ON A CLASS OF MONOMIAL IDEALS GENERATED BY s-SEQUENCES

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The symmetric algebra of classes of monomial ideals in a polynomial ring in two sets of variables is studied. In certain cases, the theory of s-sequences permits to compute standard invariants.

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INTRODUCTION

Let $R = K[X_1, \dots, X_n; Y_1, \dots, Y_m]$ be a polynomial ring in two sets of variables. Let $r, k \geq 1$ be integers, then $I_{r,s}$ is the Veronese-type ideal generated on degree r by the set $\{X_1^{a_{i_1}} \dots X_n^{a_{i_n}} \mid \sum_{j=1}^n a_{i_j} = r,\ 0 \leq a_{i_j} \leq s,$ $s \in \{1, \dots, r\}\}$ and $J_{k,s}$ is the Veronese-type ideal generated on degree k by the set $\{Y_1^{b_{i_1}} \dots Y_m^{b_{i_m}} \mid \sum_{j=1}^m b_{i_j} = k,\ 0 \leq b_{i_j} \leq s,\ s \in \{1, \dots, k\}\}$. In [10] the author introduced the Veronese bi-type ideals $L_{q,s} = \sum_{r+k=q} I_{r,s} J_{k,s}$ generated in the same degree q.

For $s = 2$ the Veronese bi-type ideals are the ideals of the walks of a bipartite graph with loops. In [9] the author studies the combinatorics of the integral closure and the normality of $L_{q,2}$. More in general, in [10] the same problem is studied for $L_{q,s}$ for all s.

In this paper we are interested to study the symmetric algebra of these classes of monomial ideals. In order to compute the standard invariants we investigate in which cases these monomial ideals are generated by s-sequences.

In [6] the notion of s-sequences has been employed to compute the invariants of the symmetric algebra of finitely generated modules. The proposal is to compute standard invariants of the symmetric algebra in terms of the corresponding invariants of special quotients of the ring R. This computation can be obtained for finitely generated R-modules generated by an s-sequence.

In Section 1 we consider the ideals of Veronese-type $I_{q,s}$. We give the conditions such that $I_{q,s}$ is generated by an s-sequence. Then we compute standard algebraic invariants of the symmetric algebra of $I_{q,s}$.

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The computation of these invariants can be obtained when the ideal $I_{q,s}$ is generated by an s-sequence in terms of its annihilator ideals.

In Section 2 we determine the subclasses of Veronese bi-type ideals $L_{q,s}$ generated by s-sequences. We establish the theorem: $L_{q,s}$ is generated by an s-sequence if and only if $q = s(n+m)-1$. The technic used to prove this result is the characterization of the monomial s-sequences by the Gröbner bases. For $s = 1$ and $q = 2, 3$ the ideals obtained are the mixed product ideals I_1J_1 and $I_1J_2+I_2J_1$, studied in [11]. Then we give the structure of the annihilator ideals of Veronese bi-type ideals generated by s-sequences and we achieve formulas for the dimension $\dim_R(\mathrm{Sym}_R(L_{q,s}))$, the multiplicity $\mathrm{e}(\mathrm{Sym}_R(L_{q,s}))$ and bounds for the Castelnuovo-Mumford regularity $\mathrm{reg}_R(\mathrm{Sym}_R(L_{q,s}))$ in terms of the annihilator ideals as described in [6]. As an application, we verify the Eisenbud-Goto inequality in the formulation given in [12], for the symmetric algebra of these ideals generated by s-sequences. In Section 3 we consider the ideals of the walks of a bipartite graph with loops $I_q(G) = L_{q,2}$. We prove that the ideals $I_q(G)$ generated by an s-sequence are a class of monomial ideals with linear quotients and we investigate some their algebraic invariants.

1. IDEALS OF VERONESE-TYPE

We recall the theory of s-sequences in order to apply it to our classes of monomial ideals. Let R be a noetherian ring, M be a finitely generated R-module and $f_1, \dots, f_t$ be the generators of M. For every $i = 1, \dots, t$, we set $M_{i-1} = Rf_1 + \cdots + Rf_{i-1}$ and let $\mathcal{I}_i = M_{i-1} :_R f_i$ be the colon ideal. Since $M_i/M_{i-1} \simeq R/\mathcal{I}_i$, so $\mathcal{I}_i$ is the annihilator of the cyclic module $R/\mathcal{I}_i$. $\mathcal{I}_i$ is called an annihilator ideal of the sequence $f_1, \dots, f_t$.

Let (a_{ij}), for $i = 1, \dots, t$, $j = 1, \dots, p$, be the relation matrix of M. The symmetric algebra $\mathrm{Sym}_R(M)$ has a presentation $R[T_1, \dots, T_t]/J$, with $J = (g_1, \dots, g_p)$ where $g_j = \sum_{i=1}^{t} a_{ij}T_i$ for $j = 1, \dots, p$. Let $S = R[T_1, \dots, T_t]$ be the polynomial ring and let $\prec$ be a monomial order on the monomials of S in the variables T_i such that $T_1 \prec T_2 \prec \cdots \prec T_t$. With respect to this term order, if $f = \sum a_\alpha \underline{T}^\alpha$, where $\underline{T}^\alpha = T_1^{\alpha_1} \cdots T_t^{\alpha_t}$ and $\alpha = (\alpha_1, \dots, \alpha_t) \in \mathbb{N}^t$, we put $\mathrm{in}_\prec(f) = a_\alpha \underline{T}^\alpha$, where $\underline{T}^\alpha$ is the largest monomial in f such that $a_\alpha \neq 0$. If we assign degree 1 to each variable T_i and degree 0 to the elements of R, we have the following facts:

1) J is a graded ideal.
2) The natural epimorphism $S \to \mathrm{Sym}_R(M)$ is a graded homomorphism of graded algebras on R.

Set the monomial ideal $\mathrm{in}_\prec(J) = (\mathrm{in}_\prec(f) | f \in J)$. In general

$$(\mathcal{I}_1T_1, \mathcal{I}_2T_2, \dots, \mathcal{I}_tT_t) \subseteq \mathrm{in}_\prec(J)$$

and the two ideal coincide in the linear case.

Definition 1.1. A sequence $f_1, \dots, f_t$ is an *s-sequence* for M if

$$(\mathcal{I}_1T_1, \mathcal{I}_2T_2, \dots, \mathcal{I}_tT_t) = \mathrm{in}_\prec(J).$$

If $\mathcal{I}_1 \subseteq \mathcal{I}_2 \subseteq \cdots \subseteq \mathcal{I}_t$, the sequence is a *strong s-sequence*.

If $R = K[X_1, \dots, X_n]$ is the polynomial ring over a field K, we can use the Gröbner bases theory to compute $\mathrm{in}_\prec(J)$. Let $\prec$ any term order on $K[X_1, \dots, X_n; T_1, \dots, T_t]$ with $T_1 \prec T_2 \prec \cdots \prec T_t$, $X_i \prec T_j$ for all i and j. Then for any Gröbner basis G of $J \subset K[X_1, \dots, X_n, T_1, \dots, T_t]$ with respect to $\prec$, we have $\mathrm{in}_\prec(J) = (\mathrm{in}_\prec(f) | f \in G)$. If the elements of G are linear in the T_i, then $f_1, \dots, f_t$ is an s-sequence for M.

Let $M = I = (f_1, \dots, f_t)$ a monomial ideal of $R = K[X_1, \dots, X_n]$. Set $f_{ij} = \frac{f_i}{[f_i, f_j]}$ for $i \neq j$, where $[f_i, f_j]$ is the greatest common divisor of the monomials f_i and f_j. J is generated by $g_{ij} = f_{ij}T_j - f_{ji}T_i$ for $1 \leq i < j \leq t$. The monomial sequence $f_1, \dots, f_t$ is an s-sequence if and only if g_{ij} for $1 \leq i < j \leq t$ is a Gröbner basis for J for any term order in $K[X_1, \dots, X_n; T_1, \dots, T_t]$ with $T_1 \prec T_2 \prec \cdots \prec T_t$, $X_i \prec T_j$ for all i,j. Notice that the annihilator ideals of the monomial sequence $f_1, \dots, f_t$ are the ideals $I_i = (f_{1i}, f_{2i}, \dots, f_{i-1,i})$ for $i = 1, \dots, t$ ([6]).

Remark 1.1 ([6, Lemma 1.4]). From the theory of Gröbner bases, if $f_1, \dots, f_t$ is a monomial s-sequence with respect to some admissible term order $\prec$, then $f_1, \dots, f_t$ is a s-sequence for any other admissible term order.

