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ON A CLASS OF NILPOTENT DISTRIBUTIONS

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Abstract. This paper presents a sufficient condition for two vector fields X and Y to have the squares noncommutative, i.e. $[X^2, Y^2] \neq 0$. We prove that if the vector fields X, Y span a nilpotent distribution with nilpotence class 2, then the squares of the vector fields do not commute.

1. Introduction

An important problem in the theory of sub-elliptic operators is to find heat kernels for subelliptic operators written as a sum of squares $\mathbf{L} = X_1^2 + \cdots + X_r^2$, where $X_1, \ldots, X_r$ are vector fields on $\mathbb{R}^m$, with $m \geq r$. A sufficient well-known condition for the hypoellipticity of the operator $\mathbf{L}$ is the bracket generating condition, see Hörmander [6]. The important notion of step arises here, see also [1], [2], [3]. The step of the operator $\mathbf{L}$ at point p is 1 plus the number of iterated brackets of vector fields $\{X_1, \ldots, X_r\}$ needed to span $\mathbb{R}^m$ at p. The step is also a measure of non-holonomy (i.e. non-integrability) of the distribution $\mathcal{D}$ generated as a subspace of $\mathbb{R}^m$ by the linear combinations of the aforementioned vector fields at each point. If the vector fields are considered linearly independent, then the rank of the distribution $\mathcal{D}$ is r, the dimension of the subspace. We notice that the bracket generating condition is equivalent with the finite step condition.

If the squares of the aforementioned vector fields commute, then the heat kernel of $\mathbf{L}$ is the product of heat kernels

$$e^{tL} = e^{tX_1^2} \cdots e^{tX_r^2}.$$

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If they do not commute the previous formula does not hold any more, and the heat kernel should be found using a different method, such as Trotter's formula, see [8]. Thus the importance of knowing when the squares of two vector fields commute. For geometric methods of computing the heat kernel one may consult [4].

In the present work we shall state a relationship between the nilpotence class of a distribution and the aforementioned non-commutativity of squares of vector fields. More precisely, we shall prove that if the nilpotence class of a distribution is equal to 2, then the squares of the vector fields do not commute. Proving this implication makes sense, since checking the non-commutativity of squares of vectors by a direct computation requires more computation than just checking the nilpotence class of two given vector fields.

2. Preliminary Notions

Let X and Y be two vector fields on $\mathbb{R}^m$. Consider the differential operator of order n obtained by iterating the same vector field k times $X^k = X \dots X$. We start by observing that if the vector fields X and Y commute, then the operators X^k and Y^k also commute. This can be written as the following set inclusion

$$\{p; [X, Y]_p = 0\} \subseteq \{p; [X^k, Y^k]_p = 0\}. \tag{2.1}$$

Definition 2.1. Two vector fields X and Y satisfy the condition $(\mathfrak{N}_k)$ at the point p if $[X^k, Y^k]_p \neq 0$. If the above condition holds at each point, then the distribution $\mathcal{D}$ spanned by the vector fields X and Y is called a $(\mathfrak{N}_k)$-distribution. In this case we say that X and Y are $(\mathfrak{N}_k)$-vector fields.

Using the contrapositive of (2.1)

$$\{p; [X^k, Y^k]_p \neq 0\} \subset \{p; [X, Y]_p \neq 0\} \tag{2.2}$$

implies that if X and Y are $(\mathfrak{N}_k)$-vector fields, then X and Y do not commute.

For instance, the vector fields

$$X = \partial_x, \qquad Y = \partial_y + x\partial_z \tag{2.3}$$

satisfy the condition $(\mathfrak{N}_2)$ everywhere on $\mathbb{R}^3$. This follows from the relations

$$\begin{aligned} [X, Y] &= \partial_z \neq 0 \\ X^2 &= \partial_z^2 \\ Y^2 &= \partial_y^2 + 2x\partial_y\partial_z + x^2\partial_z^2 \\ [X^2, Y^2] &= 4\partial_x\partial_y\partial_z + 2\partial_z^2 + 4x\partial_x\partial_z^2 \neq 0. \end{aligned}$$

Next we recall the notion of nilpotence class of a a distribution.

Definition 2.2. A distribution $\mathcal{D} = span\{X, Y\}$ is called nilpotent if there is an integer $k \geq 1$ such that all the Lie brackets of X and Y iterated k times vanish. The smallest integer k is called the nilpotence class of $\mathcal{D}$.

