathbf{e}_1 \text{ for some } k \in \mathbf{R}\}.$$

Figure 3 shows an example where $\mathcal{L}$ is a one-dimensional subspace as above while K^∞ is a ray in this subspace. Furthermore, we set

$$\mathcal{L}^\perp = \{\mathbf{v} \in \widehat{V} \mid \int_\Omega v_1\, dx = 0\}$$

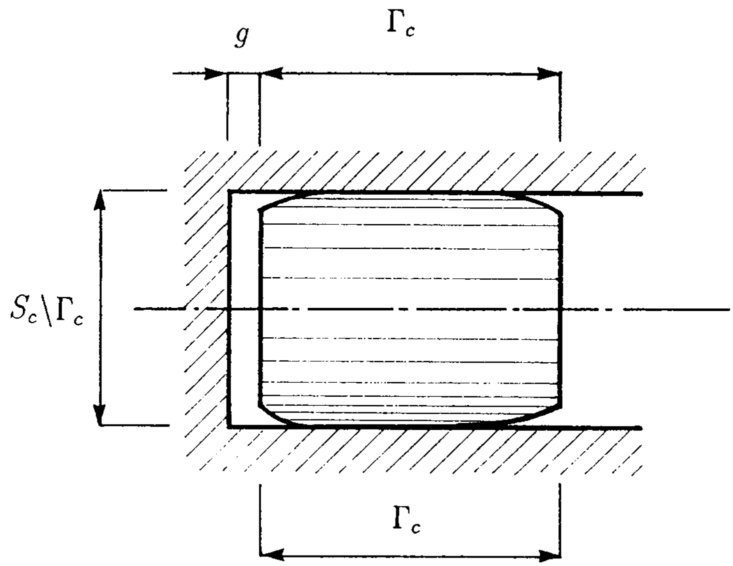

FIG. 3. An example in which $\mathcal{L}$ is a one-dimensional subspace and K^∞ is a ray in this subspace

so that $\widehat{V} = \mathcal{L}^\perp \oplus \mathcal{L}$, where the orthogonality is in the L^2-sense. Each element $\mathbf{u} \in \widehat{V}$ may then be decomposed as $\mathbf{u} = \overline{\mathbf{u}} + r\mathbf{e}_1$, with

$$\overline{\mathbf{u}} = \left(u_1 - \frac{1}{|\Omega|} \int_\Omega u_1 dx \right) \mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3 = \overline{u}_i\mathbf{e}_i \in \mathcal{L}^\perp$$

and

$$r = \frac{1}{|\Omega|} \int_\Omega u_1 \, dx.$$

We now substitute the fields $\hat{\mathbf{u}} = \overline{\mathbf{u}} + r\mathbf{e}_1$ and $\mathbf{v} = \overline{\mathbf{v}} + s\mathbf{e}_1$, where $\overline{\mathbf{v}}, \overline{\mathbf{u}} \in \mathcal{L}^\perp$, into problem $(\widehat{\mathrm{P}})$. Since $a(\hat{\mathbf{u}}, \mathbf{v}) = a(\overline{\mathbf{u}}, \overline{\mathbf{v}})$ and $\sigma_N(\hat{\mathbf{u}}) = \sigma_N(\overline{\mathbf{u}})$ one then obtains

$$\begin{aligned} a(\overline{\mathbf{u}}, \overline{\mathbf{v}} - \overline{\mathbf{u}}) - \int_{\Gamma_c} \mu\sigma_N(\overline{\mathbf{u}})(\overline{v}_1 - \overline{u}_1) \, ds - (s - r) \int_{\Gamma_c} \mu\sigma_N(\overline{\mathbf{u}}) \, ds \\ \geq \alpha(\mathbf{F}, \overline{\mathbf{v}} - \overline{\mathbf{u}}\rangle + \alpha(s - r)\langle \mathbf{F}, \mathbf{e}_1\rangle + \langle l_g, \overline{\mathbf{v}} - \overline{\mathbf{u}}\rangle \\ + (s - r)\langle l_g, \mathbf{e}_1\rangle - a(\mathbf{g}, \overline{\mathbf{v}} - \overline{\mathbf{u}}). \end{aligned}$$

Then, since s is arbitrary, we find that a solution of problem $(\widehat{\mathrm{P}})$ can be constructed from a solution of the following problem:

$$\left\{ \begin{array}{l} \text{Find } \overline{\mathbf{u}} \in \widehat{K}_0 \cap \mathcal{L}^\perp \text{ and } \alpha \in \mathbf{R} \text{ such that} \\ -\int_{\Gamma_c} \mu\sigma_N(\overline{\mathbf{u}}) \, ds = \alpha(\mathbf{F}, \mathbf{e}_1\rangle + \langle l_g, \mathbf{e}_1\rangle \\ \text{and for all } \overline{\mathbf{v}} \in \widehat{K}_0 \cap \mathcal{L}^\perp \\ a(\overline{\mathbf{u}}, \overline{\mathbf{v}} - \overline{\mathbf{u}}) - \int_{\Gamma_c} \mu\sigma_N(\overline{\mathbf{u}})(\overline{v}_1 - \overline{u}_1) \, ds \\ \geq \alpha\langle \mathbf{F}, \overline{\mathbf{v}} - \overline{\mathbf{u}}\rangle + \langle l_g, \overline{\mathbf{v}} - \overline{\mathbf{u}}\rangle - a(\mathbf{g}, \overline{\mathbf{v}} - \overline{\mathbf{u}}). \end{array} \right. \qquad (\widehat{\mathrm{P}})_d$$

If $(\overline{\mathbf{u}}, \alpha)$ is a solution of problem $(\widehat{\mathrm{P}})_d$, $(\hat{\mathbf{u}} = \overline{\mathbf{u}} + r\mathbf{e}_1, \alpha)$ is a solution of problem $(\widehat{\mathrm{P}})$ for any $r \in \mathbf{R}$. The equation of $(\widehat{\mathrm{P}})_d$ simply expresses global equilibrium in the $\mathbf{e}_1$-direction. Note that a certain nonuniqueness of solutions has appeared. The problem is indifferent with respect to a rigid body placement in the $\mathbf{e}_1$-direction.

Considering the form of $\overline{\mathbf{u}}$ it will prove useful to introduce the functionals $\overline{\mathbf{F}}$ and $\bar{l}_g$ defined by

$$\langle \overline{\mathbf{F}}, \mathbf{v} \rangle = \langle \mathbf{F}, \mathbf{v} \rangle - \langle \mathbf{F}, \mathbf{e}_1 \rangle \frac{(\mathbf{v}, \mathbf{e}_1)_{(L^2(\Omega))^3}}{\|\mathbf{e}_1\|^2_{(L^2(\Omega))^3}},$$

$$\langle \bar{l}_g, \mathbf{v} \rangle = \langle l_g, \mathbf{v} \rangle - \langle l_g, \mathbf{e}_1 \rangle \frac{(\mathbf{v}, \mathbf{e}_1)_{(L^2(\Omega))^3}}{\|\mathbf{e}_1\|^2_{(L^2(\Omega))^3}},$$

where $(\mathbf{v}, \mathbf{u})_{(L^2(\Omega))^3} = \int_\Omega v_i u_i dx$ and $\|\mathbf{e}_1\|^2_{(L^2(\Omega))^3} = (\mathbf{e}_1, \mathbf{e}_1)_{(L^2(\Omega))^3} = |\Omega|$. Then it follows easily that

$$\langle \overline{\mathbf{F}}, \overline{\mathbf{v}} \rangle = \langle \mathbf{F}, \overline{\mathbf{v}} \rangle \quad \forall \overline{\mathbf{v}} \in \widehat{K} \cap \mathcal{L}^\perp,$$

$$\langle \mathbf{F} - \overline{\mathbf{F}}, \mathbf{e}_1 \rangle = \langle \mathbf{F}, \mathbf{e}_1 \rangle = \int_\Omega f_1 dx + \int_{S_t} t_1 ds$$

and, similarly,

$$\langle \bar{l}_g, \overline{\mathbf{v}} \rangle = \langle l_g, \overline{\mathbf{v}} \rangle \quad \forall \overline{\mathbf{v}} \in \widehat{K} \cap \mathcal{L}^\perp,$$

$$\langle l_g - \bar{l}_g, \mathbf{e}_1 \rangle = \langle l_g, \mathbf{e}_1 \rangle = \int_{\Gamma_c} \mu \sigma_N(\mathbf{g}) ds.$$