The first section of this paper is dedicated to the symmetric algebra of a class of monomial modules over the polynomial ring $R = K[X_1, \dots, X_n]$ that are ideals. We recall the following definition.

Definition 1.2 ([13]). Let $R = K[X_1, \dots, X_n]$ be the polynomial ring over a field K. The *ideal of Veronese-type* of degree q is the monomial ideal $I_{q,s}$ generated by the set

$$\left\{ X_1^{a_{i_1}} \cdots X_n^{a_{i_n}} \;\Big|\; \sum_{j=1}^{n} a_{i_j} = q,\ 0 \leq a_{i_j} \leq s \right\}.$$

Remark 1.2. In general $I_{q,s} \subseteq I_q$, where I_q is the *Veronese ideal* of degree q of R which is generated by all the monomials in the variables $X_1, \dots, X_n$ of degree q: $I_q = (X_1, \dots, X_n)^q$ ([14]). If $q = 1, 2$ or $s = q$, then $I_{q,s} = I_q$.

Example 1.1. $R = K[X_1, X_2, X_3]$, $I_{3,2} = (X_1^2X_2, X_1^2X_3, X_1X_2^2, X_2^2X_3, X_1X_3^2, X_2X_3^2, X_1X_2X_3) \subset I_3$, $I_{3,3} = (X_1^3, X_2^3, X_3^3, X_1^2X_2, X_1^2X_3, X_1X_2^2, X_2^2X_3, X_1X_3^2, X_2X_3^2, X_1X_2X_3) = I_3$.

We give the condition such that the ideal $I_{q,s}$ is generated by an s-sequence. First of all we observe that the property to be an s-sequence may depend on the order of the sequence.

Example 1.2 ([6]). $R = K[X_1, X_2]$, $I_{2,2} = (X_1^2, X_1X_2, X_2^2)$; X_1^2, X_2^2, X_1X_2 is an s-sequence, but X_1^2, X_1X_2, X_2^2 is not an s-sequence. Let $\mathrm{Sym}_R(I_{2,2}) = R[T_1, T_2, T_3]/J$, $J = (X_2T_1 - X_1T_2, X_1T_2 - X_2T_3)$. Fix $T_1 \prec T_2 \prec T_3$ and $X_1 \prec X_2 \prec T_i$, we have $\mathrm{in}_\prec(J) = ((X_1^2)T_2, (X_1, X_2)T_3)$ and $\mathcal{I}_1 = (0)$, $\mathcal{I}_2 = (X_1^2)$, $\mathcal{I}_3 = (X_1, X_2)$. Hence X_1^2, X_1X_2, X_2^2 is a strong s-sequence. Instead X_1^2, X_1X_2, X_2^2 is not an s-sequence because in this case $\mathrm{in}_\prec(J) = ((X_1)T_3, (X_1)T_2, (X_2)T_1T_3)$. It means that J does not admit a linear Gröbner basis for any term order in $K[X_1, X_2; T_1, T_2, T_3]$ with $X_i \prec T_j$ for all i, j and $T_1 \prec T_2 \prec T_3$. In fact there are S-pairs $S(g_{ij}, g_{hl})$ that have not a standard expression with respect $G = \{g_{ij} = f_{ij}T_j - f_{ji}T_i \mid 1 \le i < j \le 3\}$ with remainder 0: $S(g_{12}, g_{23}) = \frac{f_{12}f_{32}}{[f_{12}, f_{23}]}T_2^2 - \frac{f_{21}f_{23}}{[f_{12}, f_{23}]}T_1T_3 = X_2T_2^2 - X_2T_1T_3$, with $f_1 = X_1^2$, $f_2 = X_1X_2$, $f_3 = X_2^2$. There is no $g_{ij} \in G$ whose initial term divides a term of $S(g_{12}, g_{23})$.

Hence in the sequel we will suppose $I_{q,s} = (f_1, f_2, \ldots, f_t)$ where $f_1 \prec f_2 \prec \cdots \prec f_t$ with respect to the monomial order $\prec_{Lex}$ on the variables X_i and $X_1 \prec X_2 \prec \cdots \prec X_n$.

LEMMA 1.1. *Let $R = K[X_1, \ldots, X_n]$ be the polynomial ring over a field K with $n > 2$ and $I_{q,s} \subset R$, $2 \le q \le sn$. If $[f_{ij}, f_{hl}] = 1$ for $i < j$, $h < l$, $i \ne h$, $j \ne l$ with $i, j, h, l \in \{1, \ldots, t\}$, then $q = sn - 1$.*

Proof. By hypotheses $q \le sn$. Let $f_1, \ldots, f_t$ be the generators of $I_{q,s}$. Set $f_{ij} = \frac{f_i}{[f_i, f_j]}$ for f_i, f_j with $i < j$. We have $f_{ij} = X_{i_1}^{a_{i_1}} \cdots X_{i_c}^{a_{i_c}}$ and $f_{hl} = X_{h_1}^{b_{h_1}} \cdots X_{h_d}^{b_{h_d}}$ for f_h, f_l with $h < l$, $i \ne h$, $j \ne l$. By the hypothesis $[f_{ij}, f_{hl}] = 1$ for $i < j$, $h < l$, $i \ne h$, $j \ne l$, we have $X_{i_j} \ne X_{h_p}$ for all $j = 1, \ldots, c$ and $p = 1, \ldots, d$. This means that there are no other generators f_h, f_l of $I_{q,s}$ such that f_{hl} contains some variables $X_{i_1}, \ldots, X_{i_c}$ (that are in f_{ij}). It follows that if a variable of f_{ij} is of degree k in the monomial f_h, with $h \ne i, j$, then such variable in the same degree k belongs to any other generators f_l for all $l > h$ and $l \ne j$. Hence we deduce the structure of the monomials that generate $I_{q,s}$ and satisfying the hypotheses of the lemma. If $q = sn - 1$ we have $f_1 = X_1^sX_2^s \cdots X_{n-1}^sX_n^{s-1}$, $f_2 = X_1^sX_2^s \cdots X_{n-1}^{s-1}X_n^s$, $f_3 = X_1^sX_2^s \cdots X_{n-2}^{s-1}X_{n-1}^sX_n^s$, $\ldots$, $f_{n-1} = X_1^sX_2^{s-1} \cdots X_{n-2}^sX_{n-1}^sX_n^s$, $f_n = X_1^{s-1}X_2^s \cdots X_{n-2}^sX_{n-1}^sX_n^s$.

We compute $f_{12} = X_{n-1}$, $f_{13} = X_{n-2}$, $\ldots$, $f_{1n} = X_1$, $f_{23} = X_{n-2}$, $\ldots$, $f_{2n} = X_1$, and so on. Hence $f_{1j} = f_{2j} = \ldots = f_{nj} = X_{n-j+1}$ for all $j = 2, \ldots, n$. Then $f_{ij} \ne f_{hl}$ because $j \ne l$ and $[f_{ij}, f_{hl}] = 1$ as required.

Let $q < sn - 1$. If we consider the generators $f_i = X_1^sX_2^s \cdots X_{n-3}^{s-2}X_{n-1}^s$, $f_j = X_1^sX_2^s \cdots X_{n-4}^{2s-2}X_n^s$, $f_h = X_1^sX_2^s \cdots X_{n-3}^{s-2}X_{n-2}^s$, $f_l = X_1^sX_2^s \cdots X_{n-4}^{2s-2}X_{n-1}^s$, then $f_{ij} = X_{n-3}^{s-2}X_{n-1}^{s}$ and $f_{hl} = X_{n-3}^{s-2}X_{n-2}^{s}$. Hence $[f_{ij}, f_{hl}] \neq 1$, a contradiction. It follows that $q = sn - 1$. $\square$

THEOREM 1.1. *Let $R = K[X_1, \ldots, X_n]$ be the polynomial ring over a field K with $n > 2$. $I_{q,s}$ is generated by an s-sequence if and only if $q = sn - 1$.*

Proof. Let $I_{q,s} = (f_1, f_2, \ldots, f_t)$ and suppose that $f_1, f_2, \ldots, f_t$ is an s-sequence. We prove that $[f_{ij}, f_{hl}] = 1$ for $i < j$, $h < l$, $i \neq h$, $j \neq l$ with $i, j, h, l \in \{1, \ldots, t\}$.

