[4] R. M. Brooks, *Boundaries for locally m-convex algebras*, Duke Math. J., 34 (1967), pp. 103–116.
[5] R. M. Crownover, *Principal ideals which are maximal ideals in Banach algebras*, Studia Math., 33 (1969), pp. 299–304.
[6] K. Hoffman, *Fundamentals of Banach Algebras*, Institutio de Mathematica da Universidad de Parana, Curitiba, 1962.
[7] M. E. Kuczma, *On a Problem of E. Michael concerning topological divisors of zero*. Colloq. Math., 19 (1968), pp. 295–299.
[8] J. A. Lindberg, Jr., *Algebraic extensions of commutative Banach algebras*, Pac. J. Math., 14 (1964), pp. 559–589.
[9] E. A. Michael, *Locally m-convex topological algebras*, Mem. Amer. Math. Soc. 11 (1952).
[10] H. Rossi, *The local maximum modulus principle*, Ann. of Math., (2) 72 (1960), pp. 1–11.
[11] W. Rudin, *Real and Complex Analysis*, New York, 1966.

S.U.N.Y.
OSWEGO, NEW YORK

Reçu par la Rédaction le 19. 9. 1970 (206)

STUDIA MATHEMATICA, T. XL. (1971)

On a class of operators on Orlicz spaces

by

J. J. UHL, Jr. (Urbana, Ill.)

Abstract. Let L^{Φ} be an Orlicz space over a σ-finite measure space. If $\mathfrak{X}$ is a Banach space and $t\colon L^{\Phi}\to\mathfrak{X}$ is a linear operator, $|||t|||_{\Phi}=\sup\sum_{i=1}^{n}\|a_i t(\chi_{E_i})\|$ where the supremum is taken over all measurable simple functions $f=\sum_{i=1}^{n}a_i\chi_{E_i}$ $\{E_i\}$ disjoint and $\|f\|_{\Phi}\leqslant 1$. Under fairly general assumptions on $\mathfrak{X}$ and Φ it is shown that $|||t|||_{\Phi}<\infty$ if and only if $t(f)=\int_{\Omega} fg\,du$ where $g\colon \Omega\to\mathfrak{X}$ is measurable and the above Bochner integral exists for all $f\in L^{\Phi}$. Consequently it is shown that such operators are compact. Finally, under moderate assumptions on Φ, it is shown that $t\colon L^{\Phi}\to L^{\Phi}$ has $|||t|||_{\Phi}<\infty$ if and only if t's adjoint is of finite double norm, thus providing a new characterization of Hilbert-Schmidt operators.

1. Introduction. Let (Ω,Σ,μ) be a sigma-finite measure space, Φ and Ψ be complementary Young's functions and $L^{\Phi}(\Omega,\Sigma,\mu)(=L^{\Phi})$ and $L^{\Psi}(\Omega,\Sigma,\mu)(=L^{\Psi})$ be the corresponding Orlicz spaces of (equivalence classes of) measurable functions on Ω. L^{Φ} is a Banach space under each of the equivalent norms N_{Φ} and $\|\cdot\|_{\Phi}$ defined for $f\in L^{\Phi}$ by $N_{\Phi}(f)=\inf\{K>0\colon \int_{\Omega}\Phi(|f|/K)\,d\mu\leqslant 1\}$ and $\|f\|_{\Phi}=\sup\{\int_{\Omega} fg\,d\mu\colon g\in L^{\Psi}\ N_{\Psi}(g)\leqslant 1\}$. If $\mathfrak{X}$ is a Banach space and t is a bounded linear operator mapping L^{Φ} into $\mathfrak{X}$, Dinculeanu has defined $|||t|||_{\Phi}$ by

$$|||t|||_{\Phi}=\sup\sum_{i=1}^{n}\|a_i t(\chi_{E_i})\|,$$

where the supremum is taken over all measurable simple functions, $f=\sum_{i=1}^{n}a_i\chi_{E_i}$, $\{E_i\}\subset\Sigma$ disjoint, such that $N_{\Phi}(f)\leqslant 1$. This norm for operators has been the subject of some study by Dinculeanu in [1], [2], and [3]. The purpose of this note centers around proving a Bochner integral representation theorem for these operators, examining their compactness properties and looking at their rather close relationship with operators of finite double norm [8].

