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On a class of N(k)-mixed generalized quasi-Einstein manifolds

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Abstract. The objective of the present paper is to study $N(k)$-mixed generalized quasi-Einstein manifolds. We prove the existence of these manifolds. Later we establish some curvature properties of $N(k)$-mixed generalized quasi-Einstein manifolds under certain conditions. In the last section, we give two examples of $N(k)$-mixed generalized quasi-Einstein manifolds.

1. Introduction

A Riemannian manifold $(M^n,\ g)$ with $n \geq 2$ is said to be an Einstein manifold if the Ricci tensor S satisfies, on M, the condition

$$S(X,\ Y) = \frac{r}{n} g(X, Y),$$

where r denotes the scalar curvature of (M^n, g). According to [1], the above equation is called the Einstein metric condition.

Chaki and Maity [3] introduced the concept of a quasi-Einstein manifold. A non-flat Riemannian manifold (M^n, g), $n \geq 2$, is said to be a quasi-Einstein manifold if the equality

$$S(X,Y) = \alpha g(X,Y) + \beta \rho(X)\rho(Y)$$

is fulfilled on M, where α and $\beta \neq 0$ are scalars, ρ is a non-zero 1-form such that $g(X,\ \xi) = \rho(X)$ for all vector fields X, and ξ is a unit vector field.

The notion of a mixed generalized quasi-Einstein manifold was introduced by Bhattacharyya and De in [2]. A non-flat Riemannian manifold is called

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a mixed generalized quasi-Einstein manifold if its non-zero Ricci tensor S of type (0,2) satisfies the condition

$$\begin{aligned} S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) + \gamma B(X)B(Y) \\ + \delta\left[A(X)B(Y) + B(X)A(Y)\right], \end{aligned} \tag{1}$$

where $\alpha, \beta, \gamma, \delta$ are non-zero scalars, $g(X,U) = A(X)$, $g(X,V) = B(X)$ and $g(U,V) = 0$, A, B being two non-zero 1-forms, and U, V are unit vector fields corresponding to the 1-forms A and B, respectively. This manifold is denoted by $MG(QE)_n$.

Let R denote the Riemannian curvature tensor of a Riemannian manifold M. The k-nullity distribution $N(k)$ of the manifold M is defined by (see [11])

$$\begin{aligned} N(k) : p \to N_p(k) = \{Z \in T_pM : R(X,Y)Z \\ = k\left[g(Y,Z)X - g(X,Z))Y\right]\}, \end{aligned} \tag{2}$$

where $X, Y \in T_pM$ and k is a smooth function. If the generators U, V of a manifold $MG(QE)_n$ belong to $N(k)$, then we say that (M^n, g) is a $N(k)$-mixed generalized quasi-Einstein manifold, and we denote it by $N(k)$-$MG(QE)_n$.

In 2007, Tripathi and Kim [12] studied $N(k)$-quasi-Einstein manifolds. They proved that an n-dimensional conformally flat quasi-Einstein manifold is an $N\left(\frac{a+b}{n-1}\right)$-quasi-Einstein manifold. Later many authors (see, for example, [10], [7], [13], [8]) have studied different types of $N(k)$-quasi-Einstein manifolds.

In this paper, we study the existence of $N(k)$-mixed generalized quasi-Einstein manifolds. Ricci-semi-symmetry, and the conharmonic and pseudo-projective curvature tensors of $N(k)$-$MG(QE)_n$ are characterized. We obtain Ricci recurrent, generalized Ricci recurrent and Ricci symmetric manifolds $N(k)$-$MG(QE)_n$. In the last section, we give two examples of $N(k)$-mixed generalized quasi-Einstein manifolds.