In $(\widehat{\mathrm{P}})_d$ we will use these substitutions of right-hand sides. It is a decomposition of the force into a part that tends to push the body in the $\mathbf{e}_1$-direction and a part that is "orthogonal" to this direction. It is straightforward to verify that $\langle \overline{\mathbf{F}}, \mathbf{v} \rangle = \int_\Omega \overline{f}_i v_i dx + \int_{S_t} \bar{t}_i v_i ds$ where $\overline{\mathbf{f}} = \overline{f}_i \mathbf{e}_i$, $\overline{f}_1 = f_1 - \frac{1}{|\Omega|}\int_\Omega f_1 dx - \frac{1}{|S_t|}\int_{S_t} t_1 ds$, $\overline{f}_2 = f_2$, $\overline{f}_3 = f_3$, and $\bar{\mathbf{t}} = \mathbf{t}$.

A further reformulation of the problem will turn out to be useful for our existence proof in Sec. 8. That is, the equality of $(\widehat{\mathrm{P}})_d$ can be merged into the inequality to result in

$$\left\{\begin{array}{l} \text{Find } (\mathbf{u}, \alpha) \in (\widehat{K}_0 \cap \mathcal{L}^\perp) \times \mathbf{R} \text{ such that for all } (\mathbf{v}, \beta) \in (\widehat{K}_0 \cap \mathcal{L}^\perp) \times \mathbf{R} \\ a(\mathbf{u}, \mathbf{v} - \mathbf{u}) - \alpha \langle \overline{\mathbf{F}}, \mathbf{v} - \mathbf{u} \rangle - \int_{\Gamma_c} \mu \sigma_N(\mathbf{u})(v_1 - u_1)\, ds \\ +\alpha \langle \mathbf{F} - \overline{\mathbf{F}}, \mathbf{e}_1 \rangle (\beta - \alpha) + \int_{\Gamma_c} \mu \sigma_N(\mathbf{u}) ds (\beta - \alpha) \\ -\langle \bar{l}_g, \mathbf{v} - \mathbf{u} \rangle + a(\mathbf{g}, \mathbf{v} - \mathbf{u}) + \langle l_g - \bar{l}_g, \mathbf{e}_1 \rangle (\beta - \alpha) \geq 0. \end{array}\right. \qquad (\widehat{\mathrm{P}}\mathrm{V})$$

Note that we have dropped the bar-sign for elements $\mathbf{u}, \mathbf{v} \in \widehat{K}_0 \cap \mathcal{L}^\perp$.

7. Functional setting. In Sec. 8 existence results will be given. For this purpose we must, however, be more specific about the choice of function spaces and about the assumptions.

First, $\Omega \subset \mathbf{R}^3$ is assumed to be an open bounded Lipschitz domain, and S_t and Γ_c relatively open subsets of $\partial\Omega$ with $\overline{S}_t \cap \overline{\Gamma}_c = \varnothing$. For the volume forces and surface traction fields $\mathbf{f}$ and $\mathbf{t}$ we require that $f_i \in L^2(\Omega)$, $t_i \in H^{-1/2}(\partial\Omega)$, and supp $\mathbf{t} \subset \overline{S}_t$. Also, let $E_{ijkl} \in L^\infty(\Omega)$ and $E_{ijkl}\xi_{ij}\xi_{kl} \geq c_0\xi_{ij}\xi_{ij}$ for some $c_0 > 0$ and for all symmetric ξ_{ij}.

The space of smooth functions $\widehat{V}$ is replaced by the Hilbert space $V = [H^1(\Omega)]^3$, with norm denoted by $|\cdot|_V$. $\mathcal{L}$ and $\mathcal{L}^\perp$ are defined as previously, but with $\widehat{V}$ replaced by V. For the gap function $\mathbf{g}$ we assume that $\mathbf{g} \in V$ and in addition that $\partial\sigma_{ij}(\mathbf{g})/\partial x_j \in L^2(\Omega)$. The sets $\widehat{K}$ and $\widehat{K}_0$ are replaced by

$$K = \{\mathbf{u} \in V \mid (u_N - g_N)|_{\Gamma_c} \leq 0\}, \qquad K_0 = \{\mathbf{u} \in V \mid u_N|_{\Gamma_c} \leq 0\}.$$

Here the gap function $g_N \in H^{1/2}(\partial\Omega) \supset L^1(\partial\Omega)$ and the inequality $(u_N - g_N)|_{\Gamma_c} \leq 0$ is to be interpreted in the following sense: $\int_{\Gamma_c}(u_N - g_N)\varphi\, ds \leq 0$, $\forall\varphi : \partial\Omega \to \mathbf{R}$ such that $\varphi \geq 0$, supp $\varphi \subset \Gamma_c$, and $\varphi \in C(\partial\Omega)$. Then K and K_0 are closed convex cones in V and $K = \mathbf{g} + K_0$.

Our purpose is now to make the different terms of $(\widehat{\mathrm{P}})_d$ and $(\widehat{\mathrm{P}}\mathrm{V})$ well defined for $\mathbf{u} \in V$. The bilinear form $a(\cdot,\cdot)$ classically satisfies this requirement. However, before we define the integrals over Γ_c properly we need several preliminaries.

Essential is the introduction of $V_A \subset V$ defined by

$$V_A = \{\mathbf{v} \in V \mid \frac{\partial\sigma_{ij}(\mathbf{v})}{\partial x_j} \in L^2(\Omega)\}$$

with the norm $\|\cdot\|_{V_A}$ given by

$$\|\mathbf{v}\|_{V_A} = \|\mathbf{v}\|_V + \left\{\int_\Omega \frac{\partial\sigma_{ij}(\mathbf{v})}{\partial x_j}\frac{\partial\sigma_{ik}(\mathbf{v})}{\partial x_k}dx\right\}^{1/2}$$

and $\sigma_{ij}(\mathbf{v}) = E_{ijkl}\frac{\partial v_k}{\partial x_l}$. Let $W = H^{1/2}(\partial\Omega)$ and $W' = H^{-1/2}(\partial\Omega)$ denote the trace space and its dual. Then, if $v \in H^1(\Omega)$, the restriction $\mathrm{tr}(v) = v|_{\partial\Omega} \in W$ is well defined. From Green's formula it also follows that $\sigma_N(\mathbf{v}) \in W'$ if $\mathbf{v} \in V_A$ and that the mapping $V_A \ni \mathbf{v} \mapsto \sigma_N(\mathbf{v}) \in W'$ is linear and bounded. In particular, for some constant $\bar{c}_{\mathrm{tr}}$, depending only on Ω and $\|E_{ijkl}\|_{L^\infty(\Omega)}$ we have

$$\|\sigma_N(\mathbf{v})\|_{W'} \leq \bar{c}_{\mathrm{tr}}\|\mathbf{v}\|_{V_A} \quad \forall\mathbf{v} \in V_A. \tag{13}$$

Similarly, the mapping $V \ni \mathbf{v} \mapsto v_1|_{\partial\Omega} \in W$ is linear and bounded, and for some constant c_{tr}, depending only on Ω, we have

$$\|v_1\|_W \leq c_{\mathrm{tr}}\|\mathbf{v}\|_V \quad \forall\mathbf{v} \in V. \tag{14}$$

Now, if $\mathbf{f} \in (L^2(\Omega))^3$ let

$$V(\mathbf{f}) := \{\mathbf{v} \in V \mid \frac{\partial\sigma_{ij}(\mathbf{v})}{\partial x_j} + f_i = 0\}.$$

$V(\mathbf{f})$ is then a closed linear manifold in V and we may introduce the orthoprojection

$$\pi_f : V \to V(\mathbf{f}).$$

For the coefficient of friction μ, we will assume that it is defined on the whole of $\partial\Omega$, $\mu : \partial\Omega \to \mathbf{R}$, $\mu \geq 0$ and that $\mu \in \mathrm{Lip}(\partial\Omega)$, i.e,

$$\|\mu\|_{\mathrm{Lip}} := \|\mu\|_{L^\infty(\partial\Omega)} + \sum_{i=1}^{3} \left\| \frac{\partial\mu}{\partial x_i} \right\|_{L^\infty(\partial\Omega)} < \infty.$$

If μ is initially defined only on Γ_c then we have to extend it to $\partial\Omega$, preferably with smallest possible Lipschitz norm $\|\mu\|_{\mathrm{Lip}}$. If $\mu(\mathbf{x}) = \mu_0$ is constant, then of course $\|\mu\|_{\mathrm{Lip}} = \mu_0$.