The s-sequence property implies that $G = \{g_{ij} = f_{ij}T_j - f_{ji}T_i \mid 1 \leq i < j \leq t\}$ is a Gröbner basis for J. In particular, $S(g_{ij}, g_{hl})$ has a standard expression with respect to G with remainder 0. We have

$$S(g_{ij}, g_{hl}) = \frac{f_{ij}f_{lh}}{[f_{ij}, f_{hl}]}T_jT_h - \frac{f_{hl}f_{ji}}{[f_{ij}, f_{hl}]}T_iT_l.$$

We suppose that $i < j$, $h < l$, $i \neq h$, $j \neq l$. As $S(g_{ij}, g_{hl})$ has a standard expression with respect to G there exists g_{rp} such that $\text{in}_{\prec}(g_{rp})$ divides $\text{in}_{\prec}(S(g_{ij}, g_{hl}))$.

I) If $l > j$ then $\text{in}_{\prec}(g_{rp}) \mid \frac{f_{hl}f_{ji}}{[f_{ij}, f_{hl}]}$.

The first case is $f_{hl} \mid \frac{f_{hl}f_{ji}}{[f_{ij}, f_{hl}]}$, then $[f_{ij}, f_{hl}] \mid f_{ji}$. But, as we have $[[f_{ij}, f_{hl}], f_{ji}] = 1$, it follows $[f_{ij}, f_{hl}] = 1$.

The second case is $f_{rl} \mid \frac{f_{hl}f_{ji}}{[f_{ij}, f_{hl}]}$, where $f_{rl} = \text{in}_{\prec}(g_{rl})$ with $r < j$ and $r < h$. We can write

$$S(g_{ij}, g_{hl}) = -\frac{f_{ji}f_{hl}}{f_{rl}[f_{ij}, f_{hl}]}g_{rl}T_i + \frac{f_{ij}f_{lh}}{[f_{ij}, f_{hl}]}T_jT_h - \frac{f_{ji}f_{hl}f_{lr}}{f_{rl}[f_{ij}, f_{hl}]}T_iT_r.$$

Then $\frac{f_{ij}f_{lh}}{[f_{ij}, f_{hl}]}T_jT_h$ is divided by f_{ij}. Hence $[f_{ij}, f_{hl}] \mid f_{lh}$. But, as we have $[[f_{ij}, f_{hl}], f_{lh}] = 1$, it follows $[f_{ij}, f_{hl}] = 1$.

II) If $l < j$ then $\text{in}_{\prec}(g_{rp}) \mid \frac{f_{ij}f_{lh}}{[f_{ij}, f_{hl}]}$.

The first case is $f_{ij} \mid \frac{f_{ij}f_{lh}}{[f_{ij}, f_{hl}]}$, then $[f_{ij}, f_{hl}] \mid f_{lh}$. But, as we have $[[f_{ij}, f_{hl}], f_{lh}] = 1$, it follows $[f_{ij}, f_{hl}] = 1$.

The second case is $f_{rh} \mid \frac{f_{ij}f_{lh}}{[f_{ij}, f_{hl}]}$, where $f_{rh} = \text{in}_{\prec}(g_{rh})$. We can write

$$S(g_{ij}, g_{hl}) = \frac{f_{ij}f_{lh}}{f_{rh}[f_{ij}, f_{hl}]}T_jg_{rh} - \frac{f_{hl}f_{ji}}{[f_{ij}, f_{hl}]}T_iT_l + \frac{f_{ij}f_{lh}f_{hr}}{f_{rh}[f_{ij}, f_{hl}]}T_jT_r.$$

Then $\frac{f_{ij}f_{lh}f_{hr}}{f_{rh}[f_{ij}, f_{hl}]}T_jT_r$ is divided by f_{ij}. Hence $f_{rh}[f_{ij}, f_{hl}] \mid f_{lh}f_{hr}$. But as we have $[[f_{ij}, f_{hl}], f_{lh}] = 1$, $f_{rh} \mid f_{lh}$ and $[f_{ij}, f_{hl}] \mid f_{hr}$. By the structure of $f_1, \ldots, f_t$, if $[f_{ij}, f_{hl}] \mid f_{hr}$ with $r < h$ then $[f_{ij}, f_{hl}] = 1$.

Hence in any case we have $[f_{ij}, f_{hl}] = 1$ in the hypothesis $i < j$, $h < l$, $i \neq h$, $j \neq l$ with $i, j, h, l \in \{1, \ldots, t\}$. It follows $q = sn - 1$ by Lemma 1.1.

Conversely, let $q = sn - 1$. $I_{q,s}$ is generated by n monomials $f_1, \dots, f_n$: $f_1 = X_1^s X_2^s \cdots X_{n-1}^s X_n^{s-1}$, $f_2 = X_1^s X_2^s \cdots X_{n-1}^{s-1} X_n^s$, $f_3 = X_1^s X_2^s \cdots X_{n-2}^{s-1} X_{n-1}^s X_n^s$, …, $f_{n-1} = X_1^s X_2^{s-1} \cdots X_{n-2}^s X_{n-1}^s X_n^s$, $f_n = X_1^{s-1} X_2^s \cdots X_{n-2}^s X_{n-1}^s X_n^s$. We compute $f_{12} = X_{n-1}$, $f_{13} = X_{n-2}$, …, $f_{1n} = X_1$; $f_{23} = X_{n-2}$, …, $f_{2n} = X_1$, and so on. It follows that $f_{ij} \neq f_{hl}$ because $j \neq l$. Then $[f_{ij}, f_{hl}] = 1$ for $i < j$, $h < l$, $i \neq h$, $j \neq l$ with $i, j, h, l \in \{1, \dots, n\}$. By [6, Proposition 1.7], $f_1, \dots, f_n$ is an s-sequence. □

Example 1.3. $R = K[X_1, X_2, X_3]$, $I_{7,3} = (X_1^3 X_2^3 X_3, X_1^3 X_2^2 X_3^2, X_1^2 X_2^3 X_3^2, X_1^3 X_2 X_3^3, X_1^2 X_2^2 X_3^3, X_1 X_2^3 X_3^3)$. Set $f_1 = X_1^3 X_2^3 X_3$, $f_2 = X_1^3 X_2^2 X_3^2$, $f_3 = X_1^2 X_2^3 X_3^2$, $f_4 = X_1^3 X_2 X_3^3$, $f_5 = X_1^2 X_2^2 X_3^3$, $f_6 = X_1 X_2^3 X_3^3$, where $f_1 \prec \cdots \prec f_6$ with respect to the Lex order and $X_1 \prec X_2 \prec X_3$. Let $G = \{g_{ij} = f_{ij} T_j - f_{ji} T_i \mid 1 \leq i < j \leq 6\}$; $f_1, \dots, f_6$ is not an s-sequence because J does not admit a linear Gröbner basis for any term order in $R[T_1, \dots, T_6]$ with $X_i \prec T_j$ for all i, j and $T_1 \prec \cdots \prec T_6$. In fact there are S-pairs $S(g_{ij}, g_{hl})$ that have not a standard expression with respect G with remainder 0: $S(g_{25}, g_{36}) = \frac{f_{25} f_{63}}{[f_{25}, f_{36}]} T_3 T_5 - \frac{f_{36} f_{52}}{[f_{25}, f_{36}]} T_2 T_6 = X_3 T_3 T_5 - X_3 T_2 T_6$. There is no $g_{ij} \in G$ whose initial term divides a term of $S(g_{25}, g_{36})$.

Remark 1.3. A particular case is $s = 1$: $I_{q,1}$ is the square-free Veronese ideal of degree q generated by all the square-free monomials in the variables $X_1, \dots, X_n$ of degree q. $I_{q,1}$ is generated by an s-sequence if and only if $q = n-1$ as proved in [11].

Now we solve the problem to compute standard algebraic invariants of the symmetric algebra of Veronese-type ideals generated by s-sequences.