2. Existence of $N(k)$-mixed generalized quasi-Einstein manifolds

Theorem 2.1. *Let μ, λ be nonzero scalars, let U, W be vector fields on M, and let $Q : T_pM \to T_pM$ be a symmetric endomorphism such that $S(X,Y) = g(QX,Y)$. If in a conformally flat Riemannian manifold (M^n, g), the Ricci tensor S satisfies the relation*

$$\begin{aligned} &\mu S(Y,W)S(X,Z) + \lambda g(Y,W)g(X,Z) \\ &\quad = [S(Y,Z)g(X,W) + g(Y,Z)S(X,W)] \\ &\qquad - [S(Y,W)g(X,Z) + S(X,Z)g(Y,W)], \end{aligned} \tag{3}$$

and the condition

$$\lambda g(X,U)Y + \mu g(QX,U)QY = 0 \tag{4}$$

holds, then (M^n, g) *is a* $N(k)$*-mixed generalized quasi-Einstein manifold.*

Proof. Let U be the vector field defined by $g(X,U) = P(X)$, $X \in \chi(M)$. Taking $X = W = U$ in (3), we get

$$S(X,Y) = \alpha g(X,Y) + \beta T(X)T(Y) + \gamma P(X)P(Y) + \delta[T(X)P(Y) + P(X)T(Y)],$$

where $\alpha = -a/u,\ a = S(U,U),\ u = g(U,U),\ \beta = \mu/u,\ \gamma = \lambda/u,\ \delta = 1/u$, and $S(U,Z) = S(Z,U) = g(QZ,U) = P(QZ) = T(Z)$. Therefore, (M^n, g) is a mixed generalized quasi-Einstein manifold.

If (M^n, g) is conformally flat, then we have

$$\begin{aligned} R(X,Y)Z = \frac{1}{n-2}&\{g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X \\ &-S(X,Z)Y\} - \frac{r}{(n-1)(n-2)}\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{5}$$

Taking $Z = U$ in (5), for any W we obtain

$$\begin{aligned} R(X,Y)U = \frac{1}{n-2}&\{P(Y)QX - P(X)QY + S(Y,U)X \\ &-S(X,U)Y\} - \frac{r}{(n-1)(n-2)}\{P(Y)X - P(X)Y\}. \end{aligned} \tag{6}$$

Taking $Z = U$ in (3), we obtain

$$[S(Y,U)g(X,W) + g(Y,U)S(X,W)] - [S(X,U)g(Y,W) + S(Y,W)g(X,U)]$$

$$= \mu S(Y,W)S(X,U) + \lambda g(Y,W)g(X,U),$$

and thus

$$g(S(Y,U)X + P(Y)QX - \mu T(X)QY - \lambda P(X)Y - S(X,U)Y - P(X)QY, W) = 0.$$

Therefore from (4) we have

$$S(Y,U)X - S(X,U)Y = P(X)QY - P(Y)QX.$$

Substituting this in (6), we get

$$R(X,Y)U = k(P(Y)X - P(X)Y),$$

where $k = -\frac{r}{(n-1)(n-2)}$. Thus $U \in N_p(k)$.

Suppose V is a vector field orthogonal to U. Then we have $V \in N_p(k)$. Hence (M^n, g) is a $N(k)$-mixed generalized quasi-Einstein manifold. □

3. Ricci curvature, eigenvectors and associated scalars of manifolds $N(k)$-$MG(QE)_n$

From (1), we deduce that

$$S(U,U) = \alpha + \beta, \quad S(V,V) = \alpha + \gamma, \quad S(U,V) = \delta = S(V,U)$$

since $g(U,V) = 0$.

It is well known that $S(X,X)$ is the Ricci curvature in the direction of a unit vector field X. Now if X is a unit vector field in the section spanned by U and V, then we have

$$1 = g(X,X) = g(aU + bV, aU + bV) = a^2 + b^2$$

since $g(U,V) = 0$ and $g(U,U) = 1$, $g(V,V) = 1$. Now

$$S(X,X) = S(aU+bV, aU+bV) = \alpha + \beta A(X)A(Y) + \gamma B(X)(Y) + 2\delta A(X)B(X).$$

Thus we can formulate the following result.