Since $\overline{S}_t \cap \overline{\Gamma}_c = \varnothing$ it is possible to find a function $\psi \in C^\infty(\mathbf{R}^3)$ such that $\psi = 1$ on a neighborhood of $\overline{\Gamma}_c$ and $\psi = 0$ on a neighborhood of $\overline{S}_t$.

We are now ready to replace the functional

$$(\mathbf{u}, \mathbf{v}) \mapsto -\int_{\Gamma_c} \mu\sigma_N(\mathbf{u}) v_1 \, ds$$

by

$$V \times V \ni (\mathbf{u}, \mathbf{v}) \mapsto j_{\alpha f}(\mathbf{u}, \mathbf{v}) = -\langle \mu\sigma_N(\pi_{\alpha f}\mathbf{u}), \psi v_1 \rangle_{W',W} \in \mathbf{R}$$

and so $-\int_{\Gamma_c} \mu\sigma_N(\overline{\mathbf{u}}) ds$ is replaced by $j_{\alpha f}(\overline{\mathbf{u}}, \mathbf{e}_1) = -\langle \mu\sigma_N(\pi_{\alpha f}\overline{\mathbf{u}}), \psi \rangle_{W',W}$. Similarly, the functional l_g is defined by

$$\langle l_g, \mathbf{v} \rangle_{V',V} = \langle \mu\sigma_N(\mathbf{g}), \psi v_1 \rangle_{W',W}.$$

Here and in the sequel, $\langle \cdot, \cdot \rangle_{B',B}$ will denote the dual pairing in a Banach space B and its dual B'. For later use we also introduce the a priori estimates

$$\|\psi u\|_W \leq c_1(\Omega) \|\psi\|_{\mathrm{Lip}} \|u\|_W \tag{15}$$

and

$$\|\psi q\|_{W'} \leq c_2(\Omega) \|\psi\|_{\mathrm{Lip}} \|q\|_{W'} \tag{16}$$

valid for all $\psi \in \mathrm{Lip}(\partial\Omega)$, $u \in W = H^{1/2}(\partial\Omega)$ and $q \in W' = H^{-1/2}(\partial\Omega)$ with the constants $c_1(\Omega)$ and $c_2(\Omega)$ depending only on Ω (the geometry).

Next, the functional $\mathbf{v} \mapsto \int_\Omega f_i v_i d\Omega + \int_{\Gamma_c} t_i v_i ds$ may be replaced by

$$V \ni \mathbf{v} \mapsto (\mathbf{f}, \mathbf{v})_{(L^2(\Omega))^3} + \langle \mathbf{t}, \mathbf{v} \rangle_{(W')^3, W^3} = \langle \mathbf{F}, \mathbf{v} \rangle_{V',V}, \quad \mathbf{F} \in V',$$

where $\mathbf{f} \in (L^2(\Omega))^3$ and $\mathbf{t} \in (W')^3 = (H^{-1/2}(\partial\Omega))^3$. As in the previous section, considering that $\overline{\mathbf{v}} = \mathbf{v} - (\mathbf{v}, \mathbf{e}_1)_{(L^2(\Omega))^3} \mathbf{e}_1 / \|\mathbf{e}_1\|^2_{(L^2(\Omega))^3}$ ($\mathbf{e}_1 \in V \subset (L^2(\Omega))^3$) we introduce $\overline{\mathbf{F}} \in V'$ and $\overline{l}_g \in V'$ defined by

$$\langle \overline{\mathbf{F}}, \mathbf{v} \rangle_{V',V} = \langle \mathbf{F}, \mathbf{v} \rangle_{V',V} - \langle \mathbf{F}, \mathbf{e}_1 \rangle_{V',V} \frac{(\mathbf{v}, \mathbf{e}_1)_{(L^2(\Omega))^3}}{\|\mathbf{e}_1\|^2_{(L^2(\Omega))^3}},$$

$$\langle \overline{l}_g, \mathbf{v} \rangle_{V',V} = \langle l_g, \mathbf{v} \rangle_{V',V} - \langle l_g, \mathbf{e}_1 \rangle_{V',V} \frac{(\mathbf{v}, \mathbf{e}_1)_{(L^2(\Omega))^3}}{\|\mathbf{e}_1\|^2_{(L^2(\Omega))^3}}.$$

The same results as in the previous section are valid; so we use $\langle \overline{\mathbf{F}}, \overline{\mathbf{v}} \rangle_{V',V}$, $\langle \mathbf{F} - \overline{\mathbf{F}}, \mathbf{e}_1 \rangle_{V',V}$, etc., as right-hand sides in the variational problem.

After these rather lengthy preliminaries we are ready to replace the problem $(\widehat{\mathrm{P}})_d$ by (we drop the bar sign in $\overline{\mathbf{u}}$ and $\overline{\mathbf{v}}$)

$$\begin{cases} \text{Find } \mathbf{u} \in K_0 \cap \mathcal{L}^{\perp} \text{ and } \alpha \in \mathbf{R} \text{ such that} \\ j_{\alpha f}(\mathbf{u}, \mathbf{e}_1) = \alpha (\mathbf{F} - \overline{\mathbf{F}}, \mathbf{e}_1 \rangle_{V',V} + \langle l_g - \bar{l}_g, \mathbf{e}_1 \rangle_{V',V} \\ \text{and for all } \mathbf{v} \in K \cap \mathcal{L}^{\perp} \\ a(\mathbf{u}, \mathbf{v} - \mathbf{u}) + j_{\alpha f}(\mathbf{u}, \mathbf{v} - \mathbf{u}) \\ \geq \alpha \langle \overline{\mathbf{F}}, \mathbf{v} - \mathbf{u} \rangle_{V',V} + \langle \bar{l}_g, \mathbf{v} - \mathbf{u} \rangle_{V',V} - a(\mathbf{g}, \mathbf{v} - \mathbf{u}) \end{cases}$$

and $(\widehat{\mathrm{P}}\mathrm{V})$ by

$$\begin{cases} \text{Find } (\mathbf{u}, \alpha) \in (K_0 \cap \mathcal{L}^{\perp}) \times \mathbf{R} \text{ such that for all } (\mathbf{v}, \beta) \in (K_0 \cap \mathcal{L}^{\perp}) \times \mathbf{R} \\ a(\mathbf{u}, \mathbf{v} - \mathbf{u}) - \alpha \langle \overline{\mathbf{F}}, \mathbf{v} - \mathbf{u} \rangle_{V',V} + j_{\alpha f}(\mathbf{u}, \mathbf{v} - \mathbf{u}) \\ +\alpha \langle \mathbf{F} - \overline{\mathbf{F}}, \mathbf{e}_1 \rangle_{V',V} (\beta - \alpha) - j_{\alpha f}(\mathbf{u}, \mathbf{e}_1)(\beta - \alpha) \\ -\langle \bar{l}_g, \mathbf{v} - \mathbf{u} \rangle_{V',V} + a(\mathbf{g}, \mathbf{v} - \mathbf{u}) + \langle l_g - \bar{l}_g, \mathbf{e}_1 \rangle_{V',V} (\beta - \alpha) \geq 0. \end{cases}$$