2. Operators with $|||t|||_\Phi < \infty$. This section is concerned with operators $t\colon L^\Phi \to \mathfrak{X}$ where $\mathfrak{X}$ is either reflexive or is a separable dual of a Banach space, which satisfy $|||t|||_\Phi < \infty$. Radon–Nikodym theorems for vector measures will be used to obtain a Bochner integral representation for these operators. The section will then conclude by looking at compactness properties of these operators. Recall that a Young's function Φ obeys the Δ_2-condition if there exists a finite constant M such that $\Phi(2x) \leqslant M\Phi(x)$ for all x.

THEOREM 1. *Let Φ obey the Δ_2-condition and let $\mathfrak{X}$ be a Banach space which is either reflexive or is a separable dual space. Then $t\colon L^\Phi \to \mathfrak{X}$ has $|||t|||_\Phi < \infty$ if and only if there exists a strongly measurable $g\colon \Omega \to \mathfrak{X}$ such that $\|g\| \in L^\Phi$ and $t(f) = \int_\Omega fg d\mu$; $f \in L^\Phi$; where the integral is the Bochner integral. In this case $|||t|||_\Phi = \|\,\|g\|\,\|_\Psi$.*

Proof. (Necessity) First assume $\mu(\Omega) < \infty$. Define $G\colon \Sigma \to \mathfrak{X}$ by $G(E) = t(\chi_E)$. Since t is bounded and linear, we find that if $E_n \to E$, $E_n \in \Sigma$, then $\|G(E) - G(E_n)\| \leqslant \|t\|\,\|\chi_E - \chi_{E_n}\|_\Phi \to 0$. Since G is clearly finitely additive the above limit shows G is countably additive and a similar computation shows G is μ-continuous. Next choose the constant $a > 0$ such that $N_\Phi(a\chi_\Omega) = 1$ and consider for any finite disjoint collection $\{E_n\} \subset \Sigma$, $\bigcup_{n=1}^m E_n = \Omega$. $a\chi_\Omega = \sum_{n=1}^m a\chi_{E_n}$. Then $a\sum_{n=1}^m \|G(E_n)\| = a\sum_{n=1}^m \|t(\chi_{E_n})\|$ $= \sum_{n=1}^m \|t(a\chi_{E_n})\| \leqslant |||t|||_\Phi < \infty$ definition of $|||t|||_\Phi$. Hence G is of bounded variation. Now since $\mathfrak{X}$ is either reflexive or a separable dual space, Phillips' Radon–Nikodym Theorem [7, p. 134], or the Dunford Pettis Theorem [4, pp. 344–45] respectively establish the existence of a strongly measurable $g\colon \Omega \to \mathfrak{X}$ such that $\|g\| \in L^1$ and

$$G(E) = \int_E g d\mu \quad \text{for } E \in \Sigma.$$

Next it will be shown that $\|g\| \in L^\Psi$. For this note that for any decomposition $\{E_n\} \subset \Sigma$ of Ω into a finite disjoint sequence of sets if follows from $|||t|||_\Phi < \infty$ and the definition of G that

$$\sum_{n=1}^m |a_n|\,|G|(E_n) \leqslant |||t|||_\Phi$$

provided $f = \sum_{n=1}^m a_n\chi_{E_n}$ satisfies $N_\Phi(f) \leqslant 1$, where $|G|(E)$ is the variation of G on $E \in \Sigma$. Now since $\int_E \|g\| d\mu = |G|(E)$, $E \in \Sigma$, one has

$$\sum_{n=1}^m \|a_n t(\chi_{E_n})\| \leqslant \sum_{n=1}^m |a_n| \int_{E_n} \|g\| d\mu \leqslant |||t|||_\Phi$$

for all $f = \sum_{n=1}^m a_n\chi_{E_n}$ as above. Taking appropriate suprema yields the following equality

$$|||t|||_\Phi = \sup\Big\{\int_\Omega |f|\,\|g\| d\mu\colon f \text{ simple};\ N_\Phi(f) \leqslant 1\Big\}.$$

From this it follows that

$$|||t|||_\Phi = \sup\Big\{\int_\Omega |f|\,\|g\| d\mu\colon N_\Phi(f) \leqslant 1\Big\}.$$