Theorem 3.1. *In $N(k)$-$MG(QE)_n$, the Ricci curvature in the direction of U is $\alpha + \beta$, and in the direction of V is $\alpha + \gamma$. The Ricci curvature in all other directions of the section of U and V is*

$$\alpha + \beta A(X)A(Y) + \gamma B(X)(Y) + 2\delta A(X)B(X).$$

Let (M^n, g) be a $N(k)$-mixed generalized quasi-Einstein manifold. Since $U, V \in N_p(k)$, we have

$$g(R(X,Y)U, W) = k\{A(Y)g(X,W) - A(X)g(Y,W)\}.$$

Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of the tangent space T_pM at any point $p \in M$. Putting $X = W = e_i$ and summing over i, $1 \leq i \leq n$, we obtain

$$S(Y,U) = k(n-1)A(X). \tag{7}$$

Similarly,

$$S(Y,V) = k(n-1)B(X). \tag{8}$$

From (1), we get

$$S(X,U) = (\alpha + \beta)A(X) + \delta B(X), \tag{9}$$

$$S(X,V) = (\alpha + \gamma)B(X) + \delta A(X). \tag{10}$$

Substracting (8) from (7) and (10) from (9), we see that

$$k(n-1) = \alpha + \beta - \delta, \tag{11}$$

$$k(n-1) = \alpha + \gamma - \delta. \tag{12}$$

Hence, adding (11) and (12), we obtain

$$k = \frac{2\alpha + \beta + \gamma - 2\delta}{2(n-1)}.$$

Therefore,

$$S(X,U)=\frac{2\alpha+\beta+\gamma-2\delta}{2}g(X,U)$$

and

$$S(X,V)=\frac{2\alpha+\beta+\gamma-2\delta}{2}g(X,V).$$

Consequently, U and V are eigenvectors corresponding to the eigenvalue $(2\alpha+\beta+\gamma-2\delta)/2$.

4. Curvature tensors of manifolds $N(k)$-$MG(QE)_n$

Let (M,g) be a Riemannian manifold of dimension n. The conharmonic curvature tensor is defined by

$$\begin{aligned}\bar{C}(X,Y)Z=R(X,Y)Z-\frac{1}{n-2}\{&S(Y,Z)X-S(X,Z)Y\\&+g(Y,Z)QX-g(X,Z)QY\},\end{aligned}\tag{13}$$

where $X,Y,Z\in\chi(M)$ and Q is the Ricci operator.

The pseudo-projective curvature tensor is defined by (see [9])

$$\begin{aligned}\bar{P}(X,Y)Z=&\,aR(X,Y)Z+b\{S(Y,Z)X-S(X,Z)Y\}\\&-\frac{r}{n}\left[\frac{a}{n-1}+b\right]\{g(Y,Z)X-g(X,Z)Y\},\end{aligned}\tag{14}$$

where $X,Y,Z\in\chi(M)$, $a,b\neq 0$ are constants, Q is the Ricci operator, and r is the scalar curvature.

Now we establish the following theorems.

Theorem 4.1. *An n-dimensional $N(k)$-mixed generalized quasi-Einstein manifold M satisfies the condition $\bar{C}(U,Y)\cdot S=0$ if and only if*

$$\begin{aligned}k(n-2)\left[n(\alpha+\beta)-r\right]-\big[&n(\alpha+\beta)^2+(n-1)\delta^2\\&-\alpha(\gamma+r)-\gamma(\gamma+\delta)-\beta(\beta+\alpha)\big]=0,\end{aligned}$$

and the condition $\bar{C}(V,Y)\cdot S=0$ if and only if

$$\begin{aligned}k(n-2)\left[n(\alpha+\gamma)-r\right]-\big[&n(\alpha+\gamma)^2+(n-1)\delta^2\\&-\alpha(\beta+r)-\beta(\beta+\delta)-\gamma(\gamma+\alpha)\big]=0,\end{aligned}$$

where r is the scalar curvature.

Proof. Since $\bar{C}(U,Y)\cdot S=0$, we have

$$S(\bar{C}(U,Y)Z,W)+S(Z,\bar{C}(U,Y)W)=0.\tag{15}$$

Then, by (2) and (13), we have that

$$k[g(Y,Z)S(U,W)-g(U,Z)S(Y,W)+g(Y,W)S(U,Z)-g(U,W)S(Y,Z)]$$

$$- \frac{1}{n-2}[g(Y,Z)S(QU,W) - g(U,Z)S(QY,W) + g(Y,W)S(QU,Z) - g(U,W)S(QY,Z)] = 0.$$

Putting $W = U$, we get

$$\begin{aligned} k[g(Y,Z)S(U,U) - g(U,Z)S(Y,U) + g(Y,U)S(U,Z) \\ - g(U,U)S(Y,Z)] - \frac{1}{n-2}[g(Y,Z)S(QU,U) - g(U,Z)S(QY,U) \\ +g(Y,U)S(QU,Z) - g(U,U)S(QY,Z)] = 0. \end{aligned} \tag{16}$$