Before proceeding we will rewrite the functional $j_{\alpha f}$ as follows. We first note that for all $\mathbf{u} \in V$

$$\pi_{\alpha f} \mathbf{u} = \pi_0 \mathbf{u} + \alpha \mathbf{u}_0$$

where $\alpha \mathbf{u}_0 = \pi_{\alpha f} \mathbf{0} = \alpha \pi_f \mathbf{0}, \mathbf{u}_0 = \pi_f \mathbf{0}$. Therefore,

$$\begin{aligned} j_{\alpha f}(\mathbf{u}, \mathbf{v}) &= \langle -\mu \sigma_N (\pi_{\alpha f} \mathbf{u}), \psi v_1 \rangle_{W',W} \\ &= \langle -\mu \sigma_N (\pi_0 \mathbf{u}), \psi v_1 \rangle_{W',W} + \alpha \langle -\mu \sigma_N (\pi_f \mathbf{0}), \psi v_1 \rangle_{W',W} \\ &= j_0(\mathbf{u}, \mathbf{v}) + \alpha \langle \mathbf{L}_{f.\mu}, \mathbf{v} \rangle_{V',V} \end{aligned}$$

where we have introduced the functionals j_0 and $\mathbf{L}_{f.\mu}$ defined by

$$j_0(\mathbf{u}, \mathbf{v}) = \langle -\mu \sigma_N (\pi_0 \mathbf{u}), \psi v_1 \rangle_{W',W},$$

$$\langle \mathbf{L}_{f,\mu}, \mathbf{v} \rangle_{V',V} = \langle -\mu \sigma_N (\mathbf{u}_0), \psi v_1 \rangle_{W',W}.$$

Moreover, $\overline{\mathbf{L}}_{f,\mu}$ is defined by

$$\langle \overline{\mathbf{L}}_{f,\mu}, \mathbf{v} \rangle_{V',V} = \langle \mathbf{L}_{f,\mu}, \mathbf{v} \rangle_{V',V} - \langle \mathbf{L}_{f,\mu}, \mathbf{e}_1 \rangle_{V',V} \frac{(\mathbf{v}, \mathbf{e}_1)_{(L^2(\Omega))^3}}{\|\mathbf{e}_1\|^2_{(L^2(\Omega))^3}}.$$

We now obtain the following problems:

$$\begin{cases} \text{Find } \mathbf{u} \in K_0 \cap \mathcal{L}^{\perp} \text{ and } \alpha \in \mathbf{R} \text{ such that} \\ j_0(\mathbf{u}, \mathbf{e}_1) = \alpha \langle \mathbf{F} - \overline{\mathbf{F}} - (\mathbf{L}_{f.\mu} - \overline{\mathbf{L}}_{f,\mu}), \mathbf{e}_1 \rangle_{V',V} + \langle l_g - \bar{l}_g, \mathbf{e}_1 \rangle_{V',V} \\ \text{and for all } \mathbf{v} \in K \cap \mathcal{L}^{\perp} \\ a(\mathbf{u}, \mathbf{v} - \mathbf{u}) + j_0(\mathbf{u}, \mathbf{v} - \mathbf{u}) \\ \geq \alpha \langle \overline{\mathbf{F}} - \overline{\mathbf{L}}_{f,\mu}, \mathbf{v} - \mathbf{u} \rangle_{V',V} + \langle \bar{l}_g, \mathbf{v} - \mathbf{u} \rangle_{V',V} - a(\mathbf{g}, \mathbf{v} - \mathbf{u}), \end{cases} \qquad (\mathrm{P})_d$$

$$\begin{cases} \text{Find } (\mathbf{u},\alpha) \in (K_0 \cap \mathcal{L}^\perp) \times \mathbf{R} \text{ such that for all } (\mathbf{v},\beta) \in (K_0 \cap \mathcal{L}^\perp) \times \mathbf{R} \\ a(\mathbf{u},\mathbf{v}-\mathbf{u}) - \alpha\langle \overline{\mathbf{F}} - \overline{\mathbf{L}}_{f,\mu}, \mathbf{v}-\mathbf{u}\rangle_{V',V} + j_0(\mathbf{u},\mathbf{v}-\mathbf{u}) \\ +\alpha\langle \mathbf{F} - \overline{\mathbf{F}} - (\mathbf{L}_{f,\mu} - \overline{\mathbf{L}}_{f,\mu}), \mathbf{e}_1\rangle_{V',V}(\beta-\alpha) - j_0(\mathbf{u},\mathbf{e}_1)(\beta-\alpha) \\ -\langle \bar{l}_g, \mathbf{v}-\mathbf{u}\rangle_{V',V} + a(\mathbf{g},\mathbf{v}-\mathbf{u}) + \langle l_g - \bar{l}_g, \mathbf{e}_1\rangle_{V',V}(\beta-\alpha) \geq 0. \end{cases} \tag{PV}$$

Introducing the notation $\mathcal{U} = (\mathbf{u},\alpha)$, $\mathcal{V} = (\mathbf{v},\beta)$, and

$$\begin{aligned} \mathcal{A}(\mathcal{U},\mathcal{V}) &= a(\mathbf{u},\mathbf{v}) + j_0(\mathbf{u},\mathbf{v}) - \alpha\langle \overline{\mathbf{F}} - \overline{\mathbf{L}}_{f,\mu}, \mathbf{v}\rangle_{V',V} \\ &\quad + \alpha\langle \mathbf{F} - \overline{\mathbf{F}} - (\mathbf{L}_{f,\mu} - \overline{\mathbf{L}}_{f,\mu}), \mathbf{e}_1\rangle_{V',V}\beta - j_0(\mathbf{u},\mathbf{e}_1)\beta, \\ \mathcal{F}(\mathcal{V}) &= \langle \bar{l}_g, \mathbf{v}\rangle_{V',V} - a(\mathbf{g},\mathbf{v}) - \langle l_g - \bar{l}_g, \mathbf{e}_1\rangle_{V',V}\beta, \end{aligned}$$

Problem (PV) takes the form

$$\begin{cases} \text{Find } \mathcal{U} \in (K_0 \cap \mathcal{L}^\perp) \times \mathbf{R} \text{ such that for all } \mathcal{V} \in (K_0 \cap \mathcal{L}^\perp) \times \mathbf{R} \\ \mathcal{A}(\mathcal{U},\mathcal{V}-\mathcal{U}) \geq \mathcal{F}(\mathcal{V}-\mathcal{U}). \end{cases} \tag{PV}$$

It is clear that $\mathcal{A}$ is a bilinear, nonsymmetric, and continuous functional on the closed convex cone $(K_0 \cap \mathcal{L}^\perp) \times \mathbf{R}$ and that $\mathcal{F}$ is a bounded linear functional on $V \times \mathbf{R}$.