A check of the definition of $|||\ |||_\Phi$ shows then that

$$|||t|||_\Phi = \|\,\|g\|\,\|_\Psi.$$

Now since $\|g\| \in L^\Psi$,

$$\int_\Omega \|fg\| d\mu < \infty \quad \text{for all} \quad f \in L^\Psi.$$

Hence $\bar{t}(f) = \int_\Omega fg d\mu$ exists for all $f \in L^\Phi$ and since $\|\bar{t}(f)\| \leqslant \int_\Omega |f|\,\|g\| d\mu$ $\leqslant \|f\|_\Phi \|\,\|g\|\,\|_\Psi$, $\bar{t}$ is bounded. But if $f = \sum_{n=1}^m a_n\chi_{E_n}$ is simple, then

$$t(f) = \sum_{n=1}^m a_n t(\chi_{E_m}) = \sum_{n=1}^m a_n G(E_m) = \sum_{n=1}^m a_n \int_{E_n} g d\mu = \int_\Omega fg d\mu = \bar{t}(f).$$

But since Φ obeys the Δ_2-condition, simple functions are dense in L^Φ's thus $t(f) = \int_\Omega fg d\mu$ for all $f \in L^\Phi$. This proves the necessity in the case of a finite measure. The σ-finite case can be proved using usual techniques.

The proof of the sufficiency follows from an application of the Hölder inequality and will be omitted. ■

The second and final result of this section is

COROLLARY 2. *If in addition to the hypothesis of Theorem 1, Ψ also obeys the Δ_2-condition, then every $t\colon L^\Phi \to \mathfrak{X}$ with $|||t|||_\Phi < \infty$ is compact.*

Proof. Let the $\mathfrak{X}$ valued strongly measurable function g satisfy

$$t(f) = \int_\Omega fg d\mu \quad (f \in L^\Phi)$$

and $\|g\| \in L^\Psi$. Choose a sequence [5, p. 117] $\{g_n\}$ of simple functions such that $\|g_n\| \leqslant 2\|g\|$ a. e. and $\lim g_n = g$ a. e. Then for any $K > 0$ $\Psi(\|g_n - g\|/K) \to 0$ a. e. Also $\Psi(\|g_n - g\|/K) \leqslant \Psi((\|g_n\| + \|g\|)/K) \leqslant \Psi(3\|g\|/K)$ which is integrable since Ψ obeys the Δ_2-condition. Hence for any $K > 0$, $\lim_n \int_\Omega \Psi(\|g_n - g\|/K) d\mu = 0$ by the dominated convergence theorem. From this it follows that $N_\Phi(\|g_n - g\|) \to 0$.

But now consider $t_n\colon L^\Phi \to \mathfrak{X}$ defined by $t_n(f) = \int_\Omega fg_n d\mu\ f \in L^\Phi$. The operators t_n are bounded, and in fact are compact since their range is contained in the span of the finite set of values of g_n for each n. Moreover

$$\|t-t_n\| = \sup_{\|f\|_\Phi\le 1} \left\| \int f(g-g_n)\,d\mu \right\| \le \sup_{\|f\|_\Phi \le 1} \int |f|\,\|g-g_n\|\,d\mu \le N_\Psi(\|g-g_n\|)$$

by the Hölder inequality. Hence $\lim_n \|t-t_n\| = 0$ and t is compact.

3. Operators of finite double norm. This section is devoted to the connection between linear operators of finite double norm $t\colon L^\Phi \to L^\Phi$ and linear operators $t\colon L^\Phi \to L^\Phi$ with $|||t|||_\Phi$ finite. It will be shown that under a fairly generous hypothesis, the two classes of operators are adjoints of each other. To this end, recall that a bounded linear operator $t\colon L^\Phi \to L^\Phi$ is of finite double norm [8, p. 177] if there exists a $\mu\times\mu$-measurable function $g\colon \Omega\times\Omega \to R$ such that

(i) the section $g(s,\cdot) \in L^\Psi$ (Ψ complementary to Φ) for almost all $s \in \Omega$;

(ii) the function $z\colon \Omega \to R$ defined by $z(s) = \|g(s,\cdot)\|_\Psi$ belongs to L^Φ, and

(iii) for each $f \in L^\Phi$ and for almost all $s \in \Omega$

$$t(f)(s) = \int_\Omega f(r)g(s,r)\mu(dr).$$

In this case the double norm of t is given by

$$|||t||| = \|z\|_\Phi = \|(\|g(\cdot,\cdot)\|_\Psi)\|_\Phi.$$

Probably the most famous operators of finite double norm are the Hilbert–Schmidt operators [5, p. 1009] which are precisely those operators of finite double norm when $\Phi(x) = |x|^2/2$; i. e. when $L^\Phi = L^\Psi = L^2$. Operators of finite double norm are discussed in some detail in [8]. The following theorem characterizes operators of finite double norm.