From (1), we have

$$QX = \alpha X + \beta A(X)U + \gamma B(X)V + \delta[A(X)V + B(X)U]. \tag{17}$$

From (16), we get

$$\begin{aligned} k[g(Y,Z)(\alpha+\beta) - g(U,Z)(\alpha g(Y,U) + \beta A(Y)\delta B(Y)) \\ + g(Y,U)(\alpha g(Z,U) + \beta A(Z) + \delta B(Z)) - S(Y,Z)] \\ - \frac{1}{n-2}[g(Y,Z)S(\alpha U + \beta U + \delta V, U) - g(U,Z)S(\alpha Y \\ + \beta A(Y)U + \gamma B(Y)V + \delta[A(Y)V + B(Y)U], U) \\ + g(Y,U)S(\alpha U + \beta U + \delta V, Z) - S(\alpha Y + \beta A(Y)U + \gamma B(Y)V \\ + \delta[A(Y)V + B(Y)U], Z)] = 0. \end{aligned} \tag{18}$$

Let $\{e_1, e_2, ..., e_n\}$ be an orthonormal basis of the tangent space T_pM at any point $p \in M$. Putting $Y = Z = e_i$ and summing over $i, 1 \leq i \leq n$, we obtain

$$\begin{aligned} k(n-2)[n(\alpha+\beta) - r] - \big[n(\alpha+\beta)^2 + (n-1)\delta^2 \\ -\alpha(\gamma + r) - \gamma(\gamma+\delta) - \beta(\beta+\alpha)] = 0. \end{aligned}$$

Similarly, we get that $\bar{C}(V,Y) \cdot S = 0$ if and only if

$$\begin{aligned} k(n-2)\,[n(\alpha+\gamma) - r] - \big[n(\alpha+\gamma)^2 + (n-1)\delta^2 \\ -\alpha(\beta + r) - \beta(\beta+\delta) - \gamma(\gamma+\alpha)] = 0, \end{aligned}$$

The theorem has been proved. □

Theorem 4.2. *A n-dimensional $N(k)$-mixed generalized quasi-Einstein manifold M satisfies the condition $\bar{P}(U,Y) \cdot S = 0$ if and only if either $ak - \frac{r}{n}\left(\frac{a}{n-1} + b\right) = 0$ or $n(\alpha+\beta) = r$, and the condition $\bar{P}(V,Y) \cdot S = 0$ if and only if either $ak - \frac{r}{n}\left(\frac{a}{n-1} + b\right) = 0$ or $n(\alpha+\gamma) = r$.*

Proof. Since $\bar{P}(U,Y) \cdot S = 0$, we have

$$S(\bar{P}(U,Y)Z, W) + S(Z, \bar{P}(U,Y)W) = 0. \tag{19}$$

By (2) and (14), we get

$$\left[ak - \frac{r}{n}(\frac{a}{n-1} + b)\right] [g(Y,Z)S(U,W) - g(U,Z)S(Y,W) + g(Y,W)S(U,Z) - g(U,W)S(Y,Z)] = 0.$$

Putting $W = U$, we obtain

$$\left[ak - \frac{r}{n}(\frac{a}{n-1} + b)\right] [g(Y,Z)(\alpha+\beta) - g(U,Z)[\alpha g(Y,U) + \beta A(Y) + \delta B(Y)] + g(Y,U)[\alpha g(Z,U) + \beta A(Z) + \delta B(Z)] - S(Y,Z)] = 0. \tag{20}$$

Let $\{e_1, e_2, ..., e_n\}$ be an orthonormal basis of the tangent space T_pM at any point $p \in M$. Putting $Y = Z = e_i$ and summing over i, $1 \leq i \leq n$, we obtain $ak - \frac{r}{n}\left(\frac{a}{n-1} + b\right) = 0$ or $n(\alpha+\beta) = r$.

Similarly, we get that $\bar{P}(V,Y)\cdot S = 0$ if and only if either $ak - \frac{r}{n}\left(\frac{a}{n-1} + b\right) = 0$ or $n(\alpha+\gamma) = r$.