Some comments are in order regarding the introduction of the projection operator $\pi_{\alpha f}$ and the function ψ. Assume that $(\mathbf{u},\alpha)$ solves problem $(\mathrm{P})_d$. It is then easy to verify that

$$a(\mathbf{u},\mathbf{v}-\mathbf{u}) + j_{\alpha f}(\mathbf{u},\mathbf{v}-\mathbf{u}) \geq \alpha\langle \mathbf{F}, \mathbf{v}-\mathbf{u}\rangle_{V',V}$$

for all $\mathbf{v} \in K$. Therefore, by Green's formula it follows that (in the sense of distributions)

$$\frac{\partial \sigma_{ij}(\mathbf{u})}{\partial x_j} + \alpha f_i = 0 \quad \text{in } \Omega, \tag{17}$$

i.e., that $\pi_{\alpha f}\mathbf{u} = \mathbf{u}$. Using this it also follows that

$$\sigma_{ij}(\mathbf{u})\nu_j = \alpha t_i \qquad \text{on } S_t, \tag{18}$$

$$\sigma_{ij}(\mathbf{u}) = 0 \qquad \text{on } \partial\Omega \backslash (\overline{S}_t \cup \overline{\Gamma}_c), \tag{19}$$

$$\sigma_N(\mathbf{u}) \leq 0,\ u_N - g \leq 0,\ \sigma_N(\mathbf{u})(u_N - g) = 0 \qquad \text{on } \Gamma_c, \tag{20}$$

$$\boldsymbol{\sigma}_T(\mathbf{u}) = -\mu\sigma_N(\mathbf{u})\mathbf{e}_1 \qquad \text{on } \Gamma_c. \tag{21}$$

The function ψ is introduced in order to get a proper definition of a functional corresponding to the integral $\int_{\Gamma_c} \mu\sigma_N(\mathbf{u})v_1 ds$. It is important to note that, by (17) and (19), the values of $j_{\alpha f}(\mathbf{u},\mathbf{v}) = -\langle \mu\sigma_N(\pi_{\alpha f}\mathbf{u}), \psi v_1\rangle_{W',W}$ and $j_{\alpha f}(\mathbf{u},\mathbf{e}_1) = -\langle \mu\sigma_N(\pi_{\alpha f}\mathbf{u}), \psi\rangle_{W',W}$ are unchanged if ψ is replaced by another function satisfying the same conditions. Therefore the class of solutions $(\mathbf{u},\alpha)$ and $(\mathrm{P})_d$ is *independent of the particular choice of* ψ.

The splitting of $\pi_{\alpha f}$ into two parts producing the functional $\mathbf{L}_{\mu,f}$ is made in order to give a formulation in terms of a bilinear form ($\mathcal{A}$). We do not attempt or find useful any physical interpretation of $\mathbf{L}_{\mu,f}$.

8. An existence result. In this section we will formulate and prove an existence theorem for the problem (PV). We first state the following lemma.

LEMMA 1. There exists a constant $c_k > 0$ such that

$$a(\mathbf{v}, \mathbf{v}) \geq c_k \|\mathbf{v}\|_V^2 \tag{22}$$

holds for all $\mathbf{v} \in K_0 \cap \mathcal{L}^\perp$ as well as for all $\mathbf{v} \in Q^\perp$, where $Q^\perp$ is the orthogonal complement of Q in the H^1-norm.

Proof. The sets $K_0 \cap \mathcal{L}^\perp \subset V$ and $Q^\perp \subset V$ are both convex closed cones of V, with vertex in the origin, and $(K_0 \cap \mathcal{L}^\perp) \cap Q = \{\mathbf{0}\}$, $Q^\perp \cap Q = \{\mathbf{0}\}$. Thus, the lemma follows from Korn's inequality; see Nečas and Hlaváček [5].

We will also need the following general result for abstract inequalities in Banach spaces, which has been formulated by Cocu [6]. A proof in a more general situation can be found in [7] or [8].

THEOREM 1. Let B be a reflexive Banach space, $C \subset B$ a nonempty closed convex subset of B, and let

$$G : C \times C \to \mathbf{R}$$

be a function such that

(i) $G(u, u) \geq 0 \quad \forall u \in C$.
(ii) The set $\{v \in C \mid G(u, v) < 0\}$ is convex for every fixed $u \in C$.
(iii) The set $\{u \in C \mid G(u, v) \geq 0\}$ is weakly sequentially closed for every fixed $v \in C$.
(iv) There exists a bounded subset $D \subset B$ and an element $v_0 \in D \cap C$ such that $G(u, v_0) < 0$ for all $u \in C \backslash D$.

Then there exists at least one vector $u_0 \in D \cap C$ satisfying the inequality

$$G(u_0, v) \geq 0 \quad \forall v \in C.$$

In order to apply this theorem we first estimate $j_0(\mathbf{u}, \mathbf{u})$. By (15) and (16)

$$\begin{aligned} |j_0(\mathbf{u}, \mathbf{u})| &= |-\langle \mu \sigma_N(\pi_0 \mathbf{u}), \psi u_1 \rangle_{W', W}| \leq \|\mu \sigma(\pi_0 \mathbf{u})\|_{W'} \|\psi u_1\|_W \\ &\leq c_1(\Omega) c_2(\Omega) \|\psi\|_{\mathrm{Lip}} \|\mu\|_{\mathrm{Lip}} \|\sigma(\pi_0 \mathbf{u})\|_{W'} \|\mathbf{u}_1\|_V. \end{aligned}$$

Now, using (13) and (14) we get

$$|j_0(\mathbf{u}, \mathbf{u})| \leq c_1(\Omega) c_2(\Omega) c_{\mathrm{tr}} \bar{c}_{\mathrm{tr}} \|\psi\|_{\mathrm{Lip}} \|\mu\|_{\mathrm{Lip}} \|\pi_0 \mathbf{u}\|_{V_A} \|\mathbf{u}\|_V.$$

By definition $\|\pi_0 \mathbf{u}\|_{V_A} = \|\pi_0 \mathbf{u}\|_V$ and, since π_0 is an orthogonal projection, $\|\pi_0 \mathbf{u}\|_V \leq \|\mathbf{u}\|_V$. Therefore,

$$|j_0(\mathbf{u}, \mathbf{u})| \leq A \|\mathbf{u}\|_V^2, \tag{23}$$

where $A = c_1(\Omega) c_2(\Omega) c_{\mathrm{tr}} \bar{c}_{\mathrm{tr}} \|\psi\|_{\mathrm{Lip}} \|\mu\|_{\mathrm{Lip}}$. Similarly, we get

$$\begin{aligned} |j_0(\mathbf{u}, \mathbf{e}_1)| &\leq c_1(\Omega) c_2(\Omega) \bar{c}_{\mathrm{tr}} \|\psi\|_{\mathrm{Lip}} \|\mu\|_{\mathrm{Lip}} \|\mathbf{u}\|_V \|\mathbf{e}_1\|_W \\ &\leq A \|\mathbf{u}\|_V \|\mathbf{e}_1\|_V = A |\Omega| \, \|\mathbf{u}\|_V, \end{aligned}$$

i.e.,

$$|j_0(\mathbf{u}, \mathbf{e}_1)| \leq A|\Omega| \, \|\mathbf{u}\|_V. \tag{24}$$

For the functional $\mathcal{A}$ we now have, using (23), (24), and Lemma 1, the following inequality for all $\mathcal{U} \in (K_0 \cap \mathcal{L}^\perp) \times \mathbf{R}$:

$$\begin{aligned}\mathcal{A}(\mathcal{U},\mathcal{U}) \geq (c_k - A)\|\mathbf{u}\|_V^2 - |\alpha| \, \|\mathbf{u}\|_V \|\overline{\mathbf{F}} - \overline{\mathbf{L}}_{f,\mu}\|_{V'} \\ + \alpha^2 \langle \mathbf{F} - \overline{\mathbf{F}} - (\mathbf{L}_{f,\mu} - \overline{\mathbf{L}}_{f,\mu}), \mathbf{e}_1 \rangle_{V',V} - A|\Omega| \, \|\mathbf{u}\|_V |\alpha|,\end{aligned}$$

i.e.,

$$\mathcal{A}(\mathcal{U},\mathcal{U}) \geq c_{11}\|\mathbf{u}\|_V^2 + c_{22}\alpha^2 - 2c_{12}|\alpha| \, \|\mathbf{u}\|_V \tag{25}$$

where we have introduced the notation

$$\begin{aligned} c_{11} &= c_k - A, \\ c_{22} &= \langle \mathbf{F} - \overline{\mathbf{F}} - (\mathbf{L}_{f,\mu} - \overline{\mathbf{L}}_{f,\mu}, \mathbf{e}_1 \rangle_{V',V}, \\ c_{12} &= \tfrac{1}{2}\{\|\overline{\mathbf{F}} - \overline{\mathbf{L}}_{f,\mu}\|_{V'} + A|\Omega|\}. \end{aligned}$$