PROPOSITION 1.1. *Let $R = K[X_1, \dots, X_n]$ be the polynomial ring over a field K and $I_{sn-1,s} = (f_1, \dots, f_n)$. Then the annihilator ideals of $f_1, \dots, f_n$ are*

$$\mathcal{I}_1 = (0), \quad \mathcal{I}_i = (X_{n-i+1}) \quad \text{for } i = 2, \dots, n.$$

Proof. Let $I_{sn-1,s} = (f_1, \dots, f_n)$ with $f_1 \prec \ldots \prec f_n$: $f_1 = X_1^s X_2^s \cdots X_{n-1}^s X_n^{s-1}$, $f_2 = X_1^s X_2^s \cdots X_{n-1}^{s-1} X_n^s$, $f_3 = X_1^s X_2^s \cdots X_{n-2}^{s-1} X_{n-1}^s X_n^s, \dots, f_{n-1} = X_1^s X_2^{s-1} \cdots X_{n-2}^s X_{n-1}^s X_n^s$, $f_n = X_1^{s-1} X_2^s \cdots X_{n-2}^s X_{n-1}^s X_n^s$. Hence we observe that $I_{sn-1,s}$ is generated by $mh_1, \dots, mh_n$, where $m = X_1^{s-1} X_2^{s-1} \cdots X_{n-1}^{s-1} X_n^{s-1}$ and $h_i = X_1 \cdots \widehat{X_{n+1-i}} \cdots X_n$ for $i = 1, \dots, n$. Then $I_{n-1} = (h_1, \dots, h_n)$ is the square free Veronese ideal of degree $n - 1$. Hence the annihilator ideals of the sequence $f_1, \dots, f_n$ are the same of the sequence $h_1, \dots, h_n$ [11, Proposition 2.1]. □

THEOREM 1.2. *Let $R = K[X_1, \dots, X_n]$ be the polynomial ring over a field K and $I_{ns-1,s}$. Then*

1) $\dim(\mathrm{Sym}_R(I_{ns-1,s})) = n + 1$;

2) $\mathrm{e}(\mathrm{Sym}_R(I_{ns-1,s})) = \sum_{j=1}^{n-2} \binom{n-1}{j} + 2$;
3) $\mathrm{reg}(\mathrm{Sym}_R(I_{ns-1,s})) \leq n-1$.

Proof. 1) By Proposition 1.1 $\mathrm{in}_\prec(J) = ((X_{n-1})T_2, (X_{n-2})T_3, \ldots, (X_1)T_n)$ and it is generated by a regular sequence. We obtain $\dim(\mathrm{Sym}_R(I_{ns-1,s})) = n + n - (n-1) = n+1$.

2) By [6, Proposition 2.4] $\mathrm{e}(\mathrm{Sym}_R(I_{ns-1,s})) = \sum_{1\leq i_1<\cdots<i_r\leq n} \mathrm{e}(R/(\mathcal{I}_{i_1}, \ldots, \mathcal{I}_{i_r}))$ with $\dim(R/(\mathcal{I}_{i_1}, \ldots, \mathcal{I}_{i_r})) = d - r$ where $d = \dim(\mathrm{Sym}_R(I_{ns-1,s})) = n+1$ and $1 \leq r \leq n$. By Proposition 1.1, the annihilator ideals of the generators of $I_{ns-1,s}$ are the same of I_{n-1}. Then by [11, Theorem 2.1], we obtain $\mathrm{e}(\mathrm{Sym}_R(I_{ns-1,s})) = \sum_{j=1}^{n-2} \binom{n-1}{j} + 2$.

3) $\mathrm{reg}(\mathrm{Sym}_R(I_{ns-1,s})) = \mathrm{reg}(R[T_1, \ldots, T_n]/J) \leq \mathrm{reg}(R[T_1, \ldots, T_n]/\mathrm{in}_\prec(J))$, where $\mathrm{in}_\prec(J) = ((X_{n-1})T_2, (X_{n-2})T_3, \ldots, (X_1)T_n)$. By Proposition 1.1 $\mathrm{in}_\prec(J)$ is generated by a regular sequence of elements of degree 2, then $R[T_1, \ldots, T_n]/\mathrm{in}_\prec(J)$ has a 2-linear resolution and projective dimension $n-1$ equal to the number of the generators of $\mathrm{in}_\prec(J)$ ([5]): $0 \to S^{b_{n-1}}(-2(n-1)) \to \cdots \to S^{b_3}(-6) \to S^{b_2}(-4) \to S^{b_1}(-2) \to S \to S/\mathrm{in}_\prec(J) \to 0$, where $S = R[T_1, \ldots, T_n]$. Then $\mathrm{reg}(R[T_1, \ldots, T_n]/\mathrm{in}_\prec(J)) = n-1$. □

2. IDEALS OF VERONESE BI-TYPE

In this section we consider the class of monomial ideals of Veronese bi-type in the polynomial ring $R = K[X_1, \ldots, X_n; Y_1, \ldots, Y_m]$.

Definition 2.1 ([10]). Let $R = K[X_1, \ldots, X_n; Y_1, \ldots, Y_m]$ be the polynomial ring over a field K in two sets of variables. We define the *ideals of Veronese bi-type* of degree q as the monomial ideals of R

$$L_{q,s} = \sum_{r+k=q} I_{r,s} J_{k,s}, \quad r, k \geq 1,$$

where $I_{r,s}$ is the ideal of Veronese-type of degree r in the variables $X_1, \ldots, X_n$ and $J_{k,s}$ is the ideal of Veronese-type of degree k in the variables $Y_1, \ldots, Y_m$.

Example 2.1. Let $R = K[X_1, X_2; Y_1, Y_2]$ be a polynomial ring. 1) $L_{2,2} = I_{1,2}J_{1,2} = I_1J_1 = (X_1Y_1, X_1Y_2, X_2Y_1, X_2Y_2)$; 2) $L_{4,2} = I_{3,2}J_{1,2} + I_{1,2}J_{3,2} + I_{2,2}J_{2,2} = I_{3,2}J_1 + I_1J_{3,2} + I_2J_2 = (X_1^2X_2Y_1, X_1^2X_2Y_2, X_1X_2^2Y_1, X_1X_2^2Y_2, X_1Y_1^2Y_2, X_2Y_1^2Y_2, X_1Y_1Y_2^2, X_2Y_1Y_2^2, X_1^2Y_1^2, X_1^2Y_1Y_2, X_1^2Y_2^2, X_2^2Y_1^2, X_2^2Y_2^2, X_2^2Y_1Y_2, X_1X_2Y_1^2, X_1X_2Y_2^2, X_1X_2Y_1Y_2)$.

Remark 2.1. For $s=1$ and $q=2,3$ we have $L_{q,1} = \sum_{r+k=q} I_{r,1}J_{k,1}$, $r, k \geq 1$, a square-free monomial ideal, more precisely a mixed product ideal ([14]).

Now our aim is to investigate in which cases $L_{q,s}$ is generated by an s-sequence. In the sequel we will suppose $L = (f_1, f_2, \ldots, f_t)$ where $f_1 \prec$

$f_2 \prec \cdots \prec f_t$ with respect to the monomial order $\prec_{Lex}$ on the variables $X_1, \ldots, X_n; Y_1, \ldots, Y_m$ and $X_1 \prec X_2 \prec \cdots \prec X_n \prec Y_1 \prec Y_2 \prec \cdots \prec Y_m$.

LEMMA 2.1. *Let* $R = K[X_1, \ldots, X_n; Y_1, \ldots, Y_m]$ *be the polynomial ring over a field* K *and* $L_{q,s} \subset R$. *If* $[f_{ij}, f_{hl}] = 1$ *for* $i < j$, $h < l$, $i \neq h$, $j \neq l$ *with* $i, j, h, l \in \{1, \ldots, t\}$, *then* $q = s(n+m) - 1$.

Proof. Let $L_{q,s} = (f_1, \ldots, f_t)$. Set $f_{ij} = \frac{f_i}{[f_i, f_j]}$ forf_i, f_j with $i < j$. We have $f_{ij} = X_{i_1}^{a_{i_1}} \cdots X_{i_w}^{a_{i_w}} Y_{i_1}^{b_{i_1}} \cdots Y_{i_z}^{b_{i_z}}$. In the same way for f_h, f_l with $h < l$, $i \neq h$, $j \neq l$ we have $f_{hl} = X_{i_1}^{c_{i_1}} \cdots X_{i_x}^{c_{i_x}} Y_{i_1}^{d_{i_1}} \cdots Y_{i_y}^{d_{i_y}}$. By the hypothesis $[f_{ij}, f_{hl}] = 1$ for $i < j$, $h < l$, $i \neq h$, $j \neq l$. Then it follows that $X_{i_j} \neq X_{h_p}$ for all $j = 1, \ldots, w$, $p = 1, \ldots, x$, and $Y_{i_j} \neq Y_{h_p}$ for all $j = 1, \ldots, z$, $p = 1, \ldots, y$. This means that there are no other generators f_h, f_l of $L_{q,s}$ such that f_{hl} contains one of the variables of f_{ij}. It follows that if a variable of f_{ij} is in degree N in the monomial f_h, with $h \neq i, j$, then such variable in degree N belongs to any other generators f_l for all $l > h$ and $l \neq j$. In the same way of the Lemma 1.1 we deduce the structure of the monomials that generate $L_{q,s}$:

$$\begin{aligned}
f_1 &= X_1^s X_2^s \cdots X_{n-2}^s X_{n-1}^s X_n^s Y_1^s Y_2^s \cdots Y_{m-1}^s Y_m^{s-1}, \\
f_2 &= X_1^s X_2^s \cdots X_{n-2}^s X_{n-1}^s X_n^s Y_1^s Y_2^s \cdots Y_{m-1}^{s-1} Y_m^s, \\
f_3 &= X_1^s X_2^s \cdots X_{n-2}^s X_{n-1}^s X_n^s Y_1^s Y_2^s \cdots Y_{m-2}^{s-1} Y_{m-1}^s Y_m^s, \\
&\cdots \\
f_{n+m-1} &= X_1^s X_2^{s-1} \cdots X_{n-2}^s X_{n-1}^s X_n^s Y_1^s Y_2^s \cdots Y_{m-1}^s Y_m^s, \\
f_{n+m} &= X_1^{s-1} X_2^s \cdots X_{n-2}^s X_{n-1}^s X_n^s Y_1^s Y_2^s \cdots Y_{m-1}^s Y_m^s. \quad \square
\end{aligned}$$