This completes the proof. □

5. Ricci-recurrent manifolds $N(k)$-$MG(QE)_n$

A manifold $N(k)$-$MG(QE)_n$ is said to be Ricci-recurrent if its Ricci tensor S of type $(0,2)$ satisfies the condition

$$(\nabla_X S)(Y,Z) = L(X)S(Y,Z), \tag{21}$$

where L is the nonzero 1-form such that $L(X) = g(X,\xi)$ holds, ξ being the associated vector field of the 1-form L.

A manifold $N(k)$-$MG(QE)_n$ is said to be generalized Ricci-recurrent if its Ricci tensor S of type $(0,2)$ satisfies the condition

$$(\nabla_X S)(Y,Z) = F(X)S(Y,Z) + G(X)g(Y,Z), \tag{22}$$

where F, G are the nonzero 1-forms such that $F(X) = g(X,\xi_1)$, $G(X) = g(X,\xi_2)$, and ξ_1, ξ_2 are associated vector fields of the 1-forms F, G, respectively.

We prove the following proposition.

Proposition 5.1. *Let F, G be nonzero 1-forms. In a generalized Ricci-recurrent manifold $N(k)$-$MG(QE)_n$, the following statements are true.*

(i) *If U is a parallel vector field, then $X(\alpha+\beta) = (\alpha+\beta)F(X) + G(X)$.*
(ii) *If V is a parallel vector field, then $X(\alpha+\gamma) = (\alpha+\gamma)F(X) + G(X)$.*

Proof. Putting $Y = Z = U$ in (22), we get

$$(\nabla_X S)(U,U) = (\alpha+\beta)F(X) + G(X).$$

On the other hand, we have

$$(\nabla_X S)(U,U) = X(\alpha+\beta) - 2\delta S(\nabla_X U, U),$$

i.e.,

$$2[(\alpha+\beta)A(\nabla_X U) + \delta B(\nabla_X U)] = X(\alpha+\beta) - (\alpha+\beta)F(X) - G(X).$$

Since U is parallel vector field, $\nabla_X U = 0$. Then from the above we get

$$X(\alpha+\beta) = (\alpha+\beta)F(X) + G(X).$$

Similarly we can show that if V is a parallel vector field, then

$$X(\alpha+\gamma) = (\alpha+\gamma)F(X) + G(X).$$

The proof is complete. □

From the previous proposition we have the following corollary.

Corollary 5.1. *Let L be a nonzero 1-form. In a Ricci-recurrent manifold $N(k)$-$MG(QE)_n$, the following statements hold.*

(i) *If U is parallel vector field, then $d(\alpha+\beta)(X) = (\alpha+\beta)L(X)$.*
(ii) *If V is parallel vector field, then $d(\alpha+\gamma)(X) = (\alpha+\gamma)L(X)$.*

6. Ricci-symmetric manifolds $N(k)$-$MG(QE)_n$

A Riemannian manifold (M^n, g) is said to be Ricci-semi-symmetric if the relation $R(X,Y)\cdot S = 0$ holds, where $R(X,Y)$ is the curvature operator and S is the Ricci tensor of type $(0,2)$.

Theorem 6.1. *An $N(k)$-mixed generalized quasi-Einstein manifold satisfies the relations $R(U,Y)\cdot S = 0$ and $R(V,Y)\cdot S = 0$ if and only if $k = 0$.*

Proof. Let (M^n, g) be a Ricci-semi-symmetric manifold $N(k)$-$MG(QE)_n$. Then we have

$$S(R(X,Y)Z,W) + S(Z,R(X,Y)W) = 0. \tag{23}$$

Putting $X = V$ in (23), we obtain

$$k\{g(Y,Z)S(V,W) - B(Z)S(Y,W) + g(Y,W)S(Z,V) - B(W)S(Z,Y)\} = 0. \tag{24}$$

Putting $W = V$, we get

$$k\{(\alpha+\gamma)g(Y,Z) - \delta A(Y)B(Z) + \delta A(Y)B(Z) - S(Y,Z)\} = 0.$$

Hence either $k = 0$ or

$$(\alpha+\gamma)g(Y,Z) - \delta A(Y)B(Z) + dA(Y)B(Z) - S(Y,Z) = 0.$$

If $k \neq 0$, then in the second case the manifold becomes an $N(k)$-mixed quasi-Einstein manifold (see [6]) which is not possible. Therefore we must have $k = 0$.