Now, a necessary and sufficient condition that there exist an $\varepsilon > 0$ such that

$$c_{11}\|\mathbf{u}\|^2 + c_{22}\alpha^2 - 2c_{12}|\alpha| \, \|\mathbf{u}\|_V \geq \varepsilon\{\|\mathbf{u}\|^2 + \alpha^2\}$$

for all $\|\mathbf{u}\|_V$, α, is that $c_{11} > 0$, $c_{22} > 0$, and $c_{11}c_{22} - c_{12}^2 > 0$, i.e., that

$$c_k - A > 0, \qquad \langle \mathbf{F} - \overline{\mathbf{F}} - (\mathbf{L}_{f,\mu} - \overline{\mathbf{L}}_{f,\mu}), \mathbf{e}_1 \rangle_{V',V} > 0 \tag{26}$$

and that

$$(c_k - A)\langle \mathbf{F} - \overline{\mathbf{F}} - (\mathbf{L}_{f,\mu} - \overline{\mathbf{L}}_{f,\mu}), \mathbf{e}_1 \rangle_{V',V} > \tfrac{1}{4}\{\|\overline{\mathbf{F}} - \overline{\mathbf{L}}_{f,\mu}\|_{V'} + A|\Omega|\}^2. \tag{27}$$

Under the assumptions (26) and (27) we have, with $\varepsilon > 0$,

$$\mathcal{A}(\mathcal{U},\mathcal{U}) \geq \varepsilon\|\mathcal{U}\|_{V\times\mathbf{R}}^2 \tag{28}$$

for all $\mathcal{U} \in (K_0 \cap \mathcal{L}^\perp) \times \mathbf{R}$.

We are now ready to formulate the main result of our paper, stating that problem (PV) has at least one solution.

THEOREM 2. Assume that our parameters satisfy the inequalities (26) and (27). Then there is at least one vector $\mathcal{U} = (\mathbf{u}, \alpha) \in (K_0 \cap \mathcal{L}^\perp) \times \mathbf{R}$ such that

$$\mathcal{A}(\mathcal{U}, \mathcal{V} - \mathcal{U}) \geq \mathcal{F}(\mathcal{V} - \mathcal{U})$$

for all $\mathcal{V} = (\mathbf{v}, \beta) \in (K_0 \cap \mathcal{L}^\perp) \times \mathbf{R}$.

Proof. Let, in Theorem 1, $B = V \times \mathbf{R}$, $C = (K_0 \cap \mathcal{L}^\perp) \times \mathbf{R}$ and let G be defined by

$$G(\mathcal{U}, \mathcal{V}) = \mathcal{A}(\mathcal{U}, \mathcal{V} - \mathcal{U}) - \mathcal{F}(\mathcal{V} - \mathcal{U}).$$

It is clear that $G(\mathcal{U},\mathcal{U}) = 0$ for all $\mathcal{U}$ so that condition (i) of Theorem 1 is satisfied. Condition (ii) is satisfied since $G(\mathcal{U},\mathcal{V})$ is linear in $\mathcal{V}$. Choosing $\mathcal{V}_0 = \bar{0}$ we have

$$G(\mathcal{U},\bar{0}) = -\mathcal{A}(\mathcal{U},\mathcal{U}) + \mathcal{F}(\mathcal{U}) \le -\varepsilon\|\mathcal{U}\|^2_{V\times\mathbf{R}} + \|\mathcal{F}\|_{V'\times\mathbf{R}}\|\mathcal{U}\|_{V\times\mathbf{R}}.$$

Taking $D = \{\mathcal{U} \in V \times \mathbf{R} \mid \|\mathcal{U}\|_{V\times\mathbf{R}} < \|\mathcal{F}\|_{V'\times\mathbf{R}}/\varepsilon\}$ we conclude that (iv) is valid. It remains to verify condition (iii). We introduce the notation $\mathcal{Q} = Q \times \mathbf{R}$. $P_{\mathcal{Q}} : V \times \mathbf{R} \to \mathcal{Q} \subset V \times \mathbf{R}$ denotes the orthogonal projection onto the subspace $\mathcal{Q}$. Now, given a vector $\mathcal{V}$ and a sequence $\{\mathcal{U}_n\}_{n=1}^{\infty} \subset (K_0 \cap \mathcal{L}^{\perp}) \times \mathbf{R}$ such that $G(\mathcal{U}_n,\mathcal{V}) = \mathcal{A}(\mathcal{U}_n,\mathcal{V}-\mathcal{U}_n) - \mathcal{F}(\mathcal{V}-\mathcal{U}_n) \ge 0$ and $\mathcal{U}_n \rightharpoonup \mathcal{U}$, we are to show that $G(\mathcal{U},\mathcal{V}) = \mathcal{A}(\mathcal{U},\mathcal{V}-\mathcal{U}) - \mathcal{F}(\mathcal{V}-\mathcal{U}) \ge 0$. Since $\mathcal{A}(\mathcal{U}_n,\mathcal{V}) \to \mathcal{A}(\mathcal{U},\mathcal{V})$ and $\mathcal{F}(\mathcal{U}_n) \to \mathcal{F}(\mathcal{U})$ it suffices to show that

$$\mathcal{A}(\mathcal{U},\mathcal{U}) \le \liminf_{n\to\infty} \mathcal{A}(\mathcal{U}_n,\mathcal{U}_n)$$

for every sequence $\{\mathcal{U}_n\}_{n=1}^{\infty} \subset (K_0 \cap \mathcal{L}^{\perp}) \times \mathbf{R}$ converging weakly towards $\mathcal{U} \in V \times \mathbf{R}$.

LEMMA 2. If $\mathcal{Q}^{\perp} = Q^{\perp} \times \{0\} \supset \{\mathcal{U}_n\}_{n=1}^{\infty}$ and if $\mathcal{U}_n \rightharpoonup \mathcal{U}$ (weakly), then

$$\mathcal{A}(\mathcal{U},\mathcal{U}) \le \liminf_{n\to\infty} \mathcal{A}(\mathcal{U}_n,\mathcal{U}_n).$$

Proof. By Lemma 1, $a(\mathbf{u}_n,\mathbf{u}_n) \ge c_k\|\mathbf{u}_n\|^2_V$. From the estimates in (23), (24), which are valid for arbitrary $\mathbf{u} \in V$, we conclude that (28) is still valid under the assumptions (26), (27), i.e.,

$$\mathcal{A}(\mathcal{U},\mathcal{U}) \ge \varepsilon\|\mathcal{U}\|^2$$

for all $\mathcal{U} \in \mathcal{Q}^{\perp}$. Now $\mathcal{Q}^{\perp} \subset V \times \mathbf{R}$ is a linear subspace, whence we conclude that Cauchy's inequality is valid for the symmetric part of the bilinear form, i.e., that

$$\tfrac{1}{2}(\mathcal{A}(\mathcal{U}_n,\mathcal{U}) + \mathcal{A}(\mathcal{U},\mathcal{U}_n)) \le \sqrt{\mathcal{A}(\mathcal{U},\mathcal{U})}\sqrt{\mathcal{A}(\mathcal{U}_n,\mathcal{U}_n)}.$$

Taking limits we get

$$\mathcal{A}(\mathcal{U},\mathcal{U}) \le \sqrt{\mathcal{A}(\mathcal{U},\mathcal{U})}\liminf_{n\to\infty}\sqrt{\mathcal{A}(\mathcal{U}_n,\mathcal{U}_n)},$$

i.e.,

$$\mathcal{A}(\mathcal{U},\mathcal{U}) \le \liminf_{n\to\infty} \mathcal{A}(\mathcal{U}_n,\mathcal{U}_n).$$