THEOREM 2.1. *Let* $R = K[X_1, \ldots, X_n; Y_1, \ldots, Y_m]$ *be the polynomial ring over a field* K. $L_{q,s}$ *is generated by an* s*-sequence if and only if* $q = s(n+m)-1$.

Proof. Let $L_{q,s} = (f_1, f_2, \ldots, f_t)$ and suppose that $f_1, f_2, \ldots, f_t$ is an s-sequence. We prove that $[f_{ij}, f_{hl}] = 1$ for $i < j$, $h < l$, $i \neq h$, $j \neq l$ with $i, j, h, l \in \{1, \ldots, t\}$.

The s-sequence property implies that $G = \{g_{ij} = f_{ij}T_j - f_{ji}T_i \mid 1 \leq i < j \leq t\}$ is a Gröbner basis for J. This means that $S(g_{ij}, g_{hl})$ has a standard expression with respect to G with remainder 0. We have

$$S(g_{ij}, g_{hl}) = \frac{f_{ij} f_{lh}}{[f_{ij}, f_{hl}]} T_j T_h - \frac{f_{hl} f_{ji}}{[f_{ij}, f_{hl}]} T_i T_l.$$

We suppose that $i < j$, $h < l$, $i \neq h$, $j \neq l$. As $S(g_{ij}, g_{hl})$ has a standard expression with respect to G, there exists g_{rp} such that $\text{in}_\prec(g_{rp})$ divides $\text{in}_\prec(S(g_{ij}, g_{hl}))$.

I) If $l > j$ then $\mathrm{in}_{\prec}(g_{rp}) \mid \frac{f_{hl}f_{ji}}{[f_{ij},f_{hl}]}$.

The first case is $f_{hl} \mid \frac{f_{hl}f_{ji}}{[f_{ij},f_{hl}]}$, then $[f_{ij}, f_{hl}] \mid f_{ji}$. But, as we have $[[f_{ij}, f_{hl}], f_{ji}] = 1$, it follows $[f_{ij}, f_{hl}] = 1$.

The second case is $f_{rl} \mid \frac{f_{hl}f_{ji}}{[f_{ij},f_{hl}]}$, where $f_{rl} = \mathrm{in}_{\prec}(g_{rl})$ with $r < j$ and $r < h$. We can write

$$S(g_{ij}, g_{hl}) = -\frac{f_{ji}f_{hl}}{f_{rl}[f_{ij}, f_{hl}]} g_{rl}T_i + \frac{f_{ij}f_{lh}}{[f_{ij}, f_{hl}]} T_jT_h - \frac{f_{ji}f_{hl}f_{lr}}{f_{rl}[f_{ij}, f_{hl}]} T_iT_r.$$

Then $\frac{f_{ij}f_{lh}}{[f_{ij},f_{hl}]}T_jT_h$ is divided by f_{ij}. Hence $[f_{ij}, f_{hl}] \mid f_{lh}$. But, as we have $[[f_{ij}, f_{hl}], f_{lh}] = 1$, it follows $[f_{ij}, f_{hl}] = 1$.

II) If $l < j$ then $\mathrm{in}_{\prec}(g_{rp}) \mid \frac{f_{ij}f_{lh}}{[f_{ij},f_{hl}]}$.

The first case is $f_{ij} \mid \frac{f_{ij}f_{lh}}{[f_{ij},f_{hl}]}$, then $[f_{ij}, f_{hl}] \mid f_{lh}$. But, as we have $[[f_{ij}, f_{hl}], f_{lh}] = 1$, it follows $[f_{ij}, f_{hl}] = 1$.

The second case is $f_{rh} \mid \frac{f_{ij}f_{lh}}{[f_{ij},f_{hl}]}$, where $f_{rh} = \mathrm{in}_{\prec}(g_{rh})$. We can write

$$S(g_{ij}, g_{hl}) = \frac{f_{ij}f_{lh}}{f_{rh}[f_{ij}, f_{hl}]} T_jg_{rh} - \frac{f_{hl}f_{ji}}{[f_{ij}, f_{hl}]} T_iT_l + \frac{f_{ij}f_{lh}f_{hr}}{f_{rh}[f_{ij}, f_{hl}]} T_jT_r.$$

Then $\frac{f_{ij}f_{lh}f_{hr}}{f_{rh}[f_{ij},f_{hl}]}T_jT_r$ is divided by f_{ij}. Hence $f_{rh}[f_{ij}, f_{hl}] \mid f_{lh}f_{hr}$. But as we have $[[f_{ij}, f_{hl}], f_{lh}] = 1$, $f_{rh} \mid f_{lh}$ and $[f_{ij}, f_{hl}] \mid f_{hr}$. By the structure of $f_1, \dots, f_t$, if $[f_{ij}, f_{hl}] \mid f_{hr}$ with $r < h$, then $[f_{ij}, f_{hl}] = 1$. Hence in any case we have $[f_{ij}, f_{hl}] = 1$ in the hypothesis $i < j$, $h < l$, $i \neq h$, $j \neq l$ with $i, j, h, l \in \{1, \dots, t\}$. It follows $q = s(n + m) - 1$ by Lemma 2.1.

Conversely, let $q = s(n + m) - 1$. The generators of $L_{q,s}$ are

$$\begin{aligned}
f_1 &= X_1^sX_2^s \cdots X_{n-2}^sX_{n-1}^sX_n^sY_1^sY_2^s \cdots Y_{m-1}^sY_m^{s-1},\\
f_2 &= X_1^sX_2^s \cdots X_{n-2}^sX_{n-1}^sX_n^sY_1^sY_2^s \cdots Y_{m-1}^{s-1}Y_m^s,\\
f_3 &= X_1^sX_2^s \cdots X_{n-2}^sX_{n-1}^sX_n^sY_1^sY_2^s \cdots Y_{m-2}^{s-1}Y_{m-1}^sY_m^s,\\
&\cdots\\
f_{n+m-1} &= X_1^sX_2^{s-1} \cdots X_{n-2}^sX_{n-1}^sX_n^sY_1^sY_2^s \cdots Y_{m-1}^sY_m^s,\\
f_{n+m} &= X_1^{s-1}X_2^s \cdots X_{n-2}^sX_{n-1}^sX_n^sY_1^sY_2^s \cdots Y_{m-1}^sY_m^s.
\end{aligned}$$

We compute $f_{12} = Y_{m-1}$, $f_{13} = Y_{m-2}$, …, $f_{1m} = Y_1$, $f_{1,m+1} = X_n$, $f_{1,m+2} = X_{n-1}$, …, $f_{1,n+m} = X_1$, $f_{23} = Y_{m-2}$, …, $f_{2m} = Y_1$, $f_{2,m+1} = X_n$, $f_{2,m+2} = X_{n-1}$, …, $f_{2,n+m} = X_1$ and so on. Hence $f_{ij} \neq f_{hl}$ because $j \neq l$. Then $[f_{ij}, f_{hl}] = 1$ for $i < j$, $h < l$, $i \neq h$, $j \neq l$ with $i, j, h, l \in \{1, \dots, n + m\}$. By [6, Proposition 1.7] it follows that $f_1, \dots, f_{n+m}$ is an s-sequence. $\square$

Example 2.2. $R = K[X_1, X_2; Y_1, Y_2]$, $L_{3,2} = (X_1^2Y_1, X_1X_2Y_1, X_2^2Y_1, X_1Y_1^2, X_2Y_1^2, X_1^2Y_2, X_1X_2Y_2, X_2^2Y_2, X_1Y_1Y_2,, X_2Y_1Y_2, X_1Y_2^2, X_2Y_2^2)$. Let $G = \{g_{ij} =$

$f_{ij}T_j - f_{ji}T_i \mid 1 \le i < j \le 12\}$; $f_1, \dots, f_{12}$ is not an s-sequence because J does not admit a linear Gröbner basis for any term order in $R[T_1, \dots, T_{12}]$ with $X_i \prec T_j$, $Y_i \prec T_j$ for all i, j and $T_1 \prec \cdots \prec T_{12}$. In fact, there are S-pairs $S(g_{ij}, g_{hl})$ that have not a standard expression with respect G with remainder 0: $S(g_{16}, g_{27}) = \frac{f_{16}f_{72}}{[f_{16},f_{27}]}T_2T_6 - \frac{f_{61}f_{27}}{[f_{16},f_{27}]}T_1T_7 = Y_2T_2T_6 - Y_2T_1T_7$. There is no $g_{ij} \in G$ whose initial term divides a term of $S(g_{16}, g_{27})$.