For instance, the vector fields (2.3) span a nilpotent distribution with nilpotence class $k = 2$

$$[X, [X, Y]] = 0, \qquad [Y, [X, Y]] = 0, \qquad \text{but} \quad [X, Y] \neq 0.$$

The *nilpotence class* of a distribution is in general a distinct notion from the *step* of a distribution. The nilpotence class describes the functional nature of the distribution (i.e. the polynomial or exponential type of the distribution), while the step describes the non-holonomy of the distribution (i.e. the degree of non-integrability). The nilpotence class and the step are equal in the case of a distribution generated by left invariant vector fields on nilpotent Lie groups. For instance this is the case of the Heisenberg distribution.

We shall consider next a of couple examples which show that the nilpotence class and the step of a distribution are distinct features. The next two examples are taken from Calin and Chang [3], p. 48.

Consider the distribution spanned by the vector fields $X = \partial_x + e^y \partial_z$ and $Y = \partial_y$ on $\mathbb{R}^3$. This is bracket generating with step 2 everywhere, but it is not nilpotent.

On the other side, the distribution spanned by the vector fields $X = \partial_x$ and $Y = \partial_y + zx\partial_z$ is nilpotent with the nilpotence class 2. However, this distribution is not bracket generating along the plane $\{z = 0\}$, i.e., it does not have a finite step there.

In the next section we shall provide a sufficient condition for a distribution to be of class $(\mathfrak{N}_2)$.

3. Main Theorem

Theorem 3.1. *Any distribution $\mathcal{D} = span\{X, Y\}$ with nilpotence class 2 is a $(\mathfrak{N}_2)$-distribution.*

Proof. Since the nilpotence class of the distribution is 2, we have

$$[X, [X, Y]] = 0, \qquad [Y, [Y, X]] = 0, \qquad [X, Y] \neq 0. \tag{3.1}$$

The first two relations of (3.1) become

$$X^2Y + YX^2 = 2XYX \tag{3.2}$$

$$Y^2X + XY^2 = 2YXY. \tag{3.3}$$

Multiplying on the right side the relation (3.2) by Y and the relation (3.3) by X yields

$$\begin{aligned} X^2Y^2 + YX^2Y &= 2(XY)^2 \\ Y^2X^2 + XY^2X &= 2(YX)^2, \end{aligned}$$

and subtracting we have

$$\begin{aligned} [X^2,Y^2] = X^2Y^2 - Y^2X^2 &= \Big(2(XY)^2 - YX^2Y\Big) - \Big(2(YX)^2 - XY^2X\Big) \\ &= 2\{(XY)^2 - (YX)^2\} + (XY^2X - YX^2Y). \end{aligned} \tag{3.4}$$

Multiplying on the left side the relation (3.2) by Y and the relation (3.3) by X yields

$$\begin{aligned} YX^2Y + Y^2X^2 &= 2(YX)^2 \\ XY^2X + X^2Y^2 &= 2(XY)^2, \end{aligned}$$

and subtracting we get

$$\begin{aligned} [X^2,Y^2] = X^2Y^2 - Y^2X^2 &= \Big(2(XY)^2 - XY^2X\Big) - \Big(2(YX)^2 - YX^2Y\Big) \\ &= 2\{(XY)^2 - (YX)^2\} + (YX^2Y - XY^2X). \end{aligned} \tag{3.5}$$

Comparing (3.4) and (3.5) yields

$$XY^2X - YX^2Y = YX^2Y - XY^2X \Longleftrightarrow XY^2X = YX^2Y, \tag{3.6}$$

and hence relations (3.4) and (3.5) become

$$[X^2,Y^2] = 2\{(XY)^2 - (YX)^2\}. \tag{3.7}$$

If let $A = XY$ and $B = YX$, using (3.6) the operators A and B commute

$$AB = (XY)(YX) = XY^2X = YX^2Y = (YX)(XY) = BA,$$

and then we can factorize the difference of squares $A^2 - B^2 = (A - B)(A + B)$. Then relation (3.7) becomes by factorization

$$[X^2,Y^2] = 2(XY - YX)(XY + YX). \tag{3.8}$$

We shall show that X and Y are $(\mathfrak{N}_2)$-fields, i.e. $[X^2,Y^2] \neq 0$. We shall pursue a proof by contradiction by assuming $[X^2,Y^2] = 0$. Then the relation (3.8) provides *either:* $XY - YX = 0$, i.e., $[X, Y] = 0$, which is in contradiction with the last of the relations (3.1).