Theorem 3. *Let Φ and its complementary function Ψ each obey the Δ_2-condition. Then a bounded linear operator $t\colon L^\Phi \to L^\Phi$ is of finite double norm if and only if its adjoint $t^*\colon L^\Psi \to L^\Psi$ satisfies $|||t^*|||_\Psi < \infty$. In this case $|||t|||_\Phi = |||t^*|||_\Psi$. In particular if $L_\Phi = L^2$, $|||t|||_\Phi < \infty$ if and only if t is a Hilbert–Schmidt operator.*

Proof. (Necessity) Suppose $t\colon L^\Phi \to L^\Phi$ is of finite double norm and that for $f \in L^\Phi$

$$t(f)(s) = \int_\Omega f(r)g(s,f)\mu(dr) \quad \text{a. e.}$$

where g satisfies (i), (ii) and (iii) above. Now if $h \in (L^\Phi)^* = L^\Psi$, since Φ obeys the Δ_2-condition, one finds

$$\begin{aligned}\int_\Omega t^*h(s)f(s)\mu(ds) &= \int_\Omega h(s)tf(s)\mu(ds)\\ &= \int_\Omega h(s)\Big(\int_\Omega f(r)g(s,r)\mu(dr)\Big)\mu(ds)\\ &= \int_\Omega f(r)\Big(\int_\Omega h(s)g(s,r)\mu(ds)\Big)\mu(dr),\end{aligned}$$

by the Fubini Theorem. Since this holds for all $h \in L^\Psi$ and for all $f \in L^\Phi$, it follows that

$$t^*(h)(r) = \int_\Omega h(s)g(s,r)\mu(ds) \quad \text{a. e.}$$

Now define the function $\bar{g}$ by $\bar{g}(s) = g(s,\cdot)$, $s \in \Omega$. By hypothesis $\bar{g}(s) \in L^\Psi$ for almost all $s \in \Omega$. Arguments entirely analogous to those of Dunford and Pettis [4, p. 336] show that $\bar{g}$ is strongly measurable as a vector-valued function. Also by (iii) above, $\|\bar{g}\|_\Psi \in L^\Phi$. Now applying [5, III. 11. 17], one finds

$$t^*h(r) = \int_\Omega h\bar{g}\,d\mu\ [r] \quad \text{a. e.}$$

and hence by Theorem 1,

$$|||t^*|||_\Psi = \|(\|\bar{g}\|_\Psi)\|_\Phi = |||t||| < \infty.$$

This proves the necessity.

To prove the sufficiency, suppose $|||t^*|||_\Psi < \infty$. Since, under the current hypothesis, L^Ψ is reflexive, Theorem 1 applies and produces a strongly measurable L^Ψ-valued g with $\|\,\|g\|_\Psi\|_\Phi < \infty$ satisfying

$$t^*(h) = \int_\Omega hg\,d\mu \quad \text{for all} \quad h \in L^\Psi.$$

Now in view of [5., III. 11. 17], which is valid for all the Orlicz spaces under consideration here, there exists a $\mu\times\mu$-measurable real valued $\tilde{g}$ on $\Omega\times\Omega$ such that

(a) $$\tilde{g}(\cdot,s) = g(s)(\cdot) \in L^\Psi \quad \text{a. e.}$$

(b) $$\int_E \tilde{g}(r,s)\mu(ds) = \int_E g(s)\mu(ds)(r) \quad \text{a. e.}$$

for all $E \in \Sigma$ of finite measure. Moreover since $\|g\|_\Psi \in L^\Phi$,