Conversely, suppose $k = 0$. Then we obtain that $R(V,Y)\cdot S = 0$.

Similarly, we get that $R(U,Y)\cdot S=0$ if and only if $k=0$, and the proof is complete. □

A manifold $N(k)$-$M(GQ)_n$ is said to be Ricci-symmetric if its Ricci tensor S of type $(0,2)$ satisfies the condition

$$(\nabla_X S)(Y,Z)=0 \tag{25}$$

for all $X,Y,Z\in\chi(M)$.

Proposition 6.1. *If a manifold $N(k)$-$MG(QE)_n$ with constant associated scalar is Ricci-symmetric with Levi-Civita connection ∇, and U is a parallel vector field, then $b(\nabla_X A)(Y)+d(\nabla_X B)(Y)=0$.*

Proof. First, putting $Z=U$ in (25), where U is a parallel vector field, we have

$$\beta(\nabla_X A)(Y)+\delta(\nabla_X B)(Y)=0.$$

Similarly, if V is a parallel vector field and M is Ricci-symmetric manifold $N(k)$-$MG(QE)_n$, then we can show that

$$\gamma(\nabla_X B)(Y)+\delta(\nabla_X A)(Y)=0,$$

which completes the proof. □

Corollary 6.1. *If a manifold $N(k)$-$MG(QE)_n$ with constant associated scalar is Ricci-symmetric with Levi-Civita connection ∇, and V is a parallel vector field, then*

$$\gamma(\nabla_X B)(Y)+\delta(\nabla_X A)(Y)=0.$$

7. Examples of manifolds $N(k)$-$MG(QE)_n$

Example 7.1. Let us consider a Riemannian metric g on R^4 determined by

$$ds^2=g_{ij}dx^i dx^j=(1+2p)[(dx^1)^2+(dx^2)^2+(dx^3)^2+(dx^4)^2],$$

where $i,j=1,2,3,4$ and $p=k^{-2}e^{x^1}$, k is constant. Then the only non-vanishing components of Christofell symbols, the curvature tensors, and the Ricci tensors are

$$\Gamma^1_{22}=\Gamma^1_{33}=\Gamma^1_{44}=-\frac{p}{1+2p},$$

$$\Gamma^1_{11}=\Gamma^2_{12}=\Gamma^3_{13}=\Gamma^4_{14}=\frac{p}{1+2p},$$

$$R_{1221}=R_{1331}=R_{1441}=\frac{p}{1+2p},$$

$$R_{2332}=R_{2442}=R_{3443}=\frac{p^2}{1+2p},$$

$$R_{11}=\frac{3p}{(1+2p)^2},\ R_{22}=R_{33}=R_{44}=\frac{p}{(1+2p)}.$$

Let us consider the associated scalars $\alpha, \beta, \gamma, \delta$ defined by

$$\alpha = \frac{p}{(1+2p)^2},\ \beta = \frac{2p}{(1+2p)^3},\ \gamma = \frac{p}{(1+2p)^3},\ \delta = -\frac{p}{2(1+2p)^2},$$

and the 1-forms

$$A_i(x) = B_i(x) = \begin{cases} \sqrt{1+2p} & \text{if } i=1, \\ 0 & \text{otherwise,} \end{cases}$$

where generators are unit vector fields. Then we have

$$\text{(i) } R_{11} = \alpha g_{11} + \beta A_1 A_1 + \gamma B_1 B_1 + \delta[A_1 B_1 + A_1 B_1],$$

$$\text{(ii) } R_{22} = \alpha g_{22} + \beta A_2 A_2 + \gamma B_2 B_2 + \delta[A_2 B_2 + A_2 B_2],$$

$$\text{(iii) } R_{33} = \alpha g_{33} + \beta A_3 A_3 + \gamma B_3 B_3 + \delta[A_3 B_3 + A_3 B_3],$$

$$\text{(iv) } R_{44} = \alpha g_{44} + \beta A_4 A_4 + \gamma B_4 B_4 + \delta[A_4 B_4 + A_4 B_4].$$

Since all the cases (i)–(iv) are trivial, we can say that

$$R_{ij} = \alpha g_{ij} + \beta A_i A_j + \gamma B_i B_j + \delta[A_i B_j + A_j B_i], \quad i,j = 1,2,3,4.$$

So, (R^4, g) is a mixed generalized quasi-Einstein manifold with non-zero and non-constant scalar curvature. We can say that (M^4, g) is an $N(\frac{p(2+p)}{3(1+2p)^3})$-mixed generalized quasi-Einstein manifold.

