

ON A CLASS OF NONCLASSICAL HYPERBOLIC EQUATIONS WITH NONLOCAL CONDITIONS

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(Received January, 1998; Revised September, 2000)

This paper proves the existence, uniqueness and continuous dependence of a solution of a class of nonclassical hyperbolic equations with nonlocal boundary and initial conditions. Results are obtained by using a functional analysis method based on an a priori estimate and on the density of the range of the linear operator corresponding to the abstract formulation of the considered problem.

Key words: Nonclassical Hyperbolic Equations, Nonlocal Boundary Conditions, A Priori Estimate, Strong Solutions.

AMS subject classifications: 35L15, 35L20, 35L99, 35B45, 35D05.

1. Introduction

In this article, we prove the existence, uniqueness and continuous dependence on a solution of a class of nonclassical hyperbolic partial differential equations with nonlocal boundary and initial conditions. Results are obtained by using a functional analysis method based on an a priori estimate and on the density of the range of the operator corresponding to the abstract formulation of the considered problem. The precise statement of the problem is of the form:

Let $b > 0$, $T_p > 0$ ($p = 1, 2$) and $Q = \{(x, t_1, t_2) \in \mathbb{R}^3: 0 < x < b, 0 < t_1 < T_1, 0 < t_2 < T_2\}$. Find a function $(x, t_1, t_2) \rightarrow v(x, t_1, t_2)$, where $(x, t_1, t_2) \in \bar{Q}$, satisfying the equation

$$\mathcal{L}v = \frac{\partial^2 v}{\partial t_1 \partial t_2} - \frac{\partial}{\partial x} (a(x, t_1, t_2) \frac{\partial v}{\partial x}) = f(x, t_1, t_2), \quad (1.1)$$

the initial conditions

$$\begin{aligned} \ell_1 v &= v(x, 0, t_2) = \Phi(x, t_2), & (x, t_2) &\in (0, b) \times (0, T_2), \\ \ell_2 v &= v(x, t_1, 0) = \Psi(x, t_1), & (x, t_1) &\in (0, b) \times (0, T_1), \end{aligned} \quad (1.2)$$

the integral condition

$$\int_0^b v(x, t_1, t_2) dx = E(t_1, t_2) \quad (t_1, t_2) \in (0, T_1) \times (0, T_2), \quad (1.3)$$

and one of the following boundary conditions:

$$\frac{\partial v(0, t_1, t_2)}{\partial x} = \mu(t_1, t_2), \quad (t_2, t_2) \in (0, T_1) \times (0, T_2), \quad (1.4a)$$

$$\int_0^b xv(x, t_1, t_2)dx = \chi(t_1, t_2), \quad (t_1, t_2) \in (0, T_1) \times (0, T_2), \quad (1.4b)$$

where a, f, Φ, Ψ, E, μ and χ are given functions.

Mixed problems with integral conditions for parabolic equations are studied by different method in Batten [1], Cannon [14], Kamynin [22], Ionkin [19], Cannon and van der Hoek [17, 18], Yurchuk [25], Benouar and Yurchuk [2], Cahlon, Kulkarni, and Shi [15], Cannon, Esteva, and van der Hoek [16], Byszewski [11], Lin [23], Shi [24], Bouziani [3-5], Bouziani-Benouar [7], Jones, Jumarhon, McKee, and Scott [19], and Jumarhon and McKee [20]. A mixed problem with integral condition for a second order pluriparabolic equation has been investigated by Bouziani [6]. Nonlocal nonlinear hyperbolic problems were studied by Byszewski [9-11], by Byszewski and Lakshmikantham [12], and by Byszewski and Papageorgiou [13]. A mixed problem with integral conditions for a second order classical hyperbolic equation has been treated under growth conditions in Bouziani and Benouar [8]. The results of the paper are generalizations of those given in [8-12]. These findings are also continuations of those obtained by the author in [8].

The paper is organized as follows. In Section 2, we state three assumptions on the functions involved in problem (1.1)-(1.4) and we reduce the posed problem to one with homogeneous boundary conditions. In addition, we present an abstract formulation of the considered problem and we define the strong solution of the problem. In Section 3, the uniqueness and continuous dependence of the solution are established. Finally, in Section 4 we prove the existence of the strong solution and offer remarks on its generalizations.

2. Preliminaries

First, we begin with the following assumptions on function a :

Assumption A1:

$$c_0 \leq a \leq c_1; \frac{\partial a}{\partial x} \leq c_2; \frac{\partial a}{\partial t_p} \leq c_3 \quad (p = 1, 2) \text{ in } \bar{Q}.$$

Assumption A2:

$$c_4 \leq \frac{\partial a}{\partial t_p}, \quad c_5 \leq \frac{\partial^2 a}{\partial t_p^2} \leq c_6, \quad \frac{\partial^2 a}{\partial t_p \partial x} \leq c_7 \quad (p = 1, 2); \quad \frac{\partial^2 a}{\partial t_2 \partial t_1}, \quad \frac{\partial^2 a}{\partial t_1 \partial t_2} \leq c_8;$$

$$\frac{\partial^3 a}{\partial t_1 \partial t_2^2}, \quad \frac{\partial^3 a}{\partial t_2 \partial t_1^2} \leq c_9 \text{ in } \bar{Q}.$$

In Assumptions A1-A2, we assume that c_i ($i = 0, \dots, 9$) are positive constants. We also, assume that the functions Φ and Ψ satisfy the following:

Assumption A3: Functions Φ and Ψ satisfy the compatibility conditions:

$$\int_0^b \Phi(x, t_2)dx = E(0, t_2), \quad \int_0^b \Psi(x, t_1)dx = E(t_1, 0),$$

$$\frac{\partial \Phi(0, t_2)}{\partial x} = \mu(0, t_2), \quad \frac{\partial \Psi(0, t_1)}{\partial x} = \mu(t_1, 0)$$

$$\left[\int_0^b x \Phi(x, t_2)dx = \chi(0, t_2), \int_0^b x \Psi(x, t_1)dx = \chi(t_1, 0), \text{ respectively} \right],$$

and

$$\Phi(x, 0) = \Psi(x, 0)$$

where $x \in (0, b)$, $t_p \in [0, T_p]$ ($p = 1, 2$).