As in Section 1 we use the theory of s-sequences to compute standard invariants of the symmetric algebra of the monomials ideals $L_{q,s}$ generated by an s-sequence.

Proposition 2.1. *Let $R = K[X_1, \dots, X_n; Y_1, \dots, Y_m]$ be a polynomial ring over a field K in two sets of variables and $L_{q,s}$ for $q = s(n+m)-1$. Then the annihilator ideals of the generators of $L_{q,s}$ are $\mathcal{I}_1 = (0)$, $\mathcal{I}_i = (Y_{m-i+1})$ for $i = 2, \dots, m$, $\mathcal{I}_i = (X_{n+m-i+1})$ for $i = m+1, \dots, m+n$.*

Proof. Let $q = s(n+m)-1$. $L_{q,s} = (f_1, \dots, f_{n+m})$ with $f_1 \prec \cdots \prec f_{n+m}$ and

$$
\begin{aligned}
f_1 &= X_1^sX_2^s\cdots X_{n-2}^sX_{n-1}^sX_n^sY_1^sY_2^s\cdots Y_{m-1}^sY_m^{s-1},\\
f_2 &= X_1^sX_2^s\cdots X_{n-2}^sX_{n-1}^sX_n^sY_1^sY_2^s\cdots Y_{m-1}^{s-1}Y_m^s,\\
f_3 &= X_1^sX_2^s\cdots X_{n-2}^sX_{n-1}^sX_n^sY_1^sY_2^s\cdots Y_{m-2}^{s-1}Y_{m-1}^sY_m^s,\\
&\cdots\\
f_{n+m-1} &= X_1^sX_2^{s-1}\cdots X_{n-2}^sX_{n-1}^sX_n^sY_1^sY_2^s\cdots Y_{m-1}^sY_m^s,\\
f_{n+m} &= X_1^{s-1}X_2^s\cdots X_{n-2}^sX_{n-1}^sX_n^sY_1^sY_2^s\cdots Y_{m-1}^sY_m^s.
\end{aligned}
$$

The annihilator ideals of the sequence $f_1, \dots, f_{n+m}$ are $\mathcal{I}_i = (f_{1i}, f_{2i}, \dots, f_{i-1,i})$ for $i = 1, \dots, n+m$. For $i = 1$ we have $\mathcal{I}_1 = (0)$. By the structure of the monomials $f_1, \dots, f_{n+m}$ we have $\mathcal{I}_2 = (f_{12}) = (Y_{m-1})$, $\mathcal{I}_3 = (f_{13}, f_{23}) = (Y_{m-2})$, $\dots$, $\mathcal{I}_m = (f_{1m}, \dots, f_{m-1,m}) = (Y_1)$, $\mathcal{I}_{m+1} = (f_{1,m+1}, \dots, f_{m,m+1}) = (X_n)$, $\dots$, $\mathcal{I}_{n+m} = (f_{1,n+m}, \dots, f_{n+m-1,n+m}) = (X_1)$. □

Theorem 2.2. *Let $R = K[X_1, \dots, X_n; Y_1, \dots, Y_m]$ be the polynomial ring over a field K in two sets of variables and $L_{q,s}$ for $q = s(n+m)-1$. Then*

1) $\dim(\mathrm{Sym}_R(L_{q,s})) = n+m+1$;
2) $\mathrm{e}(\mathrm{Sym}_R(L_{q,s})) = \sum_{j=1}^{n+m-2}\binom{n+m-1}{j} + 2$;
3) $\mathrm{reg}(\mathrm{Sym}_R(L_{q,s})) \le n+m-1$.

Proof. 1) By Proposition 2.1, we have $\mathcal{I}_1 = (0)$, $\mathcal{I}_i = (Y_{m-i+1})$ for $i = 2, \dots, m$, $\mathcal{I}_i = (X_{n+m-i+1})$ for $i = m+1, \dots, m+n$. Then $\mathrm{in}_\prec(J)$ is generated by a regular sequence. We obtain $\dim(\mathrm{Sym}_R(L_{q,s})) = n+m+n+m-(n+m-1) = n+m+1$.

2) By [6, Proposition 2.4], $\mathrm{e}(\mathrm{Sym}_R(L_{q,s})) = \sum_{1\le i_1<\cdots<i_r\le n+m} \mathrm{e}(R/(\mathcal{I}_{i_1},\dots,\mathcal{I}_{i_r}))$ with $\dim(R/(\mathcal{I}_{i_1},\dots,\mathcal{I}_{i_r})) = d-r$, where $d = \dim(\mathrm{Sym}_R(L_{q,s})) = n+m+1$ and $1 \le r \le n+m$. By Proposition 2.1, $\mathcal{H} = (\mathcal{I}_{i_1},\dots,\mathcal{I}_{i_r})$ is generated by a regular sequence of variables of R. Hence $R/\mathcal{H}$ is Cohen-Macaulay and has an 1-linear resolution, the projective dimension is equal to the number of the generators of $\mathcal{H}$ ([5]). Then $\mathrm{e}(R/\mathcal{H}) = 1$ by Huneke-Miller formula ([8]).

Set $d' = \dim(R/(\mathcal{I}_{i_1},\dots,\mathcal{I}_{i_r})) = n+m+1-r$, then $\mathrm{e}(\mathrm{Sym}_R(L_{q,s}))$ is given by the sum of the terms $\mathrm{e}(R/(0)) = 1$ for $r = 1$ and $d' = n+m$, $\sum_{j=2}^{n+m} \mathrm{e}(R/\mathcal{I}_j) = \underbrace{1+\cdots+1}_{n+m-1}$ for $r = 2$ and $d' = n+m-1$,

$\sum_{2\le i_1<i_2\le n+m} \mathrm{e}(R/(\mathcal{I}_{i_1},\mathcal{I}_{i_2})) = \underbrace{1+\cdots+1}_{\binom{n+m-1}{2}}$ for $r = 3$ and $d' = n+m-2$,

$\sum_{2\le i_1<i_2<i_3\le n+m} \mathrm{e}(R/(\mathcal{I}_{i_1},\mathcal{I}_{i_2},\mathcal{I}_{i_3})) = \underbrace{1+\cdots+1}_{\binom{n+m-1}{3}}$ for $r = 4$ and $d' = n+m-3$,

and so on up to

$\sum_{2\le i_1<\cdots<i_{n+m-2}\le n+m} \mathrm{e}(R/(\mathcal{I}_{i_1},\dots,\mathcal{I}_{i_{n+m-2}})) = \underbrace{1+\cdots+1}_{\binom{n+m-1}{n-2}}$ for $r = n+m-1$

and $d' = 2$, $\mathrm{e}(R/(\mathcal{I}_2,\mathcal{I}_3,\cdots,\mathcal{I}_{n+m})) = 1$ for $r = n+m$ and $d' = 1$.

Hence we obtain $\mathrm{e}(\mathrm{Sym}_R(L_{q,s})) = \sum_{j=1}^{n-2}\binom{n+m-1}{j} + 2$.