Example 7.2. Let us consider a Riemannian metric g on R^4 by

$$ds^2 = g_{ij}dx^i dx^j = (dx^1)^2 + e^{x^1+x^2}(dx^2)^2 + e^{x^1+x^3}(dx^3)^2 + e^{x^1+x^4}(dx^4)^2, \quad i,j = 1,2,3,4.$$

Then the only non-vanishing components of Christofell symbols, the curvature tensors, and the Ricci tensors are

$$\Gamma^1_{22} = -\frac{1}{2}e^{x^1+x^2},\ \Gamma^1_{33} = -\frac{1}{2}e^{x^1+x^3},\ \Gamma^1_{44} = -\frac{1}{2}e^{x^1+x^4},$$

$$\Gamma^2_{22} = \Gamma^3_{33} = \Gamma^4_{44} = \frac{1}{2} = \Gamma^2_{12} = \Gamma^3_{13} = \Gamma^4_{14},$$

$$R_{1221} = \frac{1}{4}e^{x^1+x^2},\ R_{1331} = \frac{1}{4}e^{x^1+x^3},\ R_{1441} = \frac{1}{4}e^{x^1+x^4},$$

$$R_{2332} = \frac{1}{4}e^{2x^1+x^2+x^3},\ R_{2442} = \frac{1}{4}e^{2x^1+x^2+x^4},\ R_{3443} = \frac{1}{4}e^{2x^1+x^3+x^4},$$

$$R_{11} = \frac{3}{4},\ R_{22} = \frac{3}{4}e^{x^1+x^2},\ R_{33} = \frac{3}{4}e^{x^1+x^3},\ R_{44} = \frac{3}{4}e^{x^1+x^4}.$$

Let us consider the associated scalars $\alpha, \beta, \gamma, \delta$ defined by

$$\alpha = \frac{3}{4},\ \beta = e^{x^1},\ \gamma = \frac{2}{e^{x^1}},\ \delta = -\frac{2}{\sqrt{2}},$$

and the 1-forms

$$A_i(x) = \begin{cases} \frac{\sqrt{2}}{\sqrt{e^{x^1}}} & \text{if } i = 1, \\ 0 & \text{otherwise,} \end{cases} \quad B_i(x) = \begin{cases} \sqrt{e^{x^1}} & \text{if } i = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where generators are unit vector fields. Then we have

$$\text{(i) } R_{11} = \alpha g_{11} + \beta A_1 A_1 + \gamma B_1 B_1 + \delta[A_1 B_1 + A_1 B_1],$$
$$\text{(ii) } R_{22} = \alpha g_{22} + \beta A_2 A_2 + \gamma B_2 B_2 + \delta[A_2 B_2 + A_2 B_2],$$
$$\text{(iii) } R_{33} = \alpha g_{33} + \beta A_3 A_3 + \gamma B_3 B_3 + \delta[A_3 B_3 + A_3 B_3],$$
$$\text{(iv) } R_{44} = \alpha g_{44} + \beta A_4 A_4 + \gamma B_4 B_4 + \delta[A_4 B_4 + A_4 B_4].$$

Since all the cases (i)–(iv) are trivial, we can say that

$$R_{ij} = \alpha g_{ij} + \beta A_i A_j + \gamma B_i B_j + \delta[A_i B_j + A_j B_i], \quad i, j = 1, 2, 3, 4.$$

So, in this case (R^4, g) is a mixed generalized quasi-Einstein manifold. We can easily see that (M^4, g) is an $N\left(\frac{2\sqrt{2}\left(e^{x^1}\right)^2 + 4\sqrt{2} + 8e^{x^1} + 3\sqrt{2}e^{x^1}}{12\sqrt{2}e^{x^1}}\right)$-mixed generalized quasi-Einstein manifold.

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