Let us reduce problem (1.1)-(1.3) and (1.4a) [respectively (1.4b)] with nonhomogeneous boundary conditions (1.3), (1.4a) [respectively (1.4b)] to an equivalent problem with homogeneous conditions by assuming that we are able to find a function $u = u(x, t_1, t_2)$ defined as follows:

$$u(x, t_1, t_2) = v(x, t_1, t_2) - U(x, t_1, t_2),$$

where

$$U(x, t_1, t_2) := \frac{x}{2b}(2b - 3x)\mu(t_1, t_2) + \frac{3x^2}{b^3}E(t_1, t_2)$$

$$[U(x, t_1, t_2) := \frac{1}{b}(-18bx^2 + 12b^2x + 1)E(t_1, t_2) - \frac{12}{b}(3x^2 - 2bx)\chi(t_1, t_2), \text{ respectively}].$$

Consequently, we have to find a function $(x, t_1, t_2) \rightarrow u(x, t_1, t_2)$, which is a solution of the problem

$$\mathcal{L}u = f(x, t_1, t_2) - \mathcal{L}U = f, \quad (2.1)$$

$$\ell_1 u = u(x, 0, t_2) = \Phi(x, t_2) - \ell_1 U = \varphi(x, t_2), \quad (2.2)$$

$$\ell_2 u = u(x, t_1, 0) = \Psi(x, t_1) - \ell_2 U = \psi(x, t_1),$$

$$\int_0^b u(x, t_1, t_2) dx = 0, \quad (2.3)$$

$$\frac{\partial u(0, t_1, t_2)}{\partial x} = 0 \quad (2.4a)$$

$$\left[\int_0^b x u(x, t_1, t_2) dx = 0, \text{ respectively} \right]. \quad (2.4b)$$

We assume that the functions φ and ψ satisfy conditions of the form (2.3), (2.4a) [respectively (2.4b)], i.e.:

Assumption A3':

$$\int_0^b \varphi(x, 0) dx = 0, \frac{\partial \varphi(0, t_2)}{\partial x} = 0 \left[\int_0^b x \varphi(x, 0) dx = 0, \text{ respectively} \right],$$

$$\int_0^b \psi(x, 0) dx = 0, \frac{\partial \psi(0, t_1)}{\partial x} = 0 \left[\int_0^b x \psi(x, 0) dx = 0, \text{ respectively} \right]$$

and

$$\varphi(x, 0) = \psi(x, 0).$$

In order to write down an abstract formulation of the problem we will need suitable function spaces. Let $B_2^1(0, b)$ be a Hilbert space defined by the author in general form in [3].

Here, we reformulate the author's definition. For this purpose, let $C_0(0, b)$ be the vector space of continuous functions with compact support in $(0, b)$. Since such functions are Lebesgue integrable with respect to dx , we can define on $C_0(0, b)$ the bilinear form $((\cdot, \cdot))$ given by

$$((u, \omega)) = \int_0^b \mathcal{T}_x u \cdot \mathcal{T}_x \omega dx, \quad (2.5)$$

where

$$\mathcal{T}_x u = \int_0^x u(\xi, t) d\xi.$$

We recall that $((\cdot, \cdot))$ is a scalar product on $C_0(0, b)$ for which $C_0(0, b)$ is not complete. We denote by $B_2^1(0, b)$ a completion of $C_0(0, b)$ for a scalar product defined by (2.5). The norm of function u in $B_2^1(0, b)$ is defined by

$$\|u\|_{B_2^1(0, b)} = ((u, u))^{1/2} = \|\mathcal{T}_x u\|_{L^2(0, b)}.$$

Lemma 1: For $x \in (0, b)$, we have

$$\|u\|_{B_2^1(0, b)}^2 \leq \frac{b^2}{2} \|u\|_{L^2(0, b)}^2.$$

Proof: See Corollary 1 for $m = 1$ in Bouziani [3].

The spaces $L^2((0, T_p), L^2(0, b))$, $L^2((0, T_p), B_2^1(0, b))$, $H^1((0, T_p), L^2(0, b))$, $H^1((0, T_p), B_2^1(0, b))$ ($p = 1, 2$), $L^2((0, T_1) \times (0, T_2), B_2^1(0, b))$ are defined as usual.

To problems (2.1)-(2.3), (2.4a) [(2.4b), respectively] we assign the operator $L = (\mathcal{L}, \ell_1, \ell_2)$ with the domain $D(L)$ consisting of functions u belonging to $L^2((0, T_1) \times (0, T_2), B_2^1(0, b))$ satisfying the condition $\frac{\partial u}{\partial t_1}, \frac{\partial u}{\partial t_2}, \frac{\partial^2 u}{\partial t_1 \partial t_2}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2} \in L^2((0, T_1) \times (0, T_2), B_2^1(0, b))$ and conditions (2.3), (2.4a) [(2.4b), respectively]. The operator L is considered from B to F , where B is the Banach space consisting of functions $u \in L^2((0, T_1) \times (0, T_2), B_2^1(0, b))$ having finite norms

$$\|u\|_B = \left(\sup_{0 \leq \tau_2 \leq T_2} \|u(\cdot, \cdot, \tau_2)\|_{H^1((0, T_1), B_2^1(0, b))}^2 + \sup_{0 \leq \tau_1 \leq T_1} \|u(\cdot, \tau_1, \cdot)\|_{H^1((0, T_2), B_2^1(0, b))}^2 \right)^{\frac{1}{2}}$$

and satisfying conditions (2.3), (2.4a) [(2.4b), respectively], and F is a Hilbert space of vector-valued functions (f, φ, ψ) with the finite norms

$$\|(f, \varphi, \psi)\|_F = \left(\|f\|_{L^2((0, T_1) \times (0, T_2), B_2^1(0, b))}^2 + \|\varphi\|_{H^1((0, T_2), L^2(0, b))}^2 + \|\Psi\|_{H^1((0, T_1), L^2(0, b))}^2 \right)^{\frac{1}{2}}.$$

Finally, we give a definition of a strong solution. Let \bar{L} be the closure of the operator L with the domain $D(\bar{L})$.

Definition: A solution of the operator equation

$$\bar{L} u = (f, \varphi, \psi)$$

is called a *strong solution* of problem (2.1)-(2.3), (2.4a) [(2.4b), respectively].

3. Uniqueness and Continuous Dependence of the Solution

In this section we will prove an a priori estimate. The uniqueness and continuous dependence of the solution upon the data presented here are direct consequences of the a priori estimate.

Theorem 1: *Under Assumption A1, the solution of problem (2.1)-(2.3), (2.4a) [respectively, (2.4b)] satisfies the following a priori estimate:*

$$\| u \|_B \leq c \| Lu \|_F, \quad (3.1)$$

where $c > 0$ is a constant independent of u .

Proof: Taking the scalar product, in $B_2^1(0, b)$, of equation (2.1) and $Mu := 2\left(\frac{\partial u}{\partial t_1} + \frac{\partial u}{\partial t_2}\right)$, we obtain

$$\begin{aligned} & \frac{\partial}{\partial t_2} \left\| \frac{\partial u}{\partial t_1} \right\|_{B_2^1(0, b)}^2 + \frac{\partial}{\partial t_1} \left\| \frac{\partial u}{\partial t_2} \right\|_{B_2^1(0, b)}^2 + \int_0^b a(x, t_1, t_2) \frac{\partial}{\partial t_1} u^2 dx \\ & + \int_0^b a(x, t_1, t_2) \frac{\partial}{\partial t_2} u^2 dx + 2 \int_0^b \frac{\partial a(x, t_1, t_2)}{\partial x} u \mathcal{T}_x \left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) dx \\ & = 2 \left(\mathcal{L}u, \frac{\partial u}{\partial t_1} + \frac{\partial u}{\partial t_2} \right)_{B_2^1(0, b)}. \end{aligned} \quad (3.2)$$