Now let $\{\mathcal{U}_n\} \subset (K_0 \cap \mathcal{L}^{\perp}) \times \mathbf{R}$ and $\mathcal{U}_n \rightharpoonup \mathcal{U}$. Take $\mathcal{Z}_n = \mathcal{U}_n - P_{\mathcal{Q}}\mathcal{U}_n$ with $\mathcal{Z}_n \in \mathcal{Q}^{\perp} = Q^{\perp} \times \{0\}$ and $P_{\mathcal{Q}}\mathcal{U}_n \in \mathcal{Q}$. Since $\mathcal{Q}$ is finite dimensional

$$P_{\mathcal{Q}}\mathcal{U}_n \to P_{\mathcal{Q}}\mathcal{U} \quad \text{(strongly)}$$

and

$$\mathcal{Z}_n \rightharpoonup \mathcal{Z} = \mathcal{U} - P_{\mathcal{Q}}\mathcal{U} \quad \text{(weakly)}.$$

Therefore, we get

$$\begin{aligned}\mathcal{A}(\mathcal{U}_n,\mathcal{U}_n) &= \mathcal{A}(\mathcal{Z}_n + P_{\mathcal{Q}}\mathcal{U}_n, \mathcal{Z}_n + P_{\mathcal{Q}}\mathcal{U}_n)\\ &= \mathcal{A}(\mathcal{Z}_n,\mathcal{Z}_n) + \mathcal{A}(P_{\mathcal{Q}}\mathcal{U}_n, P_{\mathcal{Q}}\mathcal{U}_n) + \mathcal{A}(\mathcal{Z}_n, P_{\mathcal{Q}}\mathcal{U}_n) + \mathcal{A}(P_{\mathcal{Q}}\mathcal{U}_n,\mathcal{Z}_n)\end{aligned}$$

where $\mathcal{A}(\mathcal{Z}_n, P_Q\mathcal{U}_n) \to \mathcal{A}(\mathcal{Z}, P_Q\mathcal{U})$, $\mathcal{A}(P_Q\mathcal{U}_n, \mathcal{Z}_n) \to \mathcal{A}(P_Q\mathcal{U}, \mathcal{Z})$, and $\mathcal{A}(P_Q\mathcal{U}_n, P_Q\mathcal{U}_n) \to \mathcal{A}(P_Q\mathcal{U}, P_Q\mathcal{U})$. We conclude, by Lemma 2, that

$$\begin{aligned}\liminf_{n\to\infty} \mathcal{A}(\mathcal{U}_n, \mathcal{U}_n) &= \liminf_{n\to\infty} \mathcal{A}(\mathcal{Z}_n, \mathcal{Z}_n) + \mathcal{A}(P_Q\mathcal{U}, P_Q\mathcal{U}) + \mathcal{A}(\mathcal{Z}, P_Q\mathcal{U}) + \mathcal{A}(P_Q\mathcal{U}, \mathcal{Z}) \\ &\geq \mathcal{A}(\mathcal{Z}, \mathcal{Z}) + \mathcal{A}(P_Q\mathcal{U}, P_Q\mathcal{U}) + \mathcal{A}(\mathcal{Z}, P_Q\mathcal{U}) + \mathcal{A}(P_Q\mathcal{U}, \mathcal{Z}) \\ &= \mathcal{A}(\mathcal{Z} + P_Q\mathcal{U}, \mathcal{Z} + P_Q\mathcal{U}) = \mathcal{A}(\mathcal{U}, \mathcal{U}),\end{aligned}$$

i.e., that

$$\mathcal{A}(\mathcal{U}, \mathcal{U}) \leq \liminf_{n\to\infty} \mathcal{A}(\mathcal{U}_n, \mathcal{U}_n).$$

We have proved that condition (iii) of Theorem 1 is satisfied. It follows that there exists $\mathcal{U} \in (K_0 \cap \mathcal{L}^\perp) \times \mathbf{R}$ such that for all $\mathcal{V} \in (K_0 \cap \mathcal{L}^\perp) \times \mathbf{R}$ we have

$$G(\mathcal{U}, \mathcal{V}) = \mathcal{A}(\mathcal{U}, \mathcal{V} - \mathcal{U}) - \mathcal{F}(\mathcal{V} - \mathcal{U}) \geq 0.$$

This completes the proof of Theorem 2.

From (26), (27) and the fact that A contains the factor $\|\mu\|_{\mathrm{Lip}}$ and that $\|\mathbf{L}_{f,\mu} - \overline{\mathbf{L}}_{f,\mu}\|_{V'} \leq C\|\mu\|_{\mathrm{Lip}}$ we get the following corollary.

COROLLARY 1. If $\langle \mathbf{F} - \overline{\mathbf{F}}, \mathbf{e}_1\rangle_{V',V} > 0$ and $c_k\langle \mathbf{F} - \overline{\mathbf{F}}, \mathbf{e}_1\rangle_{V',V} > \frac{1}{4}\|\overline{\mathbf{F}} - \overline{\mathbf{L}}_{f,\mu}\|_V^2$ then there exists a $\delta > 0$ such that the problem (PV) has a solution whenever $0 \leq \|\mu\|_{\mathrm{Lip}} < \delta$.

Some important comments relating to this corollary are as follows: (i) the fact that a "small" friction coefficient is needed is in agreement with previous results on frictional contact; see for instance Klarbring [9]. In fact, for large frictional coefficients it can be anticipated that the limit state corresponds to a chattering motion of stick-slip type instead of the constant velocity state considered here. (ii) The condition $c_k\langle \mathbf{F} - \overline{\mathbf{F}}, \mathbf{e}_1\rangle_{V',V} > \frac{1}{4}\|\overline{\mathbf{F}} - \overline{\mathbf{L}}_{f,\mu}\|_V^2$, which implies $\langle \mathbf{F} - \overline{\mathbf{F}}, \mathbf{e}_1\rangle_{V',V} > 0$, means, firstly, that there must be a force resultant in the $\mathbf{e}_1$-direction in order to have a sliding in this direction and, secondly, it is a condition on the direction of the forces: for given geometry and constitutive constants a large "tangential" component $\mathbf{F} - \overline{\mathbf{F}}$ promotes satisfaction of the condition, while a large "normal" component $\overline{\mathbf{F}}$ counteracts it. An illustration is given in Fig. 4.

Some comments on nonuniqueness follow. As found in Sec. 6 it is clear that if $(\mathbf{u}, \alpha)$ is a solution of problem $(\widehat{\mathrm{P}})$, then, for any r, $(\mathbf{u} + r\mathbf{e}_1, \alpha)$ is also a solution. In problem $(\widehat{\mathrm{P}})_d$ and subsequent reformulations this indeterminacy has been filtered away. However, also problem (PV) (and consequently also $(\mathrm{P})_d$) exhibit nonuniqueness. Let $\mathcal{U} = (\mathbf{u}, \alpha)$ be a solution of (PV). Then $\mathcal{U}' = (\mathbf{u} + \mathbf{u}_Q, \alpha)$, where $\mathbf{u}_Q \in Q \cap \mathcal{L}^\perp$ and $\mathbf{u} + \mathbf{u}_Q \in K_0 \cap \mathcal{L}^\perp$ is also a solution since $\mathcal{A}((\mathbf{u}_Q, 0), \mathcal{V}) = 0$ for all $\mathcal{V}$. In summary, if a solution $(\mathbf{u}, \alpha)$ of the limit-state problem has been found, all geometrically admissible displacement states that can be reached from $\mathbf{u}$ by a rigid body placement are, together with α, a new solution. Further, we have to leave as an open question whether the multitude of solutions might be even larger.