or: $XY + YX = 0$, i.e.,

$$XY = -YX. \tag{3.9}$$

The remaining of the proof deals with showing that (3.9) cannot hold. By contradiction, we assume that (3.9) holds. Then

$$[X, Y] = XY - YX = 2XY = -2YX.$$

Therefore we have

$$[X, [X, Y]] = 0 \Longrightarrow [X, XY] = 0 \Longrightarrow X^2Y = XYX \tag{3.10}$$

$$[Y, [Y, X]] = 0 \Longrightarrow [Y, YX] = 0 \Longrightarrow Y^2X = YXY. \tag{3.11}$$

Using (3.10) and (3.11) we have

$$X[X, Y] = X(XY - YX) = X^2Y - XYX = 0,$$
$$Y[X, Y] = Y(XY - YX) = YXY - Y^2X = 0.$$

Combining the last two relations we obtain

$$\begin{aligned} [X, Y]^2 &= [X, Y][X, Y] = (XY - YX)[X, Y] \\ &= X(Y[X, Y]) - Y(X, [X, Y]) \\ &= 0. \end{aligned}$$

Hence $[X, Y] = 0$, which is a contradiction. It turns out that (3.9) cannot hold.

It follows that the vector fields X and Y span a $(\mathfrak{N}_2)$-distribution. ■

Counterexample. We notice that the converse of the previous theorem does not hold, as the next counterexample shows. Let $X = \partial_x$ and $Y = e^x\partial_y$. Since

$$[X, Y] = e^x\partial_y \neq 0, \qquad [X^2, Y^2] = 4e^{2x}(\partial_y^2 + \partial_x\partial_y^2) \neq 0,$$

the distribution span$\{X, Y\}$ is a $(\mathfrak{N}_2)$-distribution. However, this distribution is not nilpotent since

$$[X, \dots [X, [X, Y]]] = e^x\partial_y.$$

Example. An important class of distributions with the nilpotence class 2 are the *Heisenberg-type distributions*. Consider the vector fields X, Y, T on $\mathbb{R}^3$. If

$$[X, Y] = T, \quad [X, T] = 0, \quad [Y, T] = 0$$

we say that $\mathcal{D} = span\{X, Y\}$ is a 3-dimensional Heisenberg-type distribution. Since

$$[[X, Y], X] = [T, X] = 0, \quad [[X, Y], Y] = [T, Y] = 0,$$

it follows that $\mathcal{D}$ is a nilpotent distribution with the nilpotence class 2. According with the previous theorem, the aforementioned distribution is a $(\mathfrak{N}_2)$-distribution, i.e. $[X^2, Y^2] \neq 0$.

One of the classical examples of vector fields with the these properties are

$$X = \partial_x + 2y\partial_z, \quad Y = \partial_y - 2x\partial_z, \quad T = -4\partial_z.$$

In this case we also have that X^2 and Y^2 do not commute. The heat kernel of the Heisenberg Laplacian $\frac{1}{2}(X^2 + Y^2)$ was computed in [5] and [7].

Further developments. One natural question is if we can generalize the theorem to the case of more than 2 vector fields. Another one is to investigate the case of $(\mathfrak{N}_k)$-distributions.

References

1. R. Beals, B. Gaveau and P. C. Greiner, On a geometric formula for the fundamental solution of subelliptic Laplacians, *Math. Nachr.*, **181** (1996), 81-163.
2. R. Beals, B. Gaveau and P. C. Greiner, Hamilton-Jacobi theory and the heat kernel on Heisenberg groups, *J. Math. Pures Appl.*, **79(7)** (2000), 633-689.
3. O. Calin and D. C. Chang, *Sub-Riemannian Geometry, General Theory and Examples*, Encyclopedia of Mathematics and Its Applications, Cambridge University Press, Vol. 126, 2009.
4. O. Calin and D. C. Chang, *Geometric Mechanics on Riemannian Manifolds: Applications to Partial Differential Equations*, Applied and Numerical Analysis. Birhäuser, Boston, 2004.
5. B. Gaveau, Systémes Dynamiques Associés a Certains Opérateurs Hypoelliptiques, *Bull. Sc. Math.*, **102** (1978), 203-229.
6. L. Hörmander, Hypo-elliptic second order differential equations, *Acta Math.*, **119** (1967), 147-171.
7. A. Hulanicki, The distribution of energy in the Brownian motion in the Gausssian field and analytic hypoellipticity of certain subelliptic operators on the Heisenberg group, *Studia Mathematica*, **56** (1976), 165-173.

8. L. S. Schulman, *Techniques and Applications of Path Integration*, Dover, 1981.

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