Integrating (3.2) over $(0, \tau_1) \times (0, \tau_2)$, where $0 < \tau_1 \leq T_1$ and $0 < \tau_2 \leq T_2$, we have

$$\begin{aligned} & \int_0^{\tau_1} \left\| \frac{\partial u(x, t_1, \tau_2)}{\partial t_1} \right\|_{B_2^1(0, b)}^2 dt_1 + \int_0^{\tau_1} \int_0^b a(x, t_1, \tau_2) (u(x, t_1, \tau_2))^2 dx dt_1 \\ & + \int_0^{\tau_2} \left\| \frac{\partial u(x, \tau_1, t_2)}{\partial t_2} \right\|_{B_2^1(0, b)}^2 dt_2 + \int_0^{\tau_2} \int_0^b a(x, \tau_1, t_2) (u(x, \tau_1, t_2))^2 dx dt_2 \\ & = 2 \int_0^{\tau_1} \int_0^{\tau_2} \left(f, \frac{\partial u}{\partial t_1} + \frac{\partial u}{\partial t_2} \right)_{B_2^1(0, b)} dt_1 dt_2 + \int_0^{\tau_1} \left\| \frac{\partial \psi(x, t_1)}{\partial t_1} \right\|_{B_2^1(0, b)}^2 dt_1 \\ & + \int_0^b \int_0^{\tau_1} a(x, t_1, 0) (\psi(x, t_1))^2 dx dt_1 + \int_0^{\tau_2} \left\| \frac{\partial \varphi(x, t_2)}{\partial t_2} \right\|_{B_2^1(0, b)}^2 dt_2 \\ & + \int_0^b \int_0^{\tau_2} a(x, 0, t_2) (\varphi(x, t_2))^2 dx dt_2 \end{aligned} \quad (3.3)$$

$$\begin{aligned}
& + \int_0^b \int_0^{\tau_1} \int_0^{\tau_2} \left[\frac{\partial a(x,t_1,t_2)}{\partial t_1} + \frac{\partial a(x,t_1,t_2)}{\partial t_2} \right] u^2 dx dt_1 dt_2 \\
& - 2 \int_0^b \int_0^{\tau_1} \int_0^{\tau_2} \frac{\partial a(x,t_1,t_2)}{\partial x} u \mathcal{T}_x \left(\frac{\partial u}{\partial t_1} + \frac{\partial u}{\partial t_2} \right) dx dt_1 dt_2.
\end{aligned}$$

Observe that

$$\begin{aligned}
2 \int_0^{\tau_1} \int_0^{\tau_2} \left(f, \frac{\partial u}{\partial t_1} + \frac{\partial u}{\partial t_2} \right)_{B_2^1(0,b)} dt_1 dt_2 & \leq 2 \int_0^{\tau_1} \int_0^{\tau_2} \| f \|_{B_2^1(0,b)}^2 dt_1 dt_2 \\
& + \int_0^{\tau_1} \int_0^{\tau_2} \left\| \frac{\partial u}{\partial t_1} \right\|_{B_2^1(0,b)}^2 dt_1 dt_2 + \int_0^{\tau_1} \int_0^{\tau_2} \left\| \frac{\partial u}{\partial t_2} \right\|_{B_2^1(0,b)}^2 dt_1 dt_2.
\end{aligned} \tag{3.4}$$

It is easy to see that

$$\begin{aligned}
& - 2 \int_0^b \int_0^{\tau_1} \int_0^{\tau_2} \frac{\partial a(x,t_1,t_2)}{\partial x} u \mathcal{T}_x \left(\frac{\partial u}{\partial t_1} + \frac{\partial u}{\partial t_2} \right) dx dt_1 dt_2 \\
& \leq 2 \int_0^b \int_0^{\tau_1} \int_0^{\tau_2} \left(\frac{\partial a(x,t_1,t_2)}{\partial x} \right)^2 u^2 dx dt_1 dt_2 \\
& + \int_0^{\tau_1} \int_0^{\tau_2} \left[\left\| \frac{\partial u}{\partial t_1} \right\|_{B_2^1(0,b)}^2 + \left\| \frac{\partial u}{\partial t_2} \right\|_{B_2^1(0,b)}^2 \right] dt_1 dt_2.
\end{aligned} \tag{3.5}$$

From (3.3)-(3.5) and from Assumption A1, we get

$$\begin{aligned}
& \int_0^{\tau_1} \left[\| u(x, t_1, \tau_2) \|_{L^2(0,b)}^2 + \left\| \frac{\partial u(x,t_1,\tau_2)}{\partial t_1} \right\|_{B_2^1(0,b)}^2 \right] dt_1 \\
& + \int_0^{\tau_2} \left[\| u(x, \tau_1, t_2) \|_{L^2(0,b)}^2 + \left\| \frac{\partial u(x,\tau_1,t_2)}{\partial t_2} \right\|_{B_2^1(0,b)}^2 \right] dt_2 \\
& \leq c_{10} \left(\int_0^{\tau_1} \int_0^{\tau_2} \| f \|_{B_2^1(0,b)}^2 dt_1 dt_2 + \int_0^{\tau_1} \left[\| \psi \|_{L^2(0,b)}^2 + \left\| \frac{\partial \psi}{\partial t_1} \right\|_{B_2^1(0,b)}^2 \right] dt_1 \right. \\
& \quad \left. + \int_0^{\tau_2} \left[\| \varphi \|_{L^2(0,b)}^2 + \left\| \frac{\partial \varphi}{\partial t_2} \right\|_{B_2^1(0,b)}^2 \right] dt_2 \right) \\
& + c_{11} \left(\int_0^{\tau_1} \int_0^{\tau_2} \left[\| u \|_{L^2(0,b)}^2 + \left\| \frac{\partial u}{\partial t_1} \right\|_{B_2^1(0,b)}^2 + \left\| \frac{\partial u}{\partial t_2} \right\|_{B_2^1(0,b)}^2 \right] dt_1 dt_2 \right),
\end{aligned} \tag{3.6}$$

where

$$c_{10} = \frac{\max(4, c_1)}{\min(c_0, 1)}$$

and

$$c_{11} = \frac{\max(2c_2, 1)}{\min(c_0, 1)}.$$

To finish the proof of Theorem 1, we will need the following result:

Lemma 2: *If f_1 and f_2 are nonnegative functions on the rectangle $(0, T_1) \times (0, T_2)$, f_1 is integrable on $(0, T_1) \times (0, T_2)$, and f_2 is nondecreasing in $(0, T_1) \times (0, T_2)$ with respect to each of its variables separately, then*

$$f_1(\tau_1, \tau_2) \leq \exp(2c(\tau_1 + \tau_2))f_2(\tau_1, \tau_2)$$

is a consequence of the inequality

$$f_1(\tau_1, \tau_2) \leq c \left(\int_0^{\tau_1} f_1(t_1, \tau_2) dt_1 + \int_0^{\tau_2} f_1(\tau_1, t_2) dt_2 \right) + f_2(\tau_1, \tau_2).$$

The proof of the above lemma is analogous to the proof of Lemma 1 in Bouziani [6].