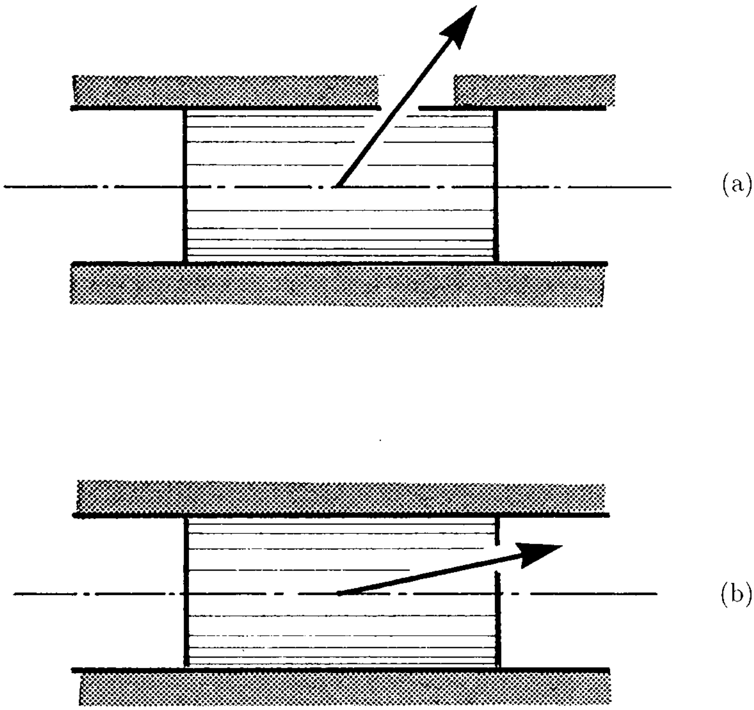

FIG. 4. The force direction in (b) is more likely to satisfy the conditions of Corollary 1 than that in (a).

9. A yield theory. It is appropriate in this context to make a reference to the so-called yield theory, see Frémond [10], for assessment of limit loads of mechanical systems. Given the strength of individual members of the system the yield theory can be used to obtain a set of external loadings that are potentially withstandable by the system. When used for members displaying a standard (i.e., associated) perfectly plastic behavior it is possible to show that a force that is potentially withstandable is also absolutely withstandable. Here we will indicate an extension of the yield theory to frictional joints. This will be done under the assumption of sufficient regularity.

Let $\widehat{\Sigma}_T$ be a set of sufficiently smooth tangential contact stresses. The set of load parameters α such that the load is potentially withstandable is denoted by $\mathcal{P}$. We have

$$\left\{\begin{array}{l} \alpha \in \mathcal{P} \text{ if and only if there exist } \mathbf{u} \in \widehat{K} \text{ and} \\ \boldsymbol{\sigma}_T \in \{\boldsymbol{\tau}_T \in \widehat{\Sigma}_T \mid |\boldsymbol{\tau}_T| \leq \mu \sigma_N(\mathbf{u}) \text{ on } S_c\} \\ \text{such that for all } \mathbf{v} \in \widehat{K} \\ a(\mathbf{u}, \mathbf{v} - \mathbf{u}) - \int_{S_c} \sigma_{Ti}(u_{Ti} - v_{Ti})\, ds \geq \alpha f(\mathbf{v} - \mathbf{u}). \end{array}\right. \tag{Y}$$

Note the typical characteristics of a yield theory that the conditions on the velocity, present in the formulation (5) through (7) of Coulomb's friction law, do not enter (Y). This is what makes the load only potentially withstandable.

Friction is a nonassociated phenomenon and therefore one expects a difference between potentially and absolutely withstandable forces. To investigate this consider again the example shown in Fig. 1. Here, an extension of the yield theory would imply that for friction coefficients above the self-locking limit the potentially withstandable forces belong to the set $(-\infty, \infty)$, while the absolutely withstandable forces belong to the set $(-\infty, 0]$. On the other hand, for friction coefficients below the self-locking limit the two sets coincide and are given as $(-\infty, 0]$. Whether the magnitude of the friction coefficient indicated by the self-locking limit has a relation to the smallness indicated in Theorem 1 is not known.

Extensions of the yield theory in the case of rigid-plastic bodies with frictional boundary conditions have been given by Drucker [11], Collins [12], and Telega [13].

10. Discussion. The paper introduces a class of limit-state problems for frictional systems. The detailed analysis is carried out for a particular problem obtained by making two partly related assumptions. The first assumption is that the limit velocity has a particular form. The second assumption is that the set of kinematically admissible rigid body displacements form a one-dimensional subspace of the configuration space. Both of these assumptions could obviously be changed in various ways without considerably changing the structure of the problem. For instance, under the same assumption on the limit velocity the set of kinematically admissible rigid body displacements could form a two-dimensional subspace. Another alternative would be letting the limit velocity be for instance a helical motion. What is typical for all such extensions, and what makes the technique of proof possible, is Lemma 1.

A related problem to the one discussed here has been considered by Gastaldi and Martins [14]. In that paper the same particular limit velocity field is considered, but the set of kinematically admissible rigid body displacements is a half-space. In this case Lemma 1 does not hold, but on the other hand, the compatibility condition, represented in the case of this paper by the equality of problem $(\mathrm{P})_d$, will not involve the displacement field: it will be represented by simple global equilibrium conditions; so α and the admissible direction of the forces are obtained as a separate equilibrium problem. Because of this Gastaldi and Martins are able to give an existence proof without access to Lemma 1. However, there is a class of intermediate situations not covered by the considerations in this paper nor the study of Gastaldi and Martins. The mathematical analysis of these problems must be considered as open questions, but can most likely be treated by an application of the general Theorem 1.

References

[1] A. Klarbring, A. Mikelić, and M. Shillor, *The rigid punch problem with friction*, Internat. J. Engrg. Sci. **29**, 751–768 (1991)

[2] C. Baiocchi, G. Buttazzo, F. Gastaldi, and F. Tomarelli, *General existence theorems for unilateral problems in continuum mechanics*, Arch. Rational Mech. Anal. **100**, 149–189 (1988)

[3] P. G. Ciarlet and J. Nečas, *Unilateral problems in nonlinear three-dimensional elasticity*, Arch. Rational Mech. Anal. **87**, 319–338 (1985)

[4] I. Hlaváček and J. Nečas, *On inequalities of Korn's type* II. *Applications to linear elasticity*, Arch. Rational Mech. Anal. **36**, 312–334 (1970)

[5] J. Nečas and I. Hlaváček, *Mathematical Theory of Elastic and Elasto-Plastic Bodies: An Introduction*, Elsevier, Amsterdam, 1981

[6] M. Cocu, *Existence of solutions of signorini problems with friction*, Internat. J. Engrg. Sci. **22**, (5), 567–575 (1984)

[7] K. Fan, *A minimax inequality and applications*, Inequalities III (ed. O. Shisha), Academic Press, 1972, pp. 103–113

[8] H. Brézis, L. Nirenberg, and G. Stampacchia, *A remark on Ky Fan's minimax principle*, Bolletino Un. Mat. Ital. **6**, no. 293(4), 293–300 (1972)

[9] A. Klarbring, *Examples of non-uniqueness and non-existence of solutions to quasistatic contact problems with friction*, Ing. Arch. **60**, 529–542 (1990)

[10] M. Fremond, *Yield theory in physics*, Topics in Nonsmooth Mechanics (eds. J. J. Moreau et al.), Birkhäuser-Verlag, Basel, 1988, pp. 187–240

[11] D. C. Drucker, *Coulomb friction, plasticity and limit loads*, J. Applied Mech., March, 71–74 (1954)

[12] I. F. Collins, *The upper bound theorem for rigid/plastic solids generalized to include Coulomb friction*, J. Mech. Phys. Solids **17**, 323–338 (1969)

[13] J. J. Telega, *Limit analysis theorems in the case of Signorini's boundary conditions and friction*, Arch. Mech. **37**, 549–562 (1985)

[14] F. Gastaldi and J. A. C. Martins, *A Noncoercive Steady-Sliding Problem with Friction*, Istituto di Analisi Numerica, Pubblicazioni N. 650, Pavia, 1988