(c) $$\|\tilde{g}(\cdot,\cdot)\|_\Psi = \|g(\cdot)\|_\Psi \in L^\Phi.$$

From (2), one has that for almost all $r \in \Omega$

$$\int_\Omega f(s)\tilde{g}(r, s)\mu(ds) = \int_\Omega f(s)g(s)\mu(ds)(r)$$

whenever $f \in L^\Psi$ is simple. Since simple functions are dense in L^Ψ, it follows that for almost all $r \in \Omega$. $t^*(h)(r) = \int_\Omega h(s)\tilde{g}(s, r)\mu(ds)$ for $h \in L^\Psi$. Arguments the same as those used in the necessity show that

$$t(f)(s) = \int_\Omega f(r)\tilde{g}(s, r)\mu(dr) \quad \text{a. e.}$$

for all $f \in L^\Phi$. The fact that t is of finite double norm follows immediately from (c). ∎

References

[1] N. Dinculeanu, *Vector Measures*, New York, 1967.
[2] — *Integral representation of linear operation*, Queen's Papers in Pure and Applied Mathematics, no. 10, (1967), pp. 2–41.
[3] — *Representation integral des operations lineaires* (roumain) I, II, Studii Cercet. Mat. 18, (1966), pp. 349–385, pp. 483–536.
[4] N. Dunford, and B. J. Pettis, *Linear operations on summable functions*, Trans. Amer. Math. Soc. 47, (1940), pp. 323–392.
[5] — and J. T. Schwartz, *Linear Operators*, New York, 1958.
[6] W. A. J. Luxemburg, *Banach function spaces*, Thesis, Delft, 1955.
[7] R. S. Phillips, *On weakly compact subsets of a Banach space*, Amer. J. Math. 65 (1943), pp. 108–136.
[8] A. C. Zaanen, *Linear Analysis*, Amsterdam, 1953.

UNIVERSITY OF ILLINOIS
URBANA, ILLINOIS

Reçu par le Rédaction le 28. 4. 1970 (203)

STUDIA MATHEMATICA, T. XL. (1971)

On shrinking basic sequences in Banach spaces*

by

DAVID W. DEAN (Columbus), IVAN SINGER (Bucharest)
and LEONARD STERNBACH (S. Carolina)

Abstract. In § 1 we prove that a Banach space E with a basis $\{x_n\}$ contains a subspace with a separable conjugate space if and only if $\{x_n\}$ admits a shrinking block basic sequence. Hence, a Banach space E contains a subspace with a separable conjugate space if and only if E contains a shrinking basic sequence. In § 2 we prove that if E has a subspace with a separable conjugate space, then E^* (the conjugate of E) has a quotient space with a basis. In § 3 we prove that if E has a basis, then every shrinking basic sequence in E has a subsequence which can be extended to a basis of E. We also raise some related unsolved problems.

Introduction. A sequence $\{x_n\}$ in a Banach space E (we shall assume, without special mention, that $\dim E = \infty$ and that the scalars are real or complex) is called a *basis* if E if for every $x \in E$ there exists a unique sequence of scalars $\{a_n\}$ such that $x = \sum_{i=1}^{\infty} a_i x_i$. A sequence $\{z_n\} \subset E$ is said to be a *basic sequence* if $\{z_n\}$ is a basis of its closed linear span $[z_n]$. A sequence $\{z_n\} \subset E$ is called a *block basic sequence* with respect to a sequence $\{y_n\} \subset E$ if it is a basic sequence of the form $z_n = \sum_{i=m_{n-1}+1}^{m_n} \beta_i y_i \neq 0$ $(n = 1, 2, \ldots)$, where $\{m_n\}$ is an increasing sequence of positive integers and $m_0 = 0$; it is well known and easy to see that if $\{y_n\}$ is a basic sequence, then $\{z_n\}$ is necessarily a basic sequence. A basic sequence $\{z_n\} \subset E$ is called *shrinking*, if $\lim_n \|\chi|_{[z_n, z_{n+1}, \ldots]}\| = 0$ for all $\chi \in [z_n]^*$. Say that a basic sequence $\{z_n\}$ *can be extended to a basis of* E if there exists a basis $\{x_n\}$ of E and a sequence of positive integers $\{k_n\}$ such that $z_n = x_{k_n}$ $(n = 1, 2, \ldots)$.

In § 1 of the present paper we shall prove some results on the existence of shrinking basic sequences. Among other results, we shall prove that if E has a basis $\{x_n\}$, then E contains a subspace G having a separable

* The work of the first author was partially supported by NSF Grant GP9037. The second author was on leave from the Institute of Mathematics of the Romanian Academy of Sciences.