Continuing the proof of Theorem 1, we apply Lemma 2 to (3.6). For this purpose, we denote the left-hand side of (3.6) by $f_1(\tau_1, \tau_2)$, and the sum of three first integrals on the right-hand side of (3.6) by $f_2(\tau_1, \tau_2)$. This procedure eliminates the last integral of the right-hand side of (3.6) and yields:

$$\begin{aligned} & \int_0^{\tau_1} \left[\| u(x, t_1, \tau_2) \|_{L^2(0,b)}^2 + \left\| \frac{\partial u(x, t_1, \tau_2)}{\partial t_1} \right\|_{B_2^1(0,b)}^2 \right] dt_1 \\ & + \int_0^{\tau_2} \left[\| u(x, \tau_1, t_2) \|_{L^2(0,b)}^2 + \left\| \frac{\partial u(x, \tau_1, t_2)}{\partial t_2} \right\|_{B_2^1(0,b)}^2 \right] dt_2 \\ & \leq c_{11} \exp(2c_{10}(\tau_1 + \tau_2)) \left(\int_0^{\tau_1} \int_0^{\tau_2} \| f \|_{B_2^1(0,b)}^2 dt_1 dt_2 \right. \\ & \quad \left. + \int_0^{\tau_1} \left[\| \psi \|_{L^2(0,b)}^2 + \left\| \frac{\partial \psi}{\partial t_1} \right\|_{B_2^1(0,b)}^2 \right] dt_1 \right. \\ & \quad \left. + \int_0^{\tau_2} \left[\| \varphi \|_{L^2(0,b)}^2 + \left\| \frac{\partial \varphi}{\partial t_2} \right\|_{B_2^1(0,b)}^2 \right] dt_2 \right). \end{aligned} \quad (3.7)$$

According to Lemma 1, we bound below the first and the third terms on the left-hand side of (3.7) and we bound above the third and the fifth terms on the right-hand side of (3.7). Consequently, we obtain

$$\begin{aligned} & \| u(x, t_1, \tau_2) \|_{H^1((0,T_1), B_2^1(0,b))}^2 + \| u(x, \tau_1, t_2) \|_{H^1((0,T_2), B_2^1(0,b))}^2 \\ & \leq c_{12} \left(\| f \|_{L^2((0,T_1) \times (0,T_2), B_2^1(0,b))}^2 + \| \psi \|_{H^1((0,T_1), L^2(0,b))}^2 \right. \\ & \quad \left. + \| \varphi \|_{H^1((0,T_2), L^2(0,b))}^2 \right), \end{aligned} \quad (3.8)$$

where

$$c_{12} = \frac{c_{11} \exp(2c_{10}(T_1+T_2)) \max\left(1, \frac{b^2}{2}\right)}{\min\left(1, \frac{2}{b^2}\right)}.$$

Since the right-hand side of inequality (3.8) does not depend on τ_1 and τ_2 , then, in the left-hand side of (3.8) we can take the supremum with respect to τ_p from 0 to T_p ($p = 1, 2$). Therefore, we obtain (3.1) with $c = c_{12}^{1/2}$. The proof of Theorem 1 is complete.

Proposition 1: *The operator L from B to F has a closure.*

The proof of the above proposition is similar to the proof of Proposition 1 in Bouziani [4].

Theorem 1 can be extended to cover strong solutions by passing to the limit.

Corollary 1: *Under the assumptions of Theorem 1, there exists a positive constant c such that*

$$\|u\|_B \leq c \|\bar{L}u\|_F,$$

where c does not depend on u .

Corollary 1 implies the following:

Corollary 2: *A strong solution of (2.1)-(2.3), (2.4a) [respectively, (2.4b)] is unique, if it exists, and it depends continuously on (f, φ, ψ) .*

Corollary 3: *The range $R(\bar{L})$ of solutions for the closure \bar{L} of L is closed in F , $R(\bar{L}) = \overline{R(L)}$ and $\bar{L}^{-1} = \overline{L^{-1}}$, where $\overline{L^{-1}}$ is the unique extension of L^{-1} (by continuity from $R(L)$ to $\overline{R(L)}$).*

4. Existence of the Solution

In this section we concentrate on the existence of the strong solution of problem (2.1)-(2.3), (2.4a) [respectively, (2.4b)]. The main idea is to demonstrate that the range $R(L)$ is dense in F .

Theorem 2: *Suppose that Assumptions A1 and A2 are satisfied. Then, for arbitrary $f \in L^2((0, T_1) \times (0, T_2), B_2^1(0, b))$, $\varphi \in H^1((0, T_2), L^2(0, b))$ and $\psi \in H^1((0, T_1), L^2(0, b))$, problem (2.1)-(2.3), (2.4a) [respectively, (2.4b)] admits a strong solution $u = \bar{L}^{-1}(f, \varphi, \psi) = \overline{L^{-1}}(f, \varphi, \psi)$.*

Proof: To prove the existence of a strong solution of problem (2.1)-(2.3), (2.4a) [respectively (2.4b)] for all $(f, \varphi, \psi) \in F$, it remains to prove that $\overline{R(L)} = F$. To this end, we establish the density in the special case, where u belongs to $D_0(L)$, which is equal to the set of all $u \in D(L)$ vanishing in a neighborhood of $t_1 = 0$ and $t_2 = 0$. Then we proceed to the general case.

Proposition 2: *Let the assumptions of Theorem 2 be satisfied. If for $g \in L^2((0, T_1) \times (0, T_2); B_2^1(0, b))$ and for all $u \in D_0(L)$, we have*

$$\int_0^{T_1} \int_0^{T_2} ((\mathcal{L}u, g)) dt_1 dt_2 = 0 \quad \forall u \in D_0(L). \quad (4.1)$$

Then g vanishes almost everywhere in Q .

Proof of Proposition 2: By the fact that relation (4.1) holds for any function $u \in D_0(L)$, we express it in a special form. Let

$$u = \begin{cases} 0, & 0 \leq t_p \leq s_p, \\ \int_{s_1}^{t_1} \int_{s_2}^{t_2} \frac{\partial^2 u}{\partial t_1 \partial t_2} d\tau_1 d\tau_2, & s_p \leq t_p \leq T_p \end{cases} \quad (4.2)$$

for $p = 1, 2$ and let $\frac{\partial^2 u}{\partial t_1 \partial t_2}$ be a solution of

$$a(\sigma, t_1, t_2) \frac{\partial^2 u}{\partial t_1 \partial t_2} = T_{t_p}^* g_p = \int_{t_p}^{T_p} g_p d\tau_p, \quad (4.3)$$

where σ is a fixed number belonging to $[0, T]$. Formulas (4.2) and (4.3) imply that u belongs to $D_{s_1, s_2}(L)$, which denotes the set of all $u \in D(L)$ vanishing in the neighborhood of $t_p = s_p$ and $t_p < s_p$ ($p = 1, 2$). Putting $s_p = 0$ ($p = 1, 2$), we have that $u \in D_0(L)$. It follows from (4.2) and (4.3) that

$$g = \sum_{p=1}^2 g_p = - \left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) \left(a(\sigma, t_1, t_2) \frac{\partial^2 u}{\partial t_1 \partial t_2} \right). \quad (4.4)$$

To prove that the function g , defined by (4.4), belongs to $L^2((s_1, T_1) \times (s_2, T_2), B_2^1(0, b))$, we apply the following result:

Lemma 3: *If the assumptions of Proposition 2 are satisfied, then the function u , defined by (4.2), has derivatives of the form $\frac{\partial^3 u}{\partial t_1 \partial t_2}$ and $\frac{\partial^3 u}{\partial t_1^2 \partial t_2}$ belonging to $L^2((s_1, T_1) \times (s_2, T_2), B_2^1(0, b))$.*

Proof: See Lemma 2 in Bouziani [3].