3) $\mathrm{reg}(\mathrm{Sym}_R(L_{q,s})) = \mathrm{reg}(R[T_1,\dots,T_{n+m}]/J) \le \mathrm{reg}(R[T_1,\dots,T_{n+m}]/\mathrm{in}_\prec(J))$, where $\mathrm{in}_\prec(J) = ((Y_{m-1})T_2,\dots,(Y_1)T_m,(X_n)T_{m+1},(X_{n-1})T_{m+2}\dots,(X_1)T_{n+m})$. By Proposition 2.1, $\mathrm{in}_\prec(J)$ is generated by a regular sequence of elements of degree 2, then $R[T_1,\dots,T_n]/\mathrm{in}_\prec(J)$ has a 2-linear resolution and projective dimension $n+m-1$ equal to the number of the generators of $\mathrm{in}_\prec(J)$ ([5]): $0 \to S^{b_{n+m-1}}(-2(n+m-1)) \to \cdots \to S^{b_3}(-6) \to S^{b_2}(-4) \to S^{b_1}(-2) \to S \to S/\mathrm{in}_\prec(J) \to 0$, where $S = R[T_1,\dots,T_{n+m}]$.

Then $\mathrm{reg}(R[T_1,\dots,T_{n+m}]/\mathrm{in}_\prec(J)) = n+m-1$. □

As an application of the previous results, now we verify the Conjecture that is formulated in [11].

Conjecture. Let $R = K[X_1,\dots,X_n]$, $\mathrm{Sym}_R(M)$ be the symmetric algebra of a graded module M generated on R by an s-sequence of elements of the same degree. Then

$$\mathrm{reg}(\mathrm{Sym}_R(M)) \le \mathrm{e}(\mathrm{Sym}_R(M)) - \mathrm{codim}(\mathrm{Sym}_R(M)).$$

Proposition 2.2. *Let $R = K[X_1,\dots,X_n]$ be the polynomial ring over a field K and $I_{sn-1,s}$. Then the Conjecture is true for the symmetric algebra of $I_{sn-1,s}$ and we have:*

$$\mathrm{reg}(\mathrm{Sym}_R(I_{sn-1,s})) < \mathrm{e}(\mathrm{Sym}_R(I_{sn-1,s})) - \mathrm{codim}(\mathrm{Sym}_R(I_{sn-1,s})).$$

Proof. Let $I_{q,s}$ for $q = sn - 1$. $\text{reg}(\text{Sym}_R(I_{q,s})) \leq n - 1 < (\sum_{j=1}^{n-2} \binom{n-1}{j} + 2) - (2n) + (n+1) = \text{e}(\text{Sym}_R(I_{q,s})) - (2n) + \dim(\text{Sym}_R(I_{q,s})) \leq \text{e}(\text{Sym}_R(I_{q,s})) - \text{embdim}(\text{Sym}_R(I_{q,s})) + \dim(\text{Sym}_R(I_{q,s}))$, as $\text{embdim}(\text{Sym}_R(I_{q,s})) \leq n+n = 2n$ and $\text{codim}(\text{Sym}_R(I_{q,s})) = \text{embdim}(\text{Sym}_R(I_{q,s})) - \dim(\text{Sym}_R(I_{q,s}))$. □

PROPOSITION 2.3. *Let $R = K[X_1, \dots, X_n; Y_1, \dots, Y_m]$ be the polynomial ring over a field K in two sets of variables and $L_{q,s}$ for $q = s(n + m) - 1$. Then the Conjecture is true for the symmetric algebra of $L_{q,s}$ and we have:*

$$\text{reg}(\text{Sym}_R(L_{q,s})) < \text{e}(\text{Sym}_R(L_{q,s})) - \text{codim}(\text{Sym}_R(L_{q,s})).$$

Proof. Let $q = s(n + m) - 1$. Then $\text{reg}(\text{Sym}_R(L_{q,s})) \leq n + m - 1 < (\sum_{j=1}^{n+m-2} \binom{n+m-1}{j} + 2) - (2n + 2m) + (n + m + 1) = \text{e}(\text{Sym}_R(L_{q,s})) - (2n + 2m) + \dim(\text{Sym}_R(L_{q,s})) \leq \text{e}(\text{Sym}_R(L_{q,s})) - \text{embdim}(\text{Sym}_R(L_{q,s})) + \dim(\text{Sym}_R(L_{q,s})) = \text{e}(\text{Sym}_R(L_{q,s})) - \text{codim}(\text{Sym}_R(L_{q,s}))$. □

3. APPLICATIONS

For $s = 2$, the ideals $L_{q,s}$ are associated to the walks of length $q - 1$ of the strong quasi-bipartite graphs with loops ([9]).

Definition 3.1 ([9]). A graph G with loops is a *strong quasi-bipartite* if all the vertices of V_1 are joined to all the vertices of V_2 and for each vertex of V there is a loop.

Definition 3.2. Let G be a strong quasi-bipartite graph. A *walk of length* q in G is an alternating sequence $w = \{v_{i_0}, l_{i_1}, v_{i_1}, l_{i_2}, \dots, v_{i_{q-1}}, l_{i_q}, v_{i_q}\}$, where v_{i_j} is a vertex of G and $l_{i_j} = \{v_{i_{j-1}}, v_{i_j}\}$ is the edge joining $v_{i_{j-1}}$ and v_{i_j} or a loop if $v_{i_{j-1}} = v_{i_j}$, $1 \leq i_1 \leq i_2 \leq \dots \leq i_q \leq n$.

Example 3.1. Let G be a strong quasi-bipartite graph on vertices $\{x_1, x_2; y_1, y_2\}$. A walk of length 2 is

$$w = \{x_1, l_1, x_1, l_2, y_1\},$$

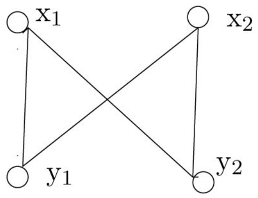
where $l_1 = \{x_1, x_1\}$ is the loop on x_1 and $l_2 = \{x_1, y_1\}$ is the edge joining x_1 and y_1. (A walk w in G cannot have the edges $\{x_i, x_j\}$, with $i \neq j$ and $\{y_s, y_t\}$ with $s \neq t$, because G is bipartite.)

Let G be a quasi-bipartite graph on vertex set $\{x_1, \dots, x_n; y_1, \dots, y_m\}$. The *generalized ideal* $I_q(G)$ associated with G is the ideal of the polynomial ring $R = K[X_1, \dots, X_n; Y_1, \dots, Y_m]$ generated by the monomials of degree q corresponding to the walks of length $q-1$. Hence the generalized ideal $I_q(G)$ is generated by all the monomials of degree $q \geq 3$ corresponding to the walks of length $q-1$ and the variables in each generator of $I_q(G)$ have at most degree 2.

Therefore,

$$I_q(G) = L_{q,2} = \sum_{r+s=q} I_{r,2}J_{s,2}, \qquad \text{for } q \geq 3 \text{ ([9])}.$$

Example 3.2. Let $R = K[X_1, X_2; Y_1, Y_2]$ be a polynomial ring over a field K and G be the strong quasi-bipartite graph on vertices x_1, x_2, y_1, y_2:



$I_3(G) = I_1J_2 + I_2J_1 = (X_1Y_1Y_2, X_2Y_1Y_2, X_1Y_1^2, X_2Y_1^2, X_1Y_2^2, X_2Y_2^2, X_1X_2Y_1, X_1X_2Y_2, X_1^2Y_1, X_1^2Y_2, X_2^2Y_1, X_2^2Y_2)$.
$I_4(G) = I_{3,2}J_1+I_1J_{3,2}+I_2J_2 = (X_1^2X_2Y_1, X_1^2X_2Y_2, X_1X_2^2Y_1, X_1X_2^2Y_2, X_1Y_1^2Y_2, X_2Y_1^2Y_2, X_1Y_1Y_2^2, X_2Y_1Y_2^2, X_1^2Y_1^2, X_1^2Y_1Y_2, X_1^2Y_2^2, X_2^2Y_1^2, X_2^2Y_2^2, X_2^2Y_1Y_2, X_1X_2Y_1^2, X_1X_2Y_2^2, X_1X_2Y_1Y_2)$.

Remark 3.1. For $q = 2$ the ideal $L_{q,2}$ does not describe the edge ideal $I(G) = I_2(G)$ of a strong quasi-bipartite graph. In fact, if we consider the strong quasi-bipartite graph on vertices x_1, x_2, y_1, y_2 then $I(G) = (X_1Y_1, X_1Y_2, X_2Y_1, X_2Y_2, X_1^2, X_2^2, Y_1^2, Y_2^2)$, but $L_{2,2} = (X_1Y_1, X_1Y_2, X_2Y_1, X_2Y_2)$. Hence $I(G) \neq L_{2,2}$.