Relation (4.4) implies that (4.1) can be written in the form

$$\begin{aligned} & - \int_0^{T_1} \int_0^{T_2} \left(\left(\frac{\partial^2 u}{\partial t_1 \partial t_2}, \frac{\partial}{\partial t_1} \left(a(\sigma, t_1, t_2) \frac{\partial^2 u}{\partial t_1 \partial t_2} \right) \right) \right) dt_1 dt_2 \\ & - \int_0^{T_1} \int_0^{T_2} \left(\left(\frac{\partial^2 u}{\partial t_1 \partial t_2}, \frac{\partial}{\partial t_2} \left(a(\sigma, t_1, t_2) \frac{\partial^2 u}{\partial t_1 \partial t_2} \right) \right) \right) dt_1 dt_2 \quad (4.5) \\ & + \int_0^{T_1} \int_0^{T_2} \left(\left(\frac{\partial}{\partial x} \left(a(x, t_1, t_2) \frac{\partial u}{\partial x} \right), \frac{\partial}{\partial t_1} \left(a(\sigma, t_1, t_2) \frac{\partial^2 u}{\partial t_1 \partial t_2} \right) \right) \right) dt_1 dt_2 \\ & + \int_0^{T_1} \int_0^{T_2} \left(\left(\frac{\partial}{\partial x} \left(a(x, t_1, t_2) \frac{\partial u}{\partial x} \right), \frac{\partial}{\partial t_2} \left(a(\sigma, t_1, t_2) \frac{\partial^2 u}{\partial t_1 \partial t_2} \right) \right) \right) dt_1 dt_2 = 0. \end{aligned}$$

Integrating by parts, each term of (4.5), we obtain

$$\begin{aligned} & \int_0^b \int_{s_2}^{T_2} a(\sigma, s_1, t_2) \left(T_x \frac{\partial^2 u(x, s_1, t_2)}{\partial t_1 \partial t_2} \right)^2 dx dt_2 \\ & = \int_0^b \int_{s_1}^{T_1} \int_{s_2}^{T_2} \frac{\partial a(\sigma, t_1, t_2)}{\partial t_1} \left(T_x \frac{\partial^2 u}{\partial t_1 \partial t_2} \right)^2 dx dt_1 dt_2 \quad (4.6) \\ & - 2 \int_0^{T_1} \int_0^{T_2} \left(\left(\frac{\partial^2 u}{\partial t_1 \partial t_2}, \frac{\partial}{\partial t_1} \left(a(\sigma, t_1, t_2) \frac{\partial^2 u}{\partial t_1 \partial t_2} \right) \right) \right) dt_1 dt_2, \end{aligned}$$

$$\begin{aligned}
& \int_0^b \int_{s_1}^{T_1} a(\sigma, t_1, s_2) \left(\mathcal{T}_x \frac{\partial^2 u(x, t_1, s_2)}{\partial t_1 \partial t_2} \right)^2 dx dt_1 \\
&= \int_0^b \int_{s_1}^{T_1} \int_{s_2}^{T_2} \frac{\partial a(\sigma, t_1, t_2)}{\partial t_2} \left(\mathcal{T}_x \frac{\partial^2 u}{\partial t_1 \partial t_2} \right)^2 dx dt_1 dt_2 \\
&- 2 \int_0^{T_1} \int_0^{T_2} \left(\left(\frac{\partial^2 u}{\partial t_1 \partial t_2}, \frac{\partial}{\partial t_2} \left(a(\sigma, t_1, t_2) \frac{\partial^2 u}{\partial t_1 \partial t_2} \right) \right) \right) dt_1 dt_2,
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
& \int_0^b \int_{s_1}^{T_1} \left[\frac{\partial^2 a(x, t_1, T_2)}{\partial t_1^2} a(\sigma, t_1, T_2) + \frac{\partial a(x, t_1, T_2)}{\partial t_1} \frac{\partial a(\sigma, t_1, T_2)}{\partial t_1} \right] (u(x, t_1, T_2))^2 dx dt_1 \\
&+ \int_0^b \int_{s_1}^{T_1} a(x, t_1, T_2) a(\sigma, t_1, T_2) \left(\frac{\partial u(x, t_1, T_2)}{\partial t_1} \right)^2 dx dt_1
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
&= \int_0^b \int_{s_1}^{T_1} \int_{s_2}^{T_2} \left[\frac{\partial^3 a(x, t_1, t_2)}{\partial t_2 \partial t_1^2} a(\sigma, t_1, t_2) + \frac{\partial^2 a(x, t_1, t_2)}{\partial t_1^2} \frac{\partial a(\sigma, t_1, t_2)}{\partial t_2} \right. \\
&+ \left. \frac{\partial a(x, t_1, t_2)}{\partial t_2 \partial t_1} \frac{\partial a(\sigma, t_1, t_2)}{\partial t_1} + \frac{\partial a(x, t_1, t_2)}{\partial t_1} \frac{\partial^2 a(\sigma, t_1, t_2)}{\partial t_2 \partial t_1} \right] u^2 dx dt_1 dt_2 \\
&- 2 \int_0^b \int_{s_2}^{T_2} \frac{\partial a(x, t_1, t_2)}{\partial t_1} a(\sigma, T_1, t_2) u(x, T_1, t_2) \frac{\partial u(x, T_1, t_2)}{\partial t_2} dx dt_2 \\
&+ \int_0^b \int_{s_1}^{T_1} \int_{s_2}^{T_2} \left[\frac{\partial a(x, t_1, t_2)}{\partial t_2} a(\sigma, t_1, t_2) + a(x, t_1, t_2) \frac{\partial a(\sigma, t_1, t_2)}{\partial t_2} \right] \left(\frac{\partial u}{\partial t_1} \right)^2 dx dt_1 dt_2 \\
&- 2 \int_0^b \int_{s_1}^{T_1} \int_{s_2}^{T_2} \frac{\partial a(x, T_1, t_2)}{\partial t_1} a(\sigma, t_1, t_2) \frac{\partial u}{\partial t_1} \frac{\partial u}{\partial t_2} dx dt_1 dt_2 \\
&- 2 \int_0^b \int_{s_1}^{T_1} \int_{s_2}^{T_2} \left[\frac{\partial^2 a(x, t_1, t_2)}{\partial t_1 \partial x} u + \frac{\partial a(x, t_1, t_2)}{\partial x} \frac{\partial u}{\partial t_1} \right] a(\sigma, t_1, t_2) \mathcal{T}_x \frac{\partial^2 u}{\partial t_1 \partial t_2} dx dt_1 dt_2 \\
&+ 2 \int_0^{T_1} \int_0^{T_2} \left(\left(\frac{\partial}{\partial x} \left(a(x, t_1, t_2) \frac{\partial u}{\partial x} \right), \frac{\partial}{\partial t_1} \left(a(\sigma, t_1, t_2) \frac{\partial^2 u}{\partial t_1 \partial t_2} \right) \right) \right) dt_1 dt_2,
\end{aligned}$$

$$\begin{aligned}
& \int_0^b \int_{s_2}^{T_2} \left[\frac{\partial a(x, T_1, t_2)}{\partial t_2^2} a(\sigma, T_1, t_2) + \frac{\partial a(x, T_1, t_2)}{\partial t_2} \frac{\partial a(\sigma, T_1, t_2)}{\partial t_2} \right] (u(x, T_1, t_2))^2 dx dt_2 \\
&+ \int_0^b \int_{s_2}^{T_2} a(x, T_1, t_2) a(\sigma, T_1, t_2) \left(\frac{\partial u(x, T_1, t_2)}{\partial t_2} \right)^2 dx dt_2
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
&= \int_0^b \int_{s_1}^{T_1} \int_{s_2}^{T_2} \left[\frac{\partial^3 a(x, t_1, t_2)}{\partial t_1 \partial t_2^2} a(\sigma, t_1, t_2) + \frac{\partial^2 a(x, t_1, t_2)}{\partial t_2^2} \frac{\partial a(\sigma, t_1, t_2)}{\partial t_1} \right. \\
&+ \left. \frac{\partial^2 a(x, t_1, t_2)}{\partial t_1 \partial t_2} \frac{\partial a(\sigma, t_1, t_2)}{\partial t_2} + \frac{\partial a(x, t_1, t_2)}{\partial t_2} \frac{\partial^2 a(\sigma, t_1, t_2)}{\partial t_1 \partial t_2} \right] u^2 dx dt_1 dt_2
\end{aligned}$$

$$\begin{aligned}
& + \int_0^b \int_{s_1}^{T_1} \int_{s_2}^{T_2} \left[\frac{\partial a(x, t_1, t_2)}{\partial t_1} a(\sigma, t_1, t_2) + a(x, t_1, t_2) \frac{\partial a(\sigma, t_1, t_2)}{\partial t_1} \right] \left(\frac{\partial u}{\partial t_2} \right)^2 dx dt_1 dt_2 \\
& - 2 \int_0^b \int_{s_1}^{T_1} \frac{\partial a(x, t_1, T_2)}{\partial t_2} a(\sigma, t_1, T_2) u(x, t_1, T_2) \frac{\partial u(x, t_1, T_2)}{\partial t_1} dx dt_1 \\
& - 2 \int_0^b \int_{s_1}^{T_1} \int_{s_2}^{T_2} \frac{\partial a(x, t_1, t_2)}{\partial t_2} a(\sigma, t_1, t_2) \frac{\partial u}{\partial t_1} \frac{\partial u}{\partial t_2} dx dt_1 dt_2 \\
& - 2 \int_0^b \int_{s_1}^{T_1} \int_{s_2}^{T_2} \left[\frac{\partial^2 a(x, t_1, t_2)}{\partial t_2 \partial x} u + \frac{\partial a(x, t_1, t_2)}{\partial x} \frac{\partial u}{\partial t_2} \right] a(\sigma, t_1, t_2) \mathcal{I}_x \frac{\partial^2 u}{\partial t_1 \partial t_2} dx dt_1 dt_2 \\
& + 2 \int_0^{T_1} \int_0^{T_2} \left(\left(\frac{\partial}{\partial x} \left(a(x, t_1, t_2) \frac{\partial u}{\partial x} \right), \frac{\partial}{\partial t_2} \left(a(\sigma, t_1, t_2) \frac{\partial^2 u}{\partial t_1 \partial t_2} \right) \right) \right) dt_1 dt_2.
\end{aligned}$$

According to Assumptions A1 and A2, we get

$$\begin{aligned}
c_0 \int_{s_2}^{T_2} \left\| \frac{\partial^2 u(x, s_1, t_2)}{\partial t_1 \partial t_2} \right\|_{B_2^1(0, b)}^2 dt_2 & \leq c_3 \int_{s_1}^{T_1} \int_{s_2}^{T_2} \left\| \frac{\partial^2 u}{\partial t_1 \partial t_2} \right\|_{B_2^1(0, b)}^2 dt_1 dt_2 \\
& - 2 \int_0^{T_1} \int_0^{T_2} \left(\left(\frac{\partial^2 u}{\partial t_1 \partial t_2}, \frac{\partial}{\partial t_1} \left(a(\sigma, t_1, t_2) \frac{\partial^2 u}{\partial t_1 \partial t_2} \right) \right) \right) dt_1 dt_2,
\end{aligned} \tag{4.10}$$

$$\begin{aligned}
c_0 \int_{s_1}^{T_1} \left\| \frac{\partial^2 u(x, t_1, s_2)}{\partial t_1 \partial t_2} \right\|_{B_2^1(0, b)}^2 dt_1 & \leq c_3 \int_{s_1}^{T_1} \int_{s_2}^{T_2} \left\| \frac{\partial^2 u}{\partial t_1 \partial t_2} \right\|_{B_2^1(0, b)}^2 dt_1 dt_2 \\
& - 2 \int_0^{T_1} \int_0^{T_2} \left(\left(\frac{\partial^2 u}{\partial t_1 \partial t_2}, \frac{\partial}{\partial t_2} \left(a(\sigma, t_1, t_2) \frac{\partial^2 u}{\partial t_1 \partial t_2} \right) \right) \right) dt_1 dt_2,
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
(c_5 c_0 + c_4^2) \int_{s_1}^{T_1} \| u(x, t_1, T_2) \|_{L^2(0, b)}^2 dt_1 & + c_0^2 \int_{s_1}^{T_1} \left\| \frac{\partial u(x, t_1, T_2)}{\partial t_1} \right\|_{L^2(0, b)}^2 dt_1 \\
& \leq (c_9 c_1 + c_6 c_3 + 2c_8 c_3 + 2c_7^2) \int_{s_1}^{T_1} \int_{s_2}^{T_2} \| u \|_{L^2(0, b)}^2 dt_1 dt_2
\end{aligned} \tag{4.12}$$

$$+ (2c_1 c_3 + c_3^2 + 2c_2^2) \int_{s_1}^{T_1} \int_{s_2}^{T_2} \left\| \frac{\partial u}{\partial t_1} \right\|_{L^2(0, b)}^2 dt_1 dt_2$$

$$+ \frac{2c_3^2 c_1^2}{c_0^2} \int_{s_2}^{T_2} \| u(x, T_1, t_2) \|_{L^2(0, b)}^2 dt_2 + \frac{c_1^2}{2} \int_{s_2}^{T_2} \left\| \frac{\partial u(x, T_1, t_2)}{\partial t_2} \right\|_{L^2(0, b)}^2 dt_2$$