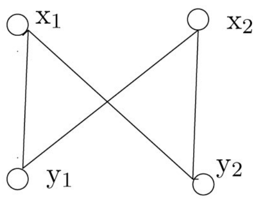
The following result classifies the ideals $I_q(G)$ that are generated by an s-sequence.

THEOREM 3.1. *Let G be a quasi-bipartite graph on the vertex set $\{x_1, \ldots, x_n; y_1, \ldots, y_m\}$. The generalized ideal $I_q(G)$ is generated by an s-sequence if and only if $q = 2(n + m) - 1$.*

Proof. One has $I_q(G) = L_{q,2}$ ([9]), then by Theorem 2.1 the proof is complete. □

Remark 3.2. The generators of $I_q(G)$ that form an s-sequence correspond in the quasi-bipartite graph G to the walks of length $2(n+m)-2$. We observe that the maximal length of the walks on G is $2(m + n) - 1$.

Example 3.3. Let $R = K[X_1, X_2; Y_1, Y_2]$ be a polynomial ring over a field K and G be the strong quasi-bipartite graph on the vertices x_1, x_2, y_1, y_2:



The walks of length 6 correspond to the generators of the generalized ideal of degree $q = 7$:

$I_7(G) = I_{4,2}J_{3,2}+I_{3,2}J_{4,2} = (X_1^2X_2^2Y_1^2Y_2, X_1^2X_2^2Y_1Y_2^2, X_1^2X_2Y_1^2Y_2^2, X_1X_2^2Y_1^2Y_2^2)$

that is the generalized ideal of G generated by an s-sequence.

Now we give a good property for the ideals $I_q(G)$ generated by a s-sequence. More precisely we prove that they have linear quotients.

Definition 3.3. Let L be a monomial ideal of $R = K[X_1, \dots, X_n; Y_1, \dots, Y_m]$ and $G(L)$ be its unique set of minimal generators. L has *linear quotients* if there is an ordering $u_1, \dots, u_t$ of monomials belonging to $G(L)$ with $\deg u_1 \leq \cdots \leq \deg u_t$ such that for each $2 \leq j \leq t$ the colon ideal $(u_1, \dots, u_{j-1}) : u_j$ is generated by a subset of $\{X_1, \dots, X_n; Y_1, \dots, Y_m\}$.

It is known that if a monomial ideal generated in the same degree has linear quotients, then it has a linear resolution ([2]).

Definition 3.4. Let L be a monomial ideal of R with linear quotients with respect to the ordering $u_1, \dots, u_t$ of the monomials of $G(L)$. We denote by $q_j(L)$ the number of the variables which is required to generate the ideal $(u_1, \dots, u_{j-1}) : u_j$. Set $q(L) = \max_{2\leq j\leq t} q_j(L)$.

The integer $q(L)$ is independent of the choice of the ordering of the generators that gives linear quotients ([4]).

Definition 3.5. Let L be a monomial ideal of R. A *vertex cover* of L is a subset W of $\{X_1, \dots, X_n; Y_1, \dots, Y_m\}$ such that each $u \in G(L)$ is divided by some variables of W. Denote by $h(I)$ the minimal cardinality of the vertex covers of L.

Theorem 3.2. *Let G be a quasi-bipartite graph on the vertex set $\{x_1, \dots, x_n; y_1, \dots, y_m\}$. The generalized ideals $I_q(G)$, for $q = 2(n+m)-1$, have linear quotients.*

Proof. Let $q = s(n+m) - 1$. The generators of $I_q(G)$ are

$$
\begin{aligned}
f_1 &= X_1^2X_2^2 \cdots X_{n-2}^2X_{n-1}^2X_n^2Y_1^2Y_2^2 \cdots Y_{m-1}^{2Y_m},\\
f_2 &= X_1^2X_2^2 \cdots X_{n-2}^2X_{n-1}^2X_n^2Y_1^2Y_2^2 \cdots Y_{m-1}Y_m^2,\\
f_3 &= X_1^2X_2^2 \cdots X_{n-2}^2X_{n-1}^2X_n^2Y_1^2Y_2^2 \cdots Y_{m-2}Y_{m-1}^2Y_m^2,\\
&\cdots\\
f_{n+m-1} &= X_1^2X_2 \cdots X_{n-2}^2X_{n-1}^2X_n^2Y_1^2Y_2^2 \cdots Y_{m-1}^2Y_m^2,\\
f_{n+m} &= X_1X_2^2 \cdots X_{n-2}^2X_{n-1}^2X_n^2Y_1^2Y_2^2 \cdots Y_{m-1}^2Y_m^2.
\end{aligned}
$$

We compute

$$\begin{aligned}
\mathcal{I}_2 &= (f_1) : (f_2) = (Y_{m-1}),\\
\mathcal{I}_3 &= (f_1, f_2) : (f_3) = (Y_{m-2}),\\
&\cdots\\
\mathcal{I}_m &= (f_1, f_2, \ldots, f_{m-1}) : (f_m) = (Y_1),\\
\mathcal{I}_{m+1} &= (f_1, f_2, \ldots, f_m) : (f_{m+1}) = (X_n),\\
&\cdots\\
\mathcal{I}_{m+n} &= (f_1, f_2, \ldots, f_{m+n-1}) : (f_{m+n}) = (X_1).
\end{aligned}$$

Hence $I_q(G)$ has linear quotients. □

COROLLARY 3.1. *Let* $R = K[X_1, \ldots, X_n; Y_1, \ldots, Y_m]$ *with* $n, m > 1$ *and* $I_q(G)$ *with* $q = 2(n + m) - 1$. *Then*

1) $\dim_R(R/I_q(G)) = n + m - 1$;
2) $\mathrm{pd}_R(R/I_q(G)) = 2$;
3) $\mathrm{depth}_R(R/I_q(G)) = n + m - 2$;
4) $\mathrm{reg}_R(R/I_q(G)) = 2(n + m - 1)$.

Proof. Let $q = 2(n + m) - 1$ and $I_q(G)$ be ideal of R. By the proof of Theorem 3.2 we have $q(I_q(G)) = 1$. The minimal cardinality of the vertex covers of $I_q(G)$ is $h(I_q(G)) = 1$, $W = \{X_1\}$ being a minimal vertex cover of $I_q(G)$. It follows that

1) $\dim_R(R/I_q(G)) = n + m - 1$ ([4]).

2) The length of the minimal free resolution of $R/I_q(G)$ over R is equal to $q(I_q(G)) + 1$ ([7, Corollary 1.6]). Hence $\mathrm{pd}_R(R/I_q(G)) = 2$.

3) As a consequence of 1) and 2) we compute $\mathrm{depth}_R(R/I_q(G)) = n + m - \mathrm{pd}_R(R/I_q(G)) = n + m - 2$.

4) $I_q(G)$ is a monomial ideal generated in degree q that has linear quotients, then $I_q(G)$ has linear resolution ([2]). Hence $\mathrm{reg}_R(R/I_q(G)) = q - 1 = 2(m + m) - 1 - 1 = 2(n + m - 1)$. □

Example 3.4. Let $R = K[X_1, X_2; Y_1, Y_2]$ be a polynomial ring over a field K and G be the strong quasi-bipartite graph on the vertices x_1, x_2, y_1, y_2 (see Example 3.3). $I_7(G) = (X_1^2X_2^2Y_1^2Y_2, X_1^2X_2^2Y_1Y_2^2, X_1^2X_2Y_1^2Y_2^2, X_1X_2^2Y_1^2Y_2^2)$. Set $f_1 = X_1^2X_2^2Y_1^2Y_2$, $f_2 = X_1^2X_2^2Y_1Y_2^2$, $f_3 = X_1^2X_2Y_1^2Y_2^2$, $f_4 = X_1X_2^2Y_1^2Y_2^2$. The linear quotients are: $\mathcal{I}_2 = (f_1) : (f_2) = (Y_1)$, $\mathcal{I}_3 = (f_1, f_2) : (f_3) = (X_2)$, $\mathcal{I}_4 = (f_1, f_2, f_3) : (f_4) = (X_1)$. Then $q(I_7(G)) = \max_{2\le i\le 4}\{q_i(I_7(G))\} = 1$. The minimal cardinality of a vertex cover of $I_7(G)$ is $h(I_7(G)) = 1$ and $W = \{X_1\}$ is a such vertex cover. Then

1) $\dim_R(R/I_7(G)) = 3$;
2) $\mathrm{pd}_R(R/I_7(G)) = 2$;

3) $\operatorname{depth}_R(R/I_7(G)) = 2$;
4) $\operatorname{reg}_R(R/I_7(G)) = 6$.

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