$$\begin{aligned}
& + c_1^2 \int_{s_1}^{T_1} \int_{s_2}^{T_2} \left\| \frac{\partial^2 u}{\partial t_1 \partial t_2} \right\|_{B_2^1(0,b)}^2 dt_1 dt_2 + c_1^2 \int_{s_1}^{T_1} \int_{s_2}^{T_2} \left\| \frac{\partial u}{\partial t_2} \right\|_{L^2(0,b)}^2 dt_1 dt_2 \\
& + 2 \int_0^{T_1} \int_0^{T_2} \left(\left(\frac{\partial}{\partial x} (a(x, v, t_1, t_2)) \frac{\partial u}{\partial x} \right), \frac{\partial}{\partial t_1} \left(a(\sigma, t_1, t_2) \frac{\partial^2 u}{\partial t_1 \partial t_2} \right) \right) dt_1 dt_2, \\
& (c_5 c_0 + c_4^2) \int_{s_2}^{T_2} \| u(x, T_1, t_2) \|_{L^2(0,b)}^2 dt_2 + c_0^2 \int_{s_2}^{T_1} \left\| \frac{\partial u(x, T_1, t_2)}{\partial t_2} \right\|_{L^2(0,b)}^2 dt_2 \\
& \leq 2 \frac{c_1^2 c_3^2}{c_0^2} \int_{s_1}^{T_1} \| u(x, t_1, T_2) \|_{L^2(0,b)}^2 dt_1 + \frac{c_0^2}{2} \int_{s_1}^{T_1} \left\| \frac{\partial u(x, t_1, T_2)}{\partial t_1} \right\|^2 dx dt_1 \\
& \quad + (c_9 c_1 + c_6 c_3 + 2c_8 c_3 + 2c_7^2) \int_{s_1}^{T_1} \int_{s_2}^{T_2} \| u \|_{L^2(0,b)}^2 dt_1 dt_2 \\
& \quad + (2c_1 c_3 + c_3^2 + 2c_2^2) \int_{s_1}^{T_1} \int_{s_2}^{T_2} \left\| \frac{\partial u}{\partial t_2} \right\|_{L^2(0,b)}^2 dt_1 dt_2 \\
& \quad + c_1^2 \int_{s_1}^{T_1} \int_{s_2}^{T_2} \left\| \frac{\partial u}{\partial t_1} \right\|_{L^2(0,b)}^2 dt_1 dt_2 + c_1^2 \int_{s_1}^{T_1} \int_{s_2}^{T_2} \left\| \frac{\partial^2 u}{\partial t_1 \partial t_2} \right\|_{B_2^1(0,b)}^2 dt_1 dt_2 \\
& \quad + 2 \int_0^{T_1} \int_0^{T_0} \left(\left(\frac{\partial}{\partial x} (a(x, t_1, t_2)) \frac{\partial u}{\partial x} \right), \frac{\partial}{\partial t_2} \left(a(\sigma, t_1, t_2) \frac{\partial^2 u}{\partial t_1 \partial t_2} \right) \right) dt_1 dt_2.
\end{aligned} \tag{4.13}$$

Observe that

$$\begin{aligned}
& \frac{2c_1^2 c_3^2}{c_0^2} \int_{s_1}^{T_1} \| u(x, t_1, T_2) \|_{L^2(0,b)}^2 dt_1 \\
& \leq \frac{2c_1^2 c_3^2}{c_0^2} \int_{s_1}^{T_1} \int_{s_2}^{T_2} \left(\| u \|_{L^2(0,b)}^2 + \left\| \frac{\partial u}{\partial t_2} \right\|_{L^2(0,b)}^2 \right) dt_1 dt_2,
\end{aligned} \tag{4.14}$$

$$\begin{aligned}
& \frac{2c_1^2 c_3^2}{c_0^2} \int_{s_2}^{T_2} \| u(x, T_1, t_2) \|_{L^2(0,b)}^2 dt_2 \\
& \leq \frac{2c_1^2 c_3^2}{c_0^2} \int_{s_1}^{T_1} \int_{s_2}^{T_2} \left(\| u \|_{L^2(0,b)}^2 + \left\| \frac{\partial u}{\partial t_1} \right\|_{L^2(0,b)}^2 \right) dt_1 dt_2.
\end{aligned} \tag{4.15}$$

Adding inequalities (4.10)-(4.15) and applying (4.5), we get

$$\begin{aligned}
& \int_{s_2}^{T_2} \left[\left\| \frac{\partial^2 u(x, s_1, t_2)}{\partial t_1 \partial t_2} \right\|_{B_2^1(0, b)}^2 + \| u(x, T_1, t_2) \|_{L^2(0, b)}^2 + \left\| \frac{\partial u(x, T_1, t_2)}{\partial t_2} \right\|_{L^2(0, b)}^2 \right] dt_2 \\
& + \int_{s_1}^{T_1} \left[\left\| \frac{\partial^2 u(x, t_1, s_2)}{\partial t_1 \partial t_2} \right\|_{B_2^1(0, b)}^2 + \| u(x, t_1, T_2) \|_{L^2(0, b)}^2 + \left\| \frac{\partial u(x, t_1, T_2)}{\partial t_1} \right\|_{L^2(0, b)}^2 \right] dt_1 \\
& \leq c_{13} \int_{s_1}^{T_1} \int_{s_2}^{T_2} \left[\left\| \frac{\partial^2 u}{\partial t_1 \partial t_2} \right\|_{B_2^1(0, b)}^2 + \| u \|_{L^2(0, b)}^2 \right. \\
& \quad \left. + \left\| \frac{\partial u}{\partial t_1} \right\|_{L^2(0, b)}^2 + \left\| \frac{\partial u}{\partial t_2} \right\|_{L^2(0, b)}^2 \right] dt_1 dt_2,
\end{aligned} \tag{4.16}$$

where

$$c_{13} = \frac{\max \left(c_3, c_1 c_9 + c_3 c_6 + c_2 c_8 + 2c_7^2 + \frac{2c_1^2 c_3^2}{c_0^2}, 2c_1 c_3 + c_3^2 + 2c_2^2 + c_1^2 + \frac{2c_1^2 c_3^2}{c_0^2} \right)}{\min \left(c_0, \frac{c_0^2}{2}, \frac{1}{2}(c_0 c_5 + c_4^2) \right)}.$$

Inequality (4.16) is basic in the proof of Proposition 2. In order to apply inequality (4.16), we introduce a new function θ by the formula

$$\theta(x, t_1, t_2) = \int_{t_1}^{T_1} u_{\tau_1} d\tau_1 + \int_{t_2}^{T_2} u_{\tau_2} d\tau_2.$$

Consequently,

$$u(x, T_1, t_2) = \theta(x, s_1, t_2), \quad u(x, t_1, T_2) = \theta(x, t_1, s_2),$$

$$\frac{\partial u(x, T_1, t_2)}{\partial t_2} = \frac{\partial \theta(x, s_1, t_2)}{\partial t_2}, \quad \frac{\partial u(x, t_1, T_2)}{\partial t_1} = \frac{\partial \theta(x, t_1, s_2)}{\partial t_1}.$$

Therefore (4.16) yields

$$\begin{aligned}
& \int_{s_2}^{T_2} \left[\left\| \frac{\partial^2 u(x, s_1, t_2)}{\partial t_1 \partial t_2} \right\|_{B_2^1(0, b)}^2 \right. \\
& \quad \left. + \left(1 - \frac{3}{4} c_{13} (T_2 - s_2) \right) \left(\| \theta(x, s_1, t_2) \|_{L^2(0, b)}^2 + \left\| \frac{\partial \theta(x, s_1, t_2)}{\partial t_2} \right\|_{L^2(0, b)}^2 \right) \right] dt_2 \\
& \quad + \int_{s_1}^{T_1} \left[\left\| \frac{\partial^2 u(x, t_1, s_2)}{\partial t_1 \partial t_2} \right\|_{B_2^1(0, b)}^2 + \left(1 - \frac{3}{4} c_{13} (T_1 - s_1) \right) \right. \\
& \quad \left. \times \left(\| \theta(x, t_1, s_2) \|_{L^2(0, b)}^2 + \left\| \frac{\partial \theta(x, t_1, s_2)}{\partial t_1} \right\|_{L^2(0, b)}^2 \right) \right] dt_1 \\
& \leq \frac{3c_{13}}{4} \int_{s_1}^{T_1} \int_{s_2}^{T_2} \left[\left\| \frac{\partial^2 u}{\partial t_1 \partial t_2} \right\|_{B_2^1(0, b)}^2 + \| \theta \|_{L^2(0, b)}^2 + \left\| \frac{\partial \theta}{\partial t_2} \right\|_{L^2(0, b)}^2 + \left\| \frac{\partial \theta}{\partial t_1} \right\|_{L^2(0, b)}^2 \right] dt_1 dt_2.
\end{aligned} \tag{4.17}$$

Hence, if $s_{p_0} > 0$ satisfies $1 - \frac{3}{4}c_{13}(T_p - s_{p_0}) = 1/2$ ($p = 1, 2$), then (4.17) becomes

$$\begin{aligned}
& \int_{s_2}^{T_2} \left[\left\| \frac{\partial^2 u(x, s_1, t_2)}{\partial t_1 \partial t_2} \right\|_{B_2^1(0, b)}^2 + \|\theta(x, s_1, t_2)\|_{L^2(0, b)}^2 + \left\| \frac{\partial \theta(x, s_1, t_2)}{\partial t_2} \right\|_{L^2(0, b)}^2 \right] dt_2 \\
& + \int_{s_1}^{T_1} \left[\left\| \frac{\partial^2 u(x, t_1, s_2)}{\partial t_1 \partial t_2} \right\|_{B_2^1(0, b)}^2 + \|\theta(x, t_1, s_2)\|_{L^2(0, b)}^2 + \left\| \frac{\partial \theta(x, t_1, s_2)}{\partial t_1} \right\|_{L^2(0, b)}^2 \right] dt_1 \\
& \leq \frac{3c_{13}}{2} \int_{s_1}^{T_1} \int_{s_2}^{T_2} \left[\left\| \frac{\partial^2 u}{\partial t_1 \partial t_2} \right\|_{B_2^1(0, b)}^2 + \|\theta\|_{L^2(0, b)}^2 \right. \\
& \quad \left. + \left\| \frac{\partial \theta}{\partial t_2} \right\|_{L^2(0, b)}^2 + \left\| \frac{\partial \theta}{\partial t_1} \right\|_{L^2(0, b)}^2 \right] dt_1 dt_2.
\end{aligned} \tag{4.18}$$

Applying a variant of Lemma 2 for the case of the reversed intervals to (4.18), to eliminate the double integrals on the right-hand side in (4.18), we conclude that $g = 0$ almost everywhere in $(0, b) \times (s_1, T_1) \times (s_2, T_2)$. Proceeding in this way, step by step, we prove that $g = 0$ almost everywhere in Q . The proof of Proposition 2 is complete.

To finish the proof of Theorem 2, consider the general case. Let the element $W = (f, \varphi, \psi) \in F$ be orthogonal to $R(L)$, so that

$$\begin{aligned}
& (\mathcal{L}u, f)_{L^2((0, T_1) \times (0, T_2), B_2^1(0, b))} + (\ell_1 u, \varphi)_{H^1((0, T_2), L^2(0, b))} \\
& + (\ell_2 u, \psi)_{H^1((0, T_1), L^2(0, b))} = 0, \quad \forall u \in D(L).
\end{aligned} \tag{4.19}$$

Putting u in (4.19) equal to any element of $D_0(L)$, we get

$$(\mathcal{L}u, f)_{L^2((0, T_1) \times (0, T_2), B_2^1(0, b))} = 0, \quad \forall u \in D_0(L).$$

Proposition 2 implies that $f \equiv 0$. Hence, (4.19) implies that

$$(\ell_1 u, \varphi)_{H^1((0, T_2), L^2(0, b))} + (\ell_2 u, \psi)_{H^1((0, T_1), L^2(0, b))} = 0, \quad \forall u \in D(L).$$

Since $\ell_1 u$ and $\ell_2 u$ are independent and the ranges of values ℓ_1 and ℓ_2 are everywhere dense in $H^1((0, T_2), L^2(0, b))$, $H^1((0, T_1), L^2(0, b))$, the above equality implies that $\varphi \equiv 0$, and $\psi \equiv 0$ (recall that φ and ψ satisfy Assumption A3'). Hence $\overline{R(L)} = F$. Therefore, the proof of Theorem 2 is complete.

Remark 1: If equation (1.1) is replaced by

$$\mathcal{L}v = \frac{\partial^2 v}{\partial t_1 \partial t_2} - \frac{\partial}{\partial x} (a(x, t_1, t_2) \frac{\partial v}{\partial x}) = f(x, t_1, t_2, v)$$

then the existence, uniqueness and continuous dependence upon the data of a solution can be proved by the same method, provided that f satisfies a growth condition of the form

$$|f(x, t_1, t_2, v)| \leq \tilde{f}(x, t_1, t_2) + C|v|.$$

Remark 2: Our results can be extended to the following nonlocal hyperbolic problem:

$$\left\{ \begin{array}{l} \mathcal{L}v: = \frac{\partial^2 v}{\partial t_1 \partial t_2} + \text{sign}[(1 - |\lambda_1|^2)(1 - |\lambda_2|^2)] \\ \ell_1 v: = v|_{t_1=0} - \lambda_1 v|_{t_1=T_1} = \Phi(x, t_2), \\ \ell_2 v: = v|_{t_2=0} - \lambda_2 v|_{t_2=T_2} = \Psi(x, t_1), \\ \frac{\partial v(0, t_1, t_2)}{\partial x} = \mu_1(t_1, t_2), \\ \int_0^b v(x, t_1, t_2) dx = E(t_1, t_2), \\ \int_0^b x v(x, t_1, t_2) dx = \chi(t_1, t_2), \end{array} \right. \quad \begin{array}{l} \frac{\partial}{\partial x}(a(x, t_1, t_2) \frac{\partial v}{\partial x}) = f, (x, t_1, t_2) \in Q, \lambda_1, \lambda_2 \in C, \\ (x, t_2) \in (0, b) \times (0, T_2), \\ (x, t_1) \in (0, b) \times (0, T_1), \\ (t_1, t_2) \in (0, T_1) \times (0, T_2), \\ (t_1, t_2) \in (0, T_1) \times (0, T_2) \\ (t_1, t_2) \in (0, T_1) \times (0, T_2), \text{ respectively].} \end{array}$$

Remark 3: Our results can also be extended to the following problem:

$$\left\{ \begin{array}{l} \mathcal{L}v: = \frac{\partial^2 v}{\partial t_1 \partial t_2} - \frac{\partial^{2m}}{\partial x^{2m}}(a(x, t_1, t_2)v) = f, \quad (x, t_1, t_2) \in Q, \\ \ell_1 v: = v(x, 0, t_2) = \Phi(x, t_2), \quad (x, t_2) \in (0, b) \times (0, T_2), \\ \ell_2 v: = v(x, t_1, 0) = \Psi(x, t_1), \quad (x, t_1) \in (0, b) \times (0, T_1), \\ \frac{\partial^p v(0, t_1, t_2)}{\partial x^p} = 0, \quad 0 < p \leq m, \quad (t_1, t_2) \in (0, T_1) \times (0, T_2), \\ \int_0^b x^q v(x, t) dx = 0, \quad p \leq q \leq 2m - 1, \quad (t_1, t_2) \in (0, T_1) \times (0, T_2), \end{array} \right.$$

where m, p, q are